Frequency shift and viewing direction variations in gravitational lensing

Mikołaj Korzyński

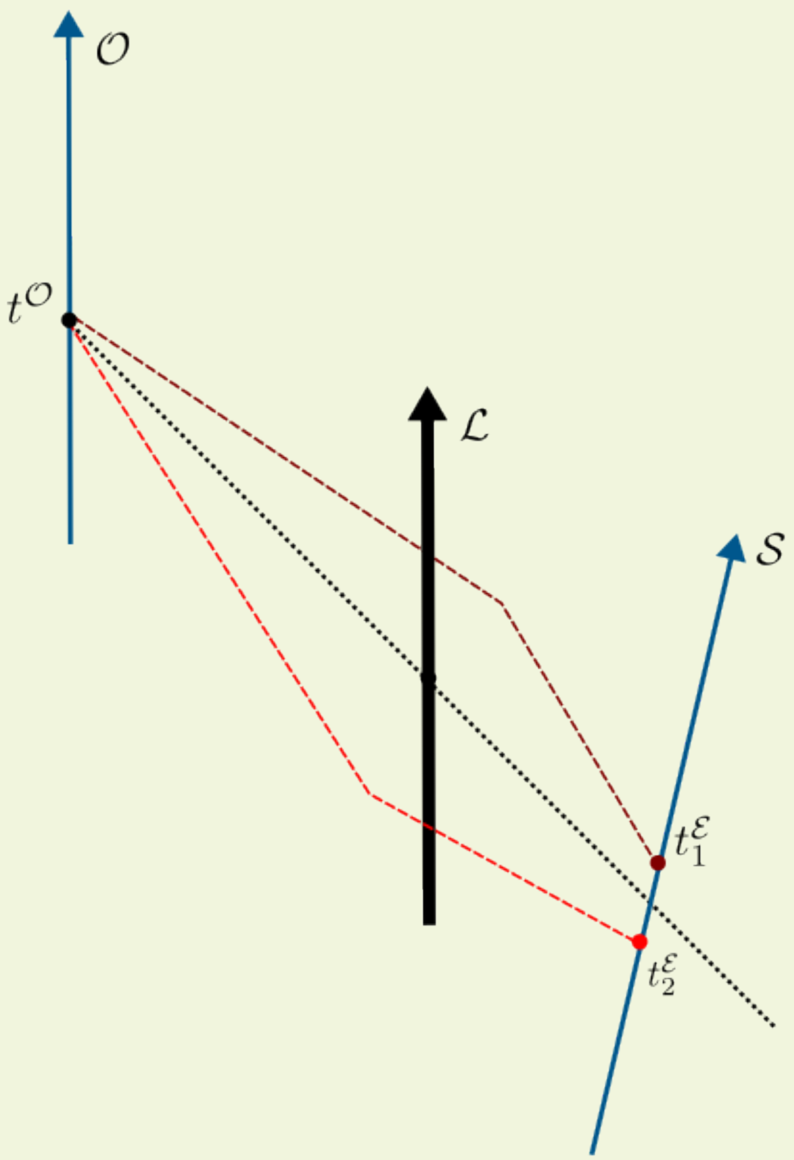
in collaboration with **Mateusz Kulejewski**

WEAVE Unisono project NCN 2024/06/Y/ST2/00190
„Lensing of electromagnetic and gravitational waves"

The 12th POTOR Conference, Będlewo, 4 July 2026

Introduction

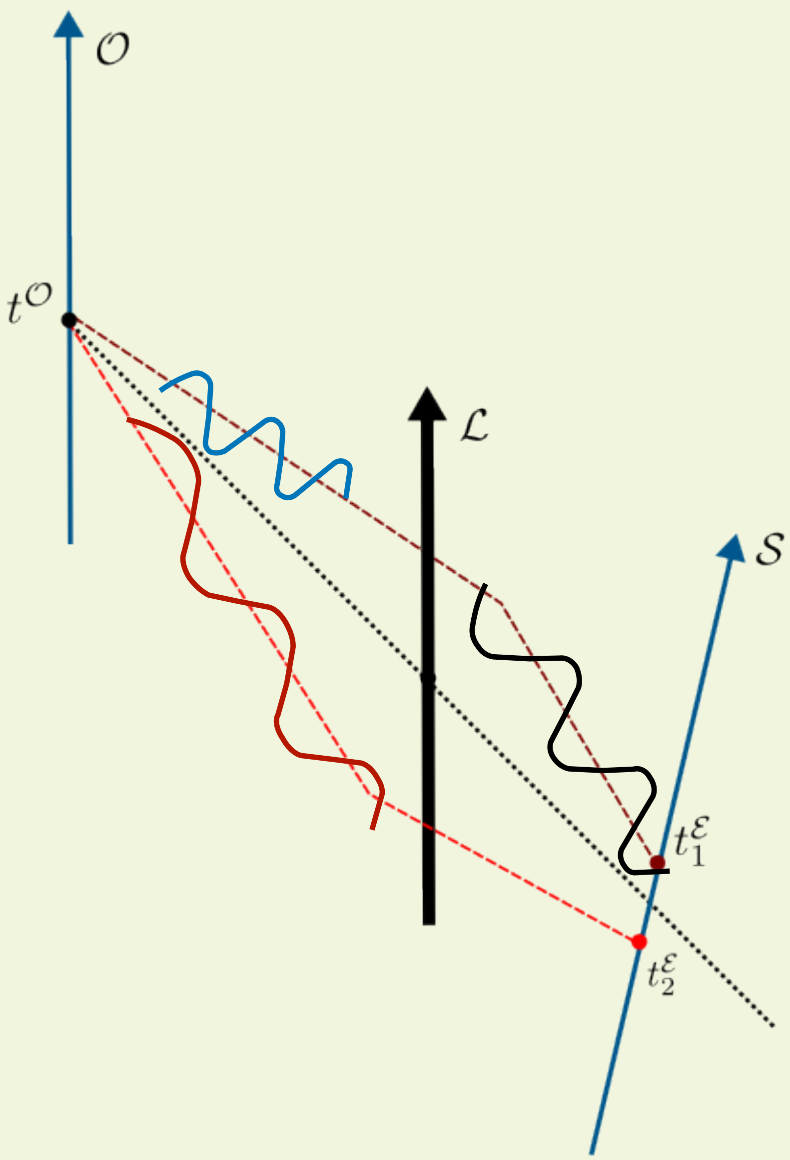
Gravitational lensing = effect of deflection of light rays by a massive body located between the source and observer



Credit: ESA/Hubble & NASA

Introduction

Gravitational lensing = effect of deflection of light rays by a massive body located between the source and observer

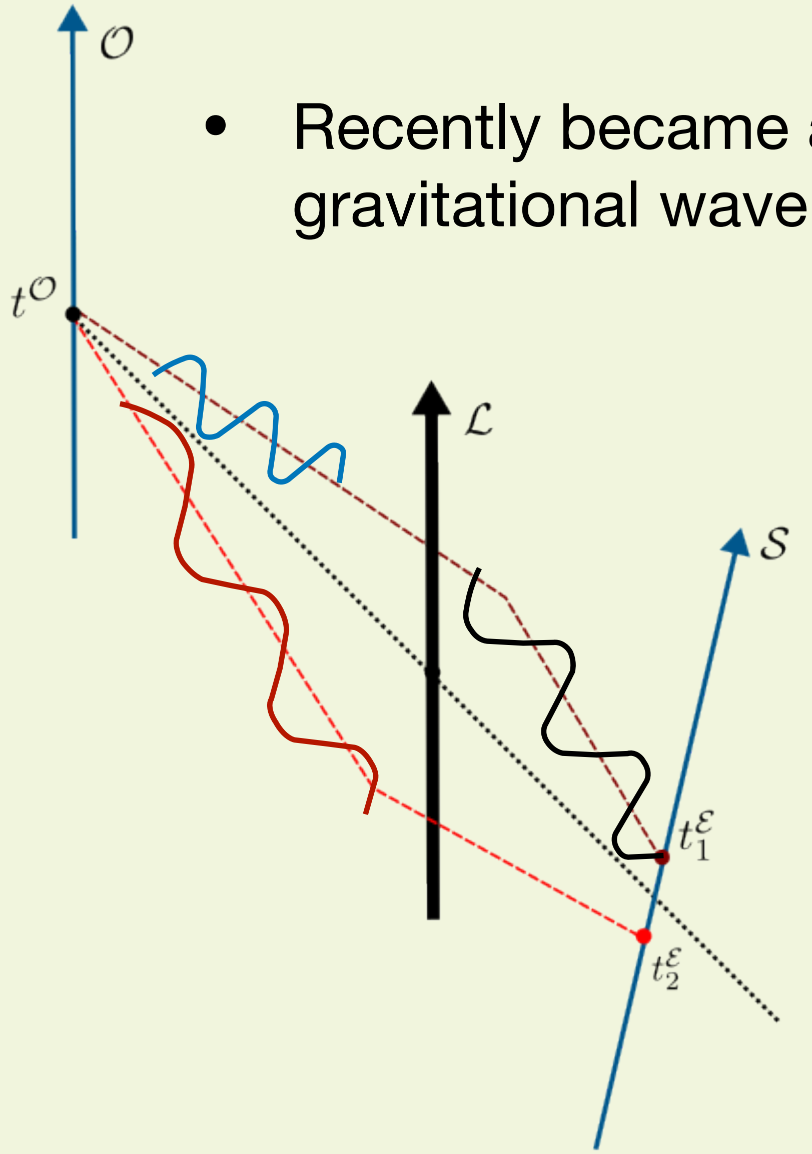


Credit: ESA/Hubble & NASA

If the lens, source and source are in relative motion, lensing causes a frequency shift as well

Frequency effect of lensing

- Requires non-radial motions
- Usually this is a very small effect
- Identical for electromagnetic and gravitational waves (within the geometric optics approximation)
- Recently became an interesting topic in both electromagnetic and gravitational wave lensing theory



- Microlensing of EM waves: frequency variation may help to break the degeneracy in microlensing observations [Rahvar 2020], signal $1 - 10\,\mathrm{cm\,s^{-1}}$
- Lensing of GW: affects the interference pattern of the two images of the waveform, can be used to measure the transverse velocities [Itoh et al. 2009, Savastano et al. 2024, Suyampraksam et al 2025, Samsing et al 2024, Ubach 2025…]

History of the problem

- First studies (small velocities, no cosmological corrections): Pyne and Birkinshaw 1993, Bertotti and Gambieri 1992

- Velocity corrections to the bending angles: Capozello *et al* 1999, Fritelli, Kling and Newman 2002: conflicting results

- …explained by Fritelli *et al* 2003 and Fritelli 2003, also discussing the frequency effect (small velocities, no cosmological corrections)

- Kopeikin and Schäfer 1999 - Lorentz covariant approach to light propagation near point masses (no cosmological corrections, only point masses)

- Wucknitz and Sperhake 2004: for small velocities and fixed position of the source

No paper containing the full formula for large velocities and with cosmological corrections

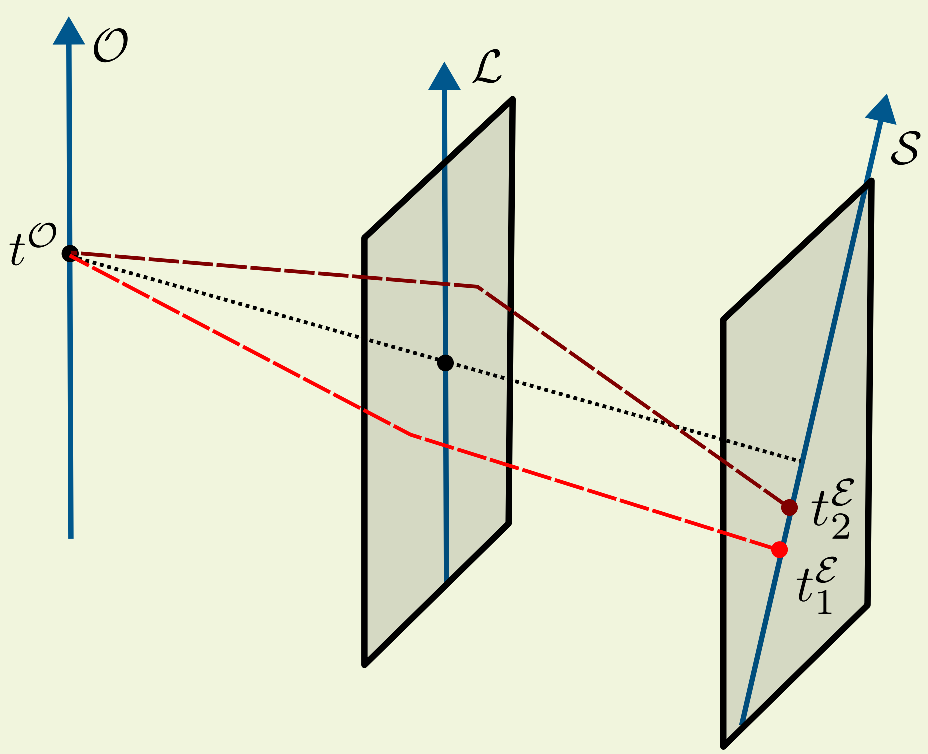
Idea of our paper

- Standard gravitational lensing formalism and thin lens approximation are applicable

- The lensing system is embedded in a FLRW Universe model, although the source, lens and observer may be moving wrt the cosmic frame

- In the vicinity of the lens the metric is approximately static **in the lens frame**, leading the the frequency being conserved **in the lens frame**

- No linearization in velocities, expressions valid also for relativistic motions



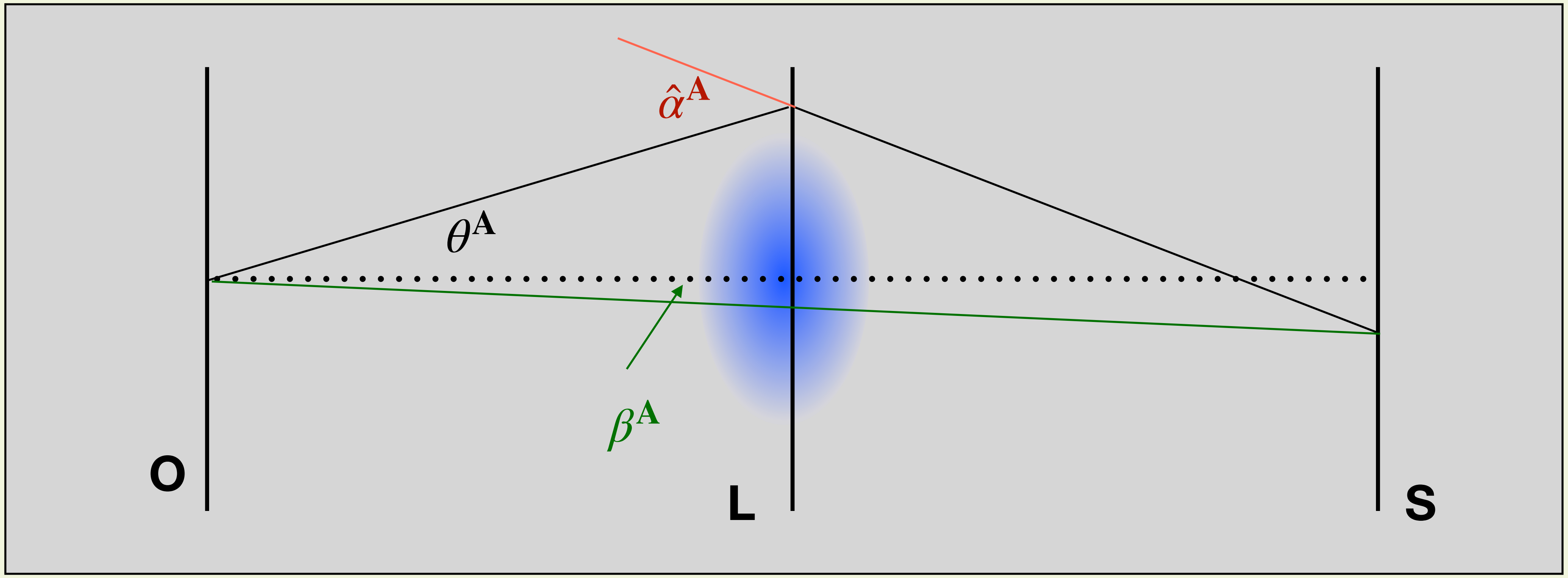
- MK, M. Kulejewski *„Frequency shift and viewing direction variations in gravitational lensing”*

arXiv:2601.13912 [astro-ph.CO]

accepted in Phys. Rev. D

Standard gravitational lensing framework

- Gravitational lensing formalism



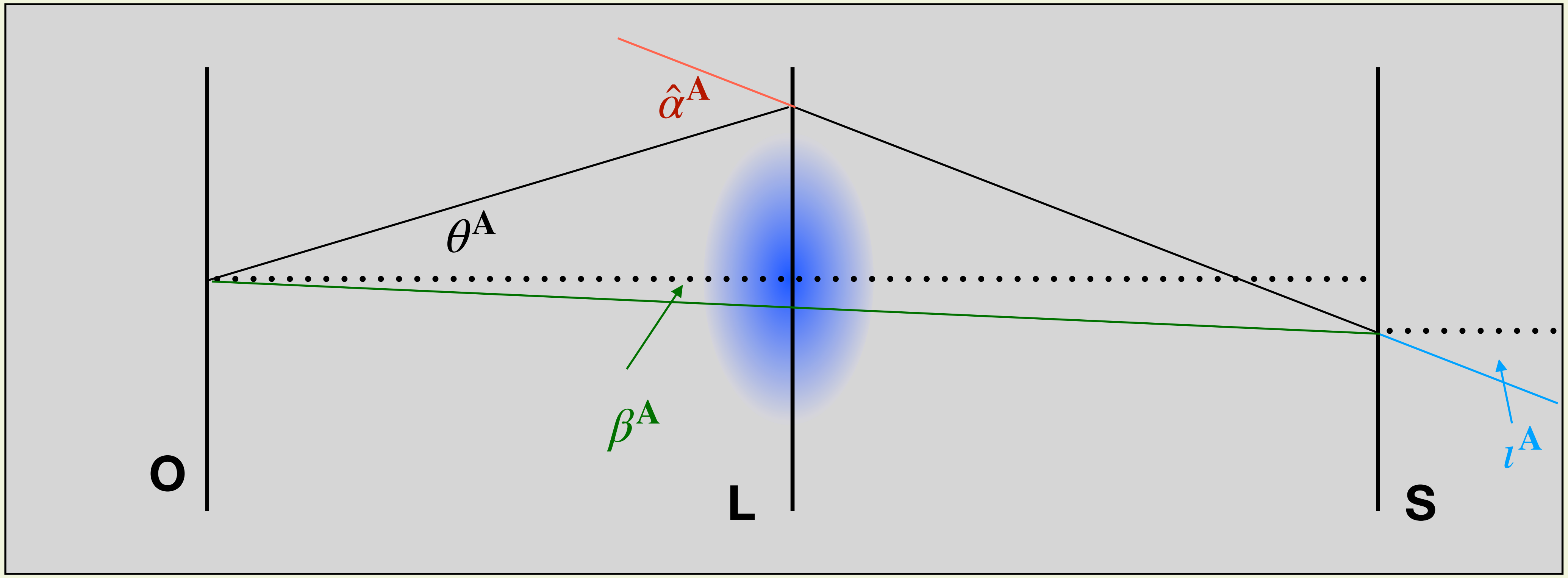
- Lensing equation $\beta^{\mathbf{A}}(\theta^{\mathbf{B}}) = \theta^{\mathbf{A}} - \alpha^{\mathbf{A}}(\theta^{\mathbf{B}})$

$$\alpha^{\mathbf{A}} = \frac{D_{LS}}{D_S}\hat{\alpha}^{\mathbf{A}} = \frac{\partial \psi}{\partial \theta^{\mathbf{A}}}$$

Given the velocities v_O, v_L, v_S, what is the frequency measured by the observer?

Standard gravitational lensing framework

- Gravitational lensing formalism

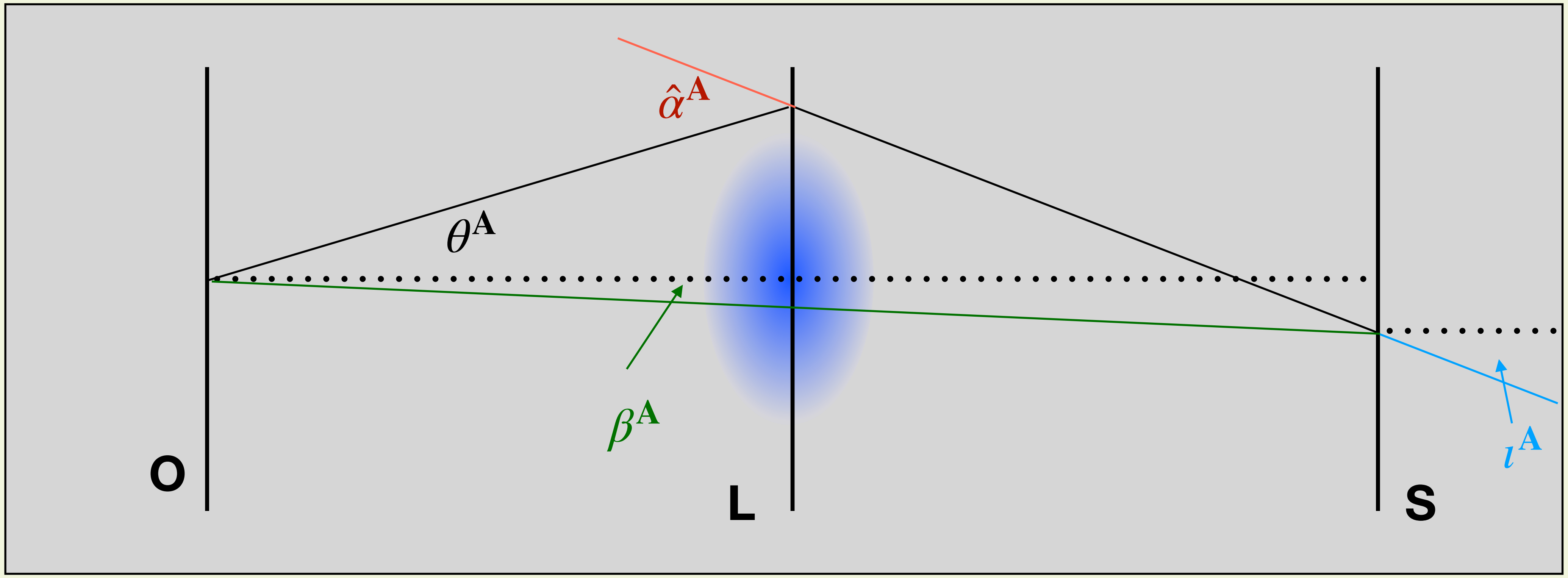


- Lensing equation $\beta^{\mathbf{A}}(\theta^{\mathbf{B}}) = \theta^{\mathbf{A}} - \alpha^{\mathbf{A}}(\theta^{\mathbf{B}})$

viewing direction variation angle $\iota^{\mathbf{A}}$: not a part of the standard formalism

Standard gravitational lensing framework

- Gravitational lensing formalism



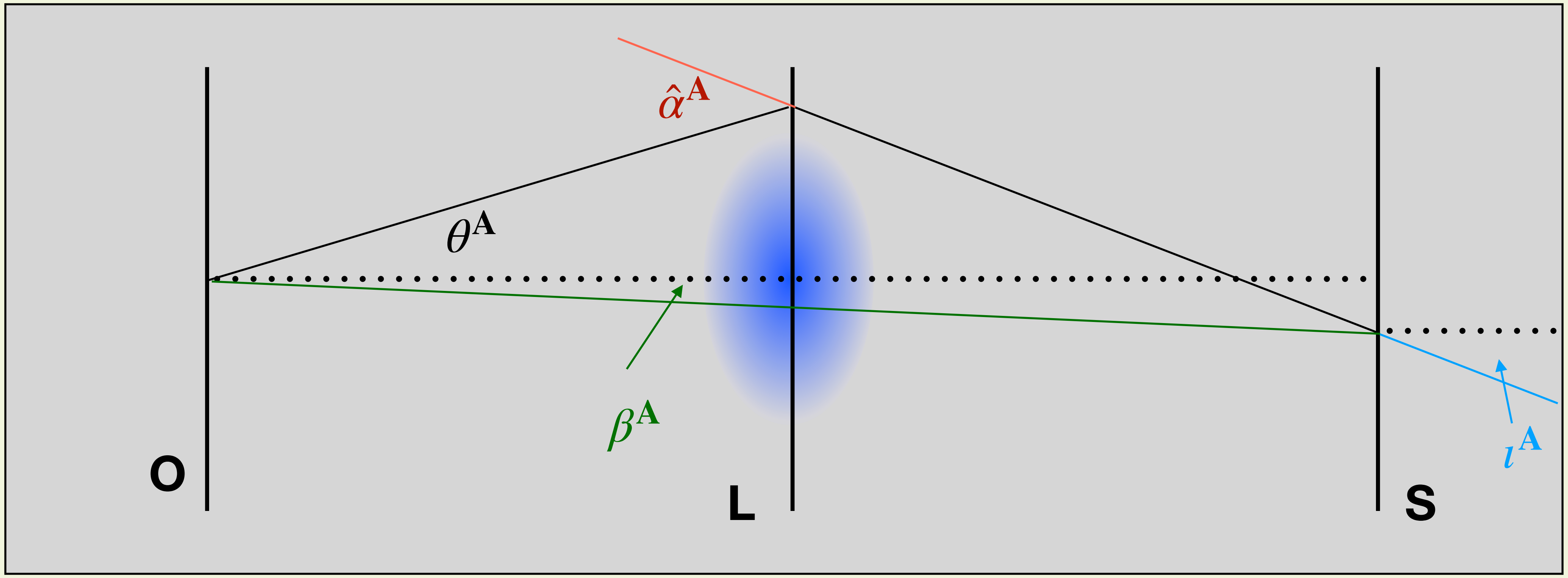
- Lensing equation $\beta^{\mathbf{A}}(\theta^{\mathbf{B}}) = \theta^{\mathbf{A}} - \alpha^{\mathbf{A}}(\theta^{\mathbf{B}})$

viewing direction variation angle $\iota^{\mathbf{A}}$: not a part of the standard formalism

necessary for the frequency effect

Standard gravitational lensing framework

- Gravitational lensing formalism



- Lensing equation $\beta^{\mathbf{A}}(\theta^{\mathbf{B}}) = \theta^{\mathbf{A}} - \alpha^{\mathbf{A}}(\theta^{\mathbf{B}})$

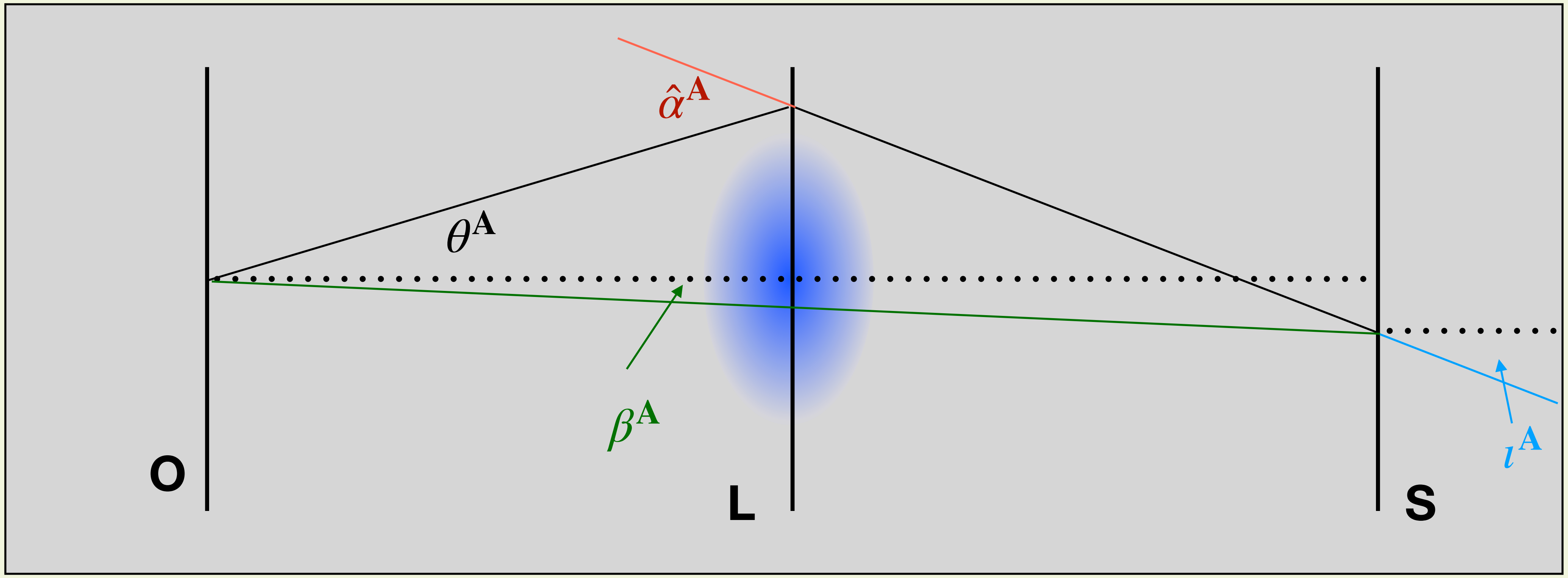
viewing direction variation angle $\iota^{\mathbf{A}}$: not a part of the standard formalism

necessary for the frequency effect

potentially important for GW sources (compact coalescing binaries)

Standard gravitational lensing framework

- Gravitational lensing formalism



- Lensing equation $\beta^{\mathbf{A}}(\theta^{\mathbf{B}}) = \theta^{\mathbf{A}} - \alpha^{\mathbf{A}}(\theta^{\mathbf{B}})$

viewing direction variation angle $\iota^{\mathbf{A}}$: not a part of the standard formalism

necessary for the frequency effect

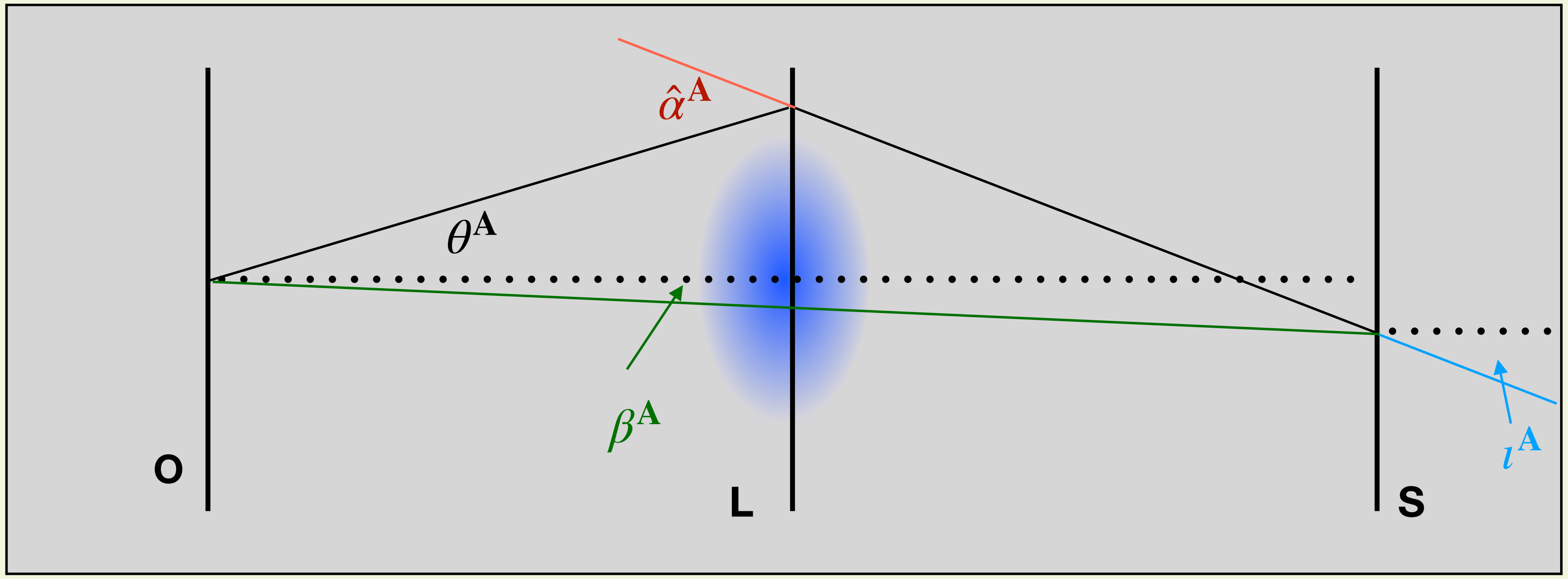
potentially important for GW sources (compact coalescing binaries)

What is $\iota^{\mathbf{A}}$ expressed in terms of $\theta^{\mathbf{A}}$, $\beta^{\mathbf{A}}$?

Derivation

How do you derive these equations? Ingredients:

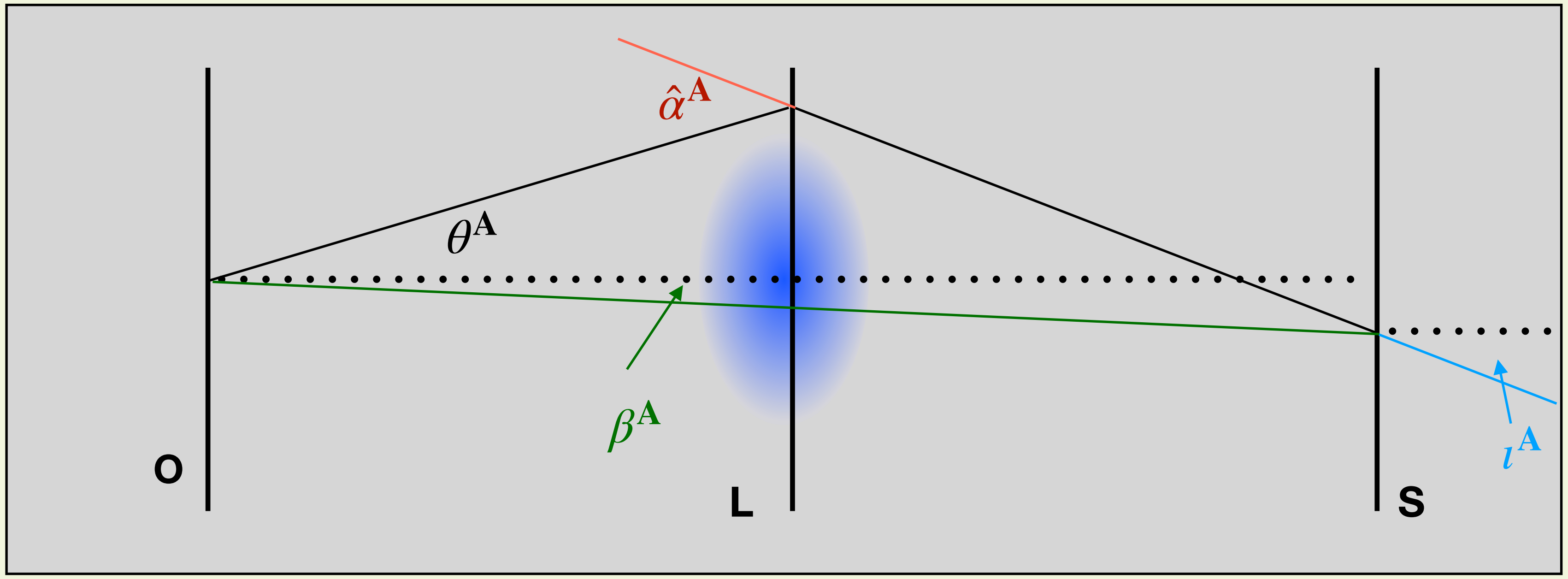
- Standard gravitational lensing theory



Derivation

How do you derive these equations? Ingredients:

- Standard gravitational lensing theory



- Geodesic deviation equation for null geodesics for light propagation

$$\nabla_l \begin{pmatrix} \xi^\mu \\ \nabla_l \xi^\mu \end{pmatrix} = \begin{pmatrix} 0 & \delta^\mu{}_\beta \\ R^\mu{}_{ll\alpha} & 0 \end{pmatrix} \begin{pmatrix} \xi^\alpha \\ \nabla_l \xi^\beta \end{pmatrix}$$

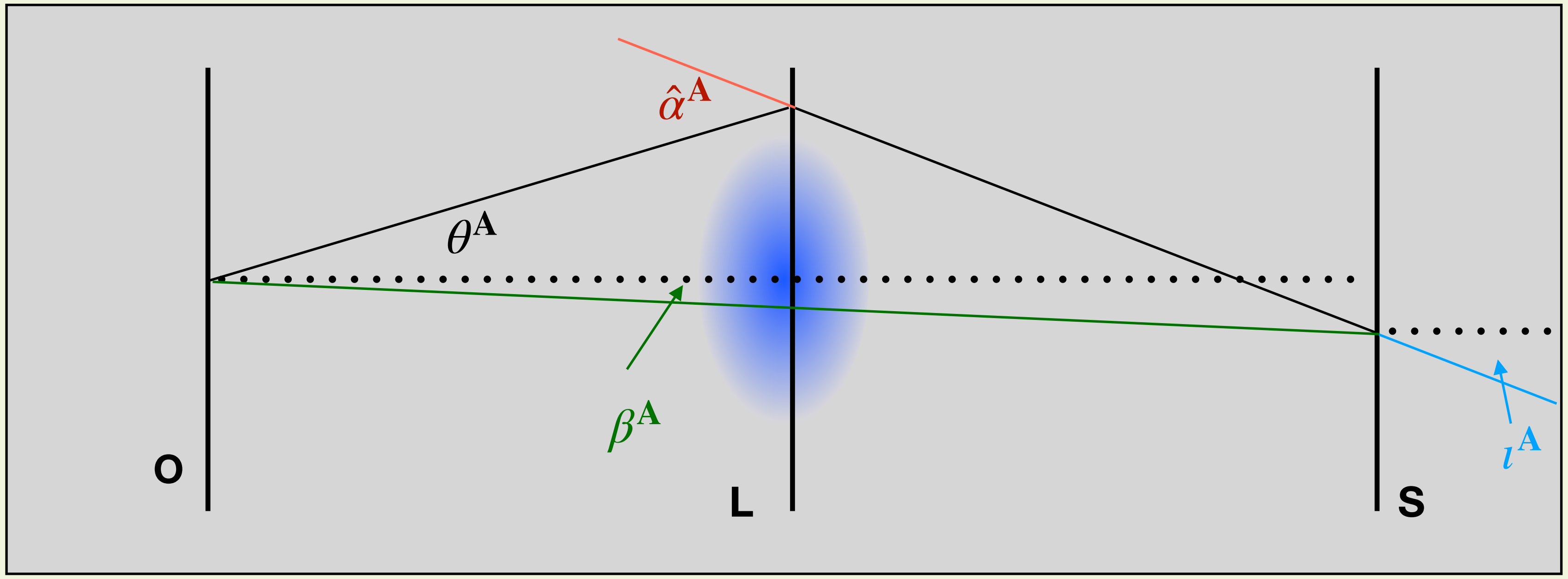
- Resolvent $$\begin{pmatrix} \xi^\mu(\lambda) \\ \nabla_l \xi^\mu(\lambda) \end{pmatrix} = \mathbf{W}(\lambda) \begin{pmatrix} \xi^\mu(0) \\ \nabla_l \xi^\mu(0) \end{pmatrix}$$

[Grasso et al 2019], [MK, Villa 2020], [MK, Uzun 2024]

Derivation

How do you derive these equations? Ingredients:

- Standard gravitational lensing theory



- Geodesic deviation equation for null geodesics for light propagation

$$\nabla_l \begin{pmatrix} \xi^\mu \\ \nabla_l \xi^\mu \end{pmatrix} = \begin{pmatrix} 0 & \delta^\mu{}_\beta \\ R^\mu{}_{ll\alpha} & 0 \end{pmatrix} \begin{pmatrix} \xi^\alpha \\ \nabla_l \xi^\beta \end{pmatrix}$$

  - Resolvent $$\begin{pmatrix} \xi^\mu(\lambda) \\ \nabla_l \xi^\mu(\lambda) \end{pmatrix} = \mathbf{W}(\lambda) \begin{pmatrix} \xi^\mu(0) \\ \nabla_l \xi^\mu(0) \end{pmatrix}$$

[Grasso et al 2019], [MK, Villa 2020], [MK, Uzun 2024]

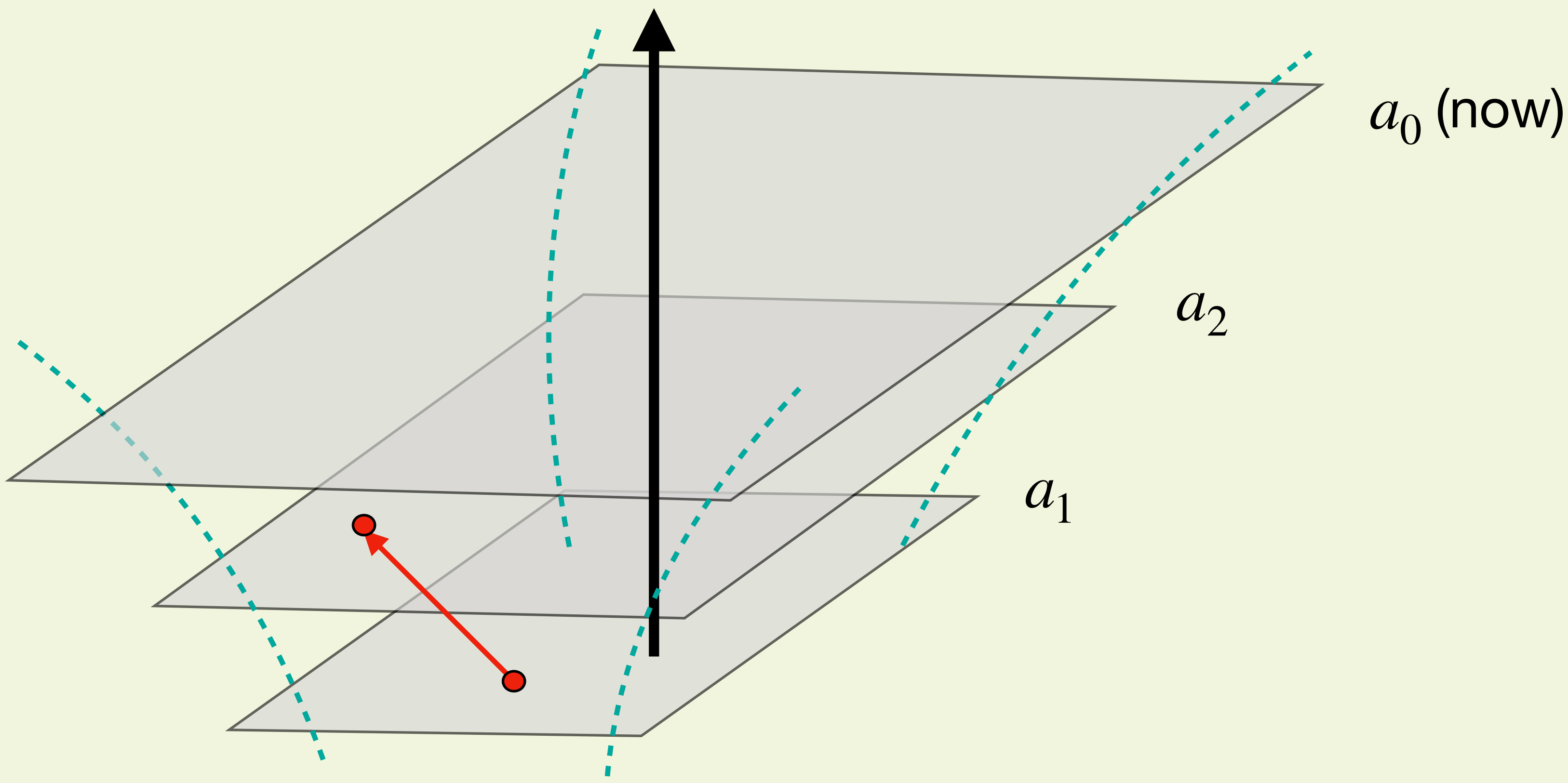
- Geometry of null tetrads and null geodesics

Main results

$$g = -\,dt^2 + a(t)^2\,\left(d\chi^2 + S_k(\chi)^2\,d\Omega^2\right)$$

FRLW metric

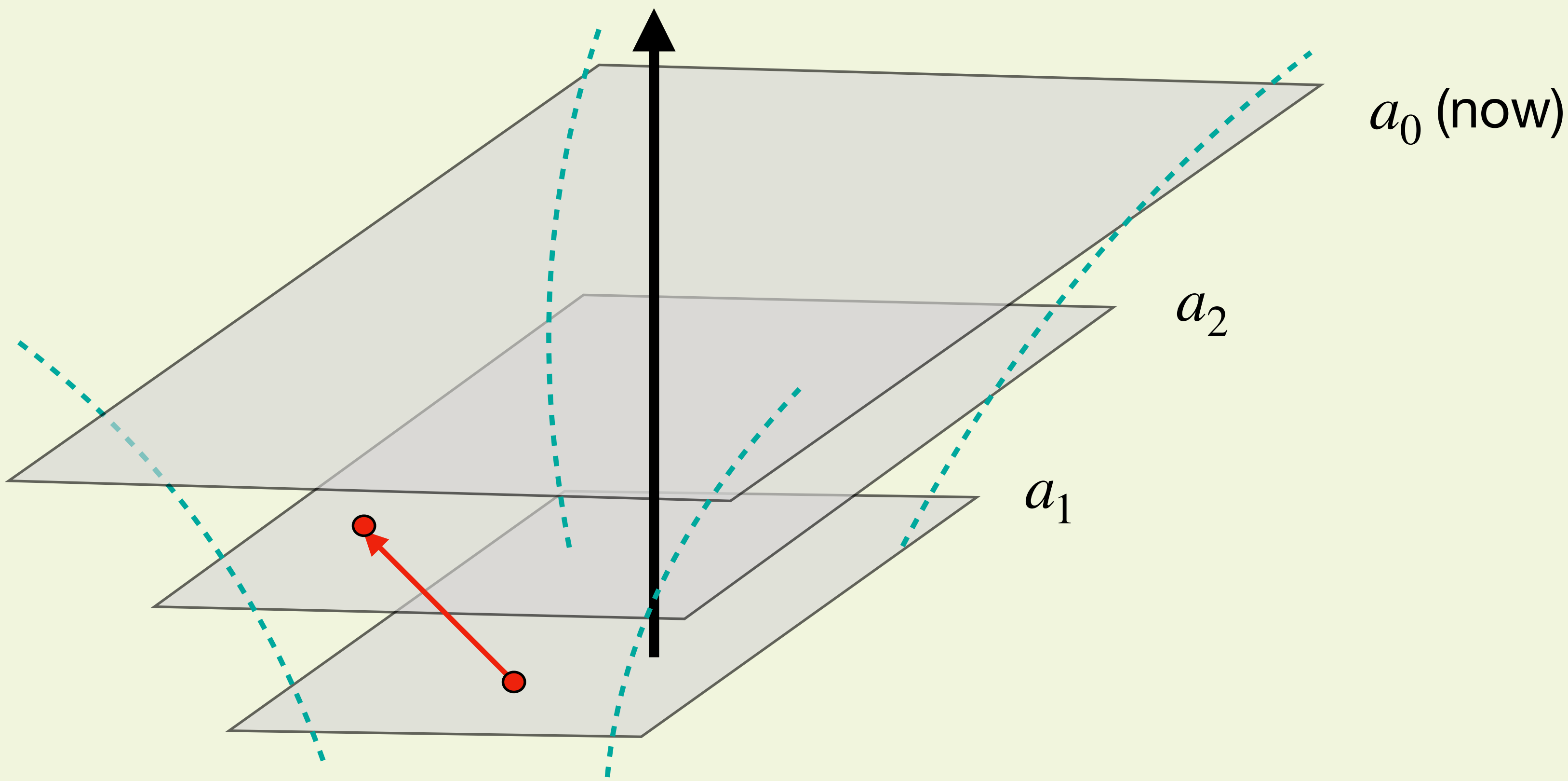
$$k = \pm\,1, 0$$



Main results

$g = -\,dt^2 + a(t)^2\,\left(d\chi^2 + S_k(\chi)^2\,d\Omega^2\right)$ **FRLW metric**

$k = \pm 1, 0$

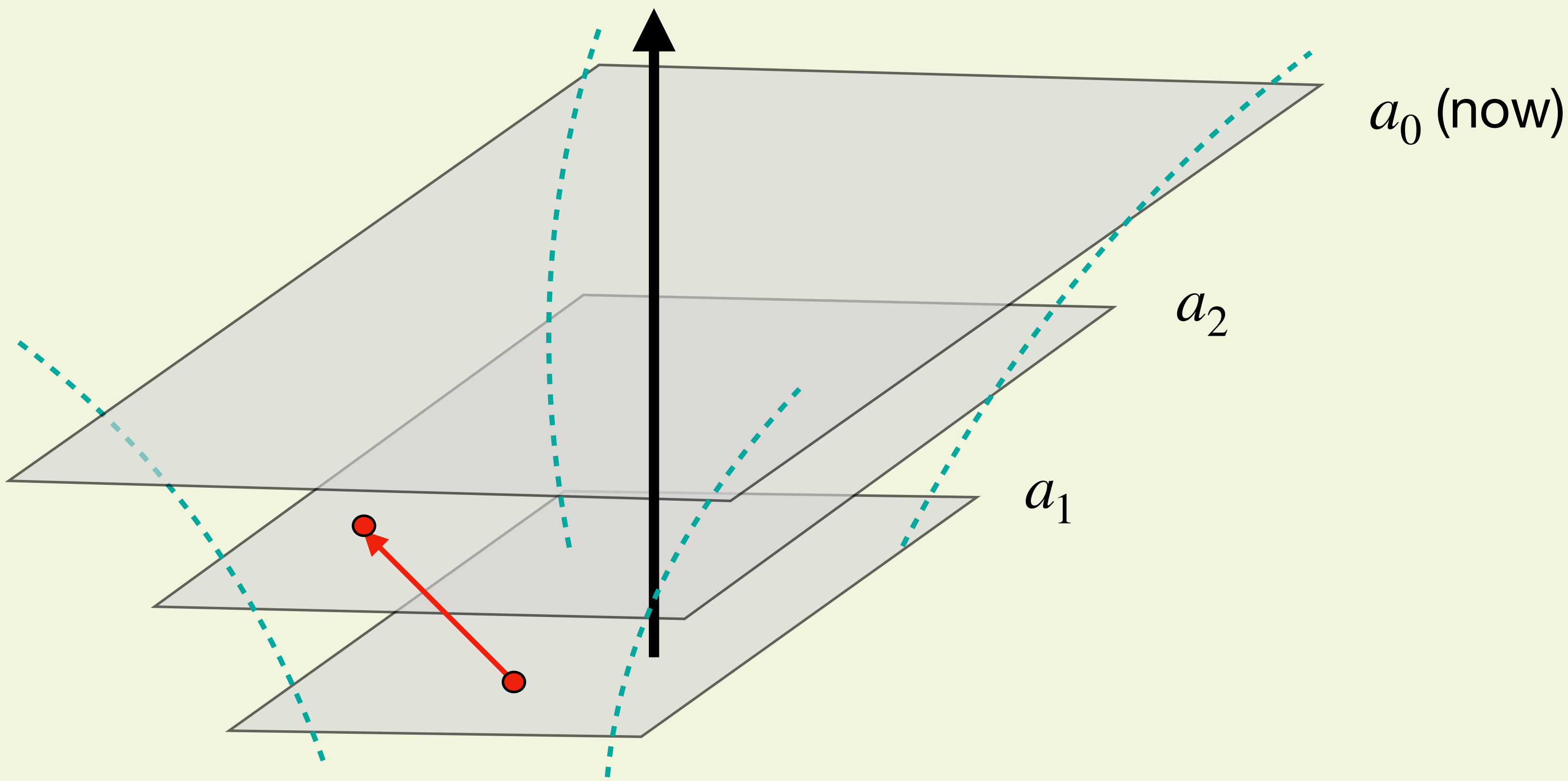


$z_1 = \dfrac{a_0}{a_1} - 1$ **redshift**

Main results

$g = -dt^2 + a(t)^2 \left(d\chi^2 + S_k(\chi)^2 d\Omega^2\right)$ **FRLW metric**

$k = \pm 1, 0$



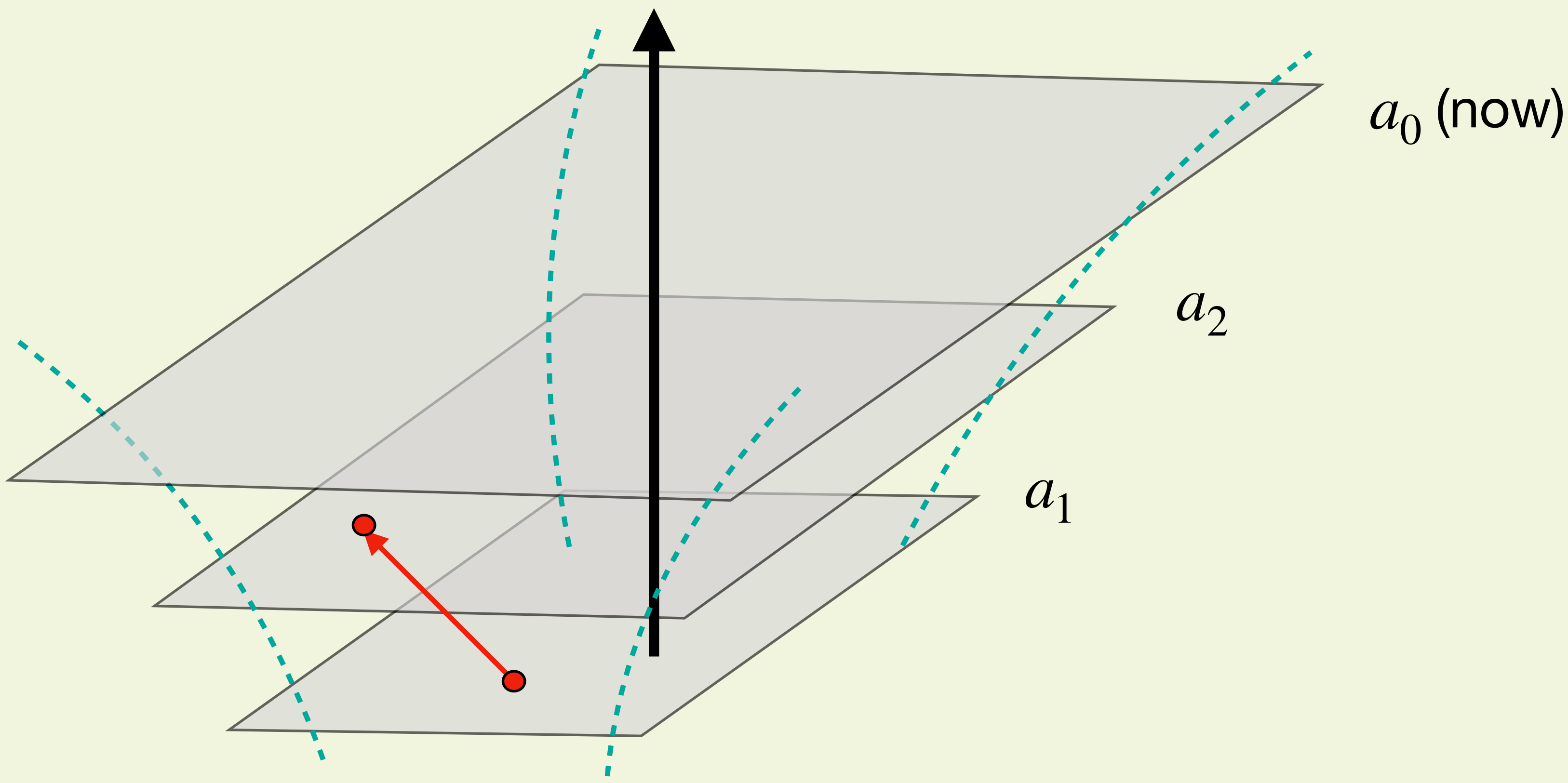
$z_1 = \dfrac{a_0}{a_1} - 1$ **redshift**

$D_{12} = a_1 S_k(\chi_{12})$ **angular diameter distance between a_1 and a_2**

Main results

$$g = -\,dt^2 + a(t)^2\,\left(d\chi^2 + S_k(\chi)^2\,d\Omega^2\right)$$ **FRLW metric**

$$k = \pm 1, 0$$



$$z_1 = \frac{a_0}{a_1} - 1$$ **redshift**

$$D_{12} = a_1\,S_k(\chi_{12})$$ **angular diameter distance between a_1 and a_2**

$$\Sigma(a_1, a_2) = \sqrt{1 - \Omega_{k,0}\,(H_0\,D_{12})^2(1+z_1)^2} - H_1\,D_{12}$$ **dimensionless coefficient, close to 1**

Main results

Covariant transverse 4-velocity difference $\mathcal{T}_{U,l}(u) := \dfrac{l \cdot U}{l \cdot u} u - U$

$$\mathcal{T}_{l,U}(u) \cdot l = 0$$

only the transverse components matter

$$\xi \cdot l = 0 \Longrightarrow \xi \cdot \mathcal{T}_{U,l}(u) = \xi_{\mathbf{A}} \mathcal{T}_{U,l}(u)^{\mathbf{A}}$$

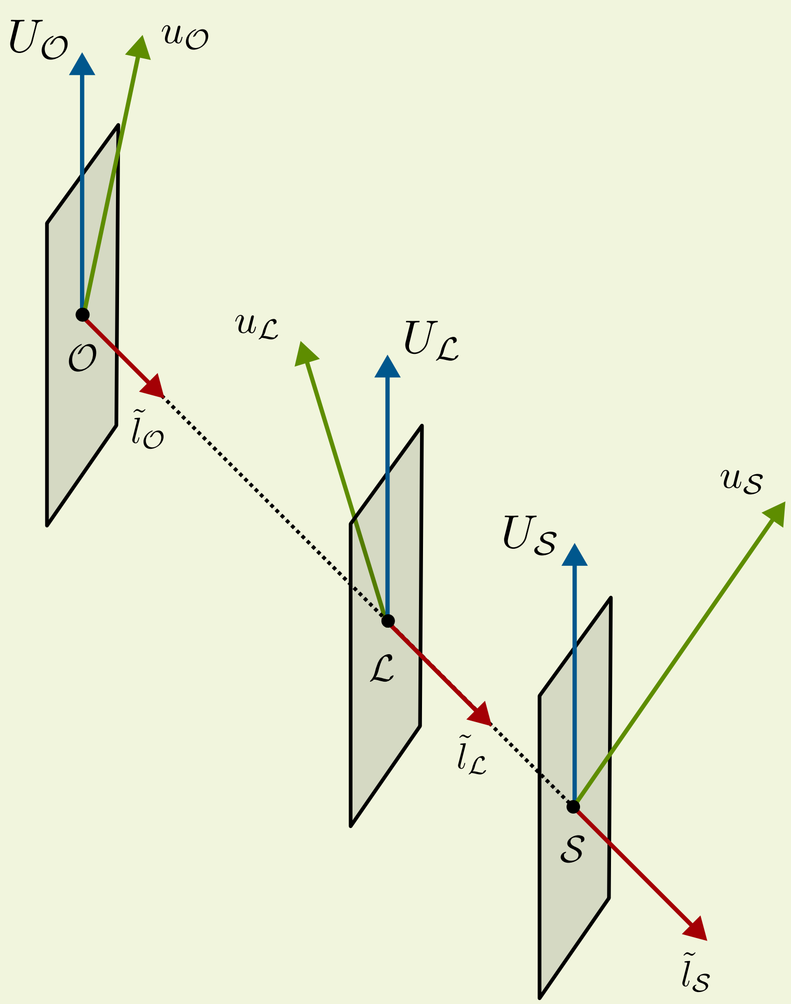
transverse velocity difference for small v

$$\mathcal{T}_{l,U}(U + v)^{\mathbf{A}} = v^{\mathbf{A}} + O(v^2)$$

Main results

Covariant transverse 4-velocity difference $\mathscr{T}_{U,l}(u) := \dfrac{l \cdot U}{l \cdot u} u - U$

$$\mathscr{T}_{l,U}(u) \cdot l = 0$$



only the transverse components matter

$$\xi \cdot l = 0 \Longrightarrow \xi \cdot \mathscr{T}_{U,l}(u) = \xi_{\mathbf{A}} \mathscr{T}_{U,l}(u)^{\mathbf{A}}$$

transverse velocity difference for small v

$$\mathscr{T}_{l,U}(U + v)^{\mathbf{A}} = v^{\mathbf{A}} + O(v^2)$$

short-hand notation:

$$\mathscr{T}_{l,U_O}(u_O)^{\mathbf{A}} =: \mathscr{T}(u_O)^{\mathbf{A}}$$

Main result: frequency variation

$$\Delta\zeta \equiv \frac{\Delta z}{1+z} = \frac{l_O \cdot u_O}{l_O \cdot U_O}\left(\theta^{\mathbf{A}}\mathscr{V}_{\mathbf{A}} + \beta^{\mathbf{A}}\mathscr{W}_{\mathbf{A}}\right)$$

$$\mathscr{V}_{\mathbf{A}} = -\,\mathscr{T}(u_O)_{\mathbf{A}} + \frac{D_{SO}}{D_{SL}}\mathscr{T}(u_L)_{\mathbf{A}} + \frac{a_S}{a_L}\frac{D_{LO}}{D_{SL}}\mathscr{T}(u_S)_{\mathbf{A}}$$

combination of all 3 transverse velocities

$$\mathscr{W}_{\mathbf{A}} = \frac{D_{SO}}{D_{SL}}\left(-\mathscr{T}(u_L)_{\mathbf{A}} + \Sigma(a_S, a_L)\,\mathscr{T}(u_S)_{\mathbf{A}}\right)$$

combination of u_L and u_S, no u_O

$$\frac{l_O \cdot u_O}{l_O \cdot U_O}$$

stellar aberration pre-factor

Main result: frequency variation

$$\Delta\zeta \equiv \frac{\Delta z}{1+z} = \frac{l_O \cdot u_O}{l_O \cdot U_O}\left(\theta^{\mathbf{A}}\mathscr{V}_{\mathbf{A}} + \beta^{\mathbf{A}}\mathscr{W}_{\mathbf{A}}\right)$$

$$\mathscr{V}_{\mathbf{A}} = -\mathscr{T}(u_O)_{\mathbf{A}} + \frac{D_{SO}}{D_{SL}}\mathscr{T}(u_L)_{\mathbf{A}} + \frac{a_S}{a_L}\frac{D_{LO}}{D_{SL}}\mathscr{T}(u_S)_{\mathbf{A}}$$

combination of all 3 transverse velocities

$$\mathscr{W}_{\mathbf{A}} = \frac{D_{SO}}{D_{SL}}\left(-\mathscr{T}(u_L)_{\mathbf{A}} + \Sigma(a_S, a_L)\,\mathscr{T}(u_S)_{\mathbf{A}}\right)$$

combination of u_L and u_S, no u_O

$$\frac{l_O \cdot u_O}{l_O \cdot U_O}$$

stellar aberration pre-factor

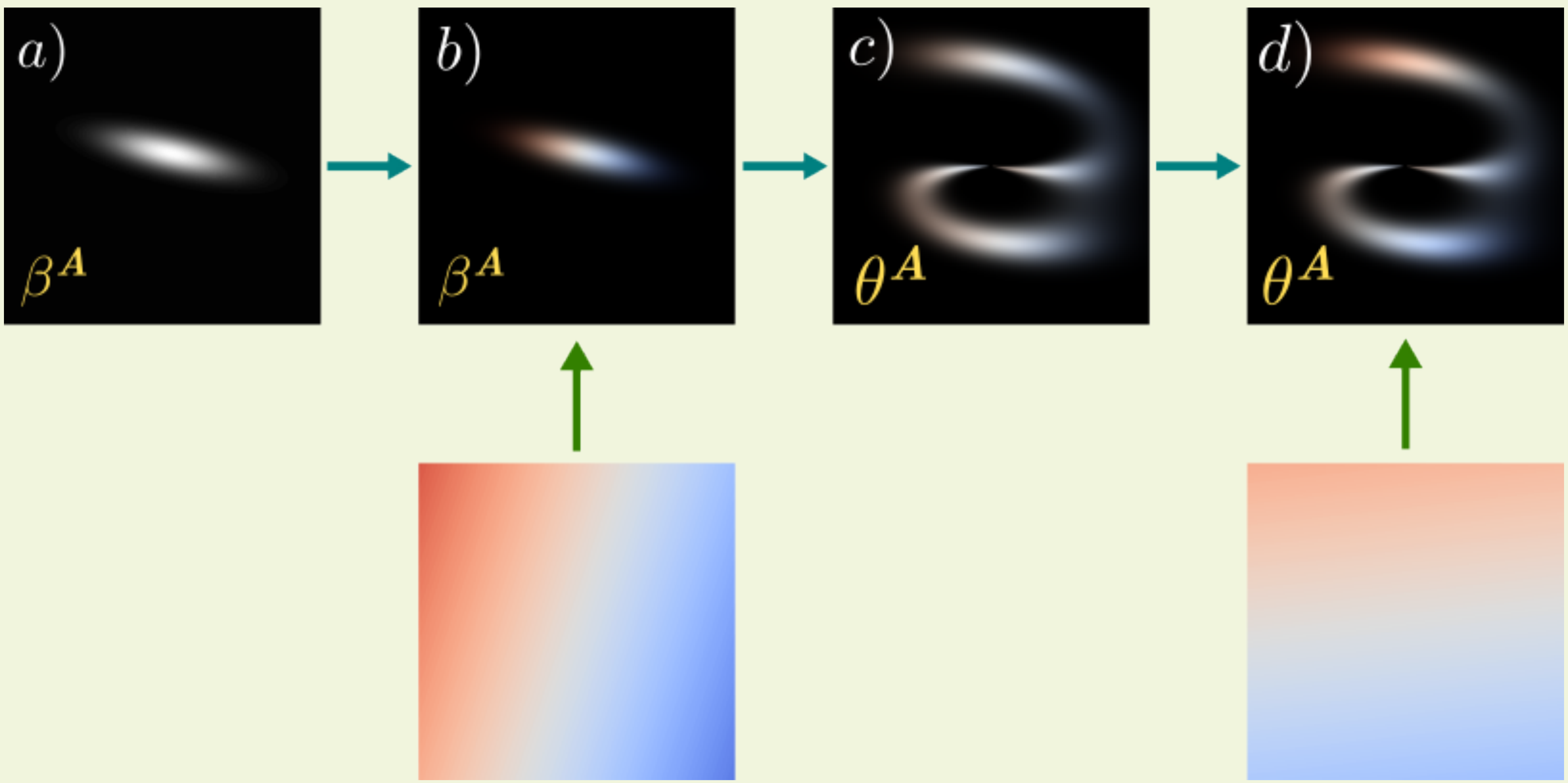
$$\frac{l_O \cdot u_O}{l_O \cdot U_O}\theta^{\mathbf{A}}$$

angles measured in the cosmic frame

$$\frac{l_O \cdot u_O}{l_O \cdot U_O} \cong 1 + \frac{v_O^r}{c} + O(v_O^2/c^2)$$

Main result: frequency variation

$$\Delta\zeta \equiv \frac{\Delta z}{1+z} = \frac{l_O \cdot u_O}{l_O \cdot U_O}\left(\theta^{\mathbf{A}}\mathscr{V}_{\mathbf{A}} + \beta^{\mathbf{A}}\mathscr{W}_{\mathbf{A}}\right)$$



Main result: frequency variation

- Equivalent formulation in terms of $\theta^{\mathbf{A}}, \alpha^{\mathbf{A}}$

$$\frac{\Delta z}{1+z} = \frac{l_O \cdot u_O}{l_O \cdot U_O} \left(\theta^{\mathbf{A}} \mathscr{U}_{\mathbf{A}} - \alpha^{\mathbf{A}} \mathscr{W}_{\mathbf{A}}\right)$$

$$\mathscr{U}_{\mathbf{A}} = -\mathscr{T}(u_O) + \Sigma(a_S, a_O)\,\mathscr{T}(u_S)$$

combination of u_O and u_S, no u_L

$$\mathscr{W}_{\mathbf{A}} = \frac{D_{SO}}{D_{SL}} \left(-\mathscr{T}(u_L) + \Sigma(a_S, a_L)\,\mathscr{T}(u_S)\right)$$

combination of u_L and u_S, no u_O

Main result: frequency variation

- Equivalent formulation in terms of $\theta^{\mathbf{A}}, \alpha^{\mathbf{A}}$

$$\frac{\Delta z}{1+z} = \frac{l_O \cdot u_O}{l_O \cdot U_O} \left(\theta^{\mathbf{A}} \mathscr{U}_{\mathbf{A}} - \alpha^{\mathbf{A}} \mathscr{W}_{\mathbf{A}}\right)$$

no dependence of the lens
term present even in the absence of lensing

$$\mathscr{U}_{\mathbf{A}} = -\mathscr{T}(u_O) + \Sigma(a_S, a_O)\,\mathscr{T}(u_S)$$

combination of u_O and u_S, no u_L

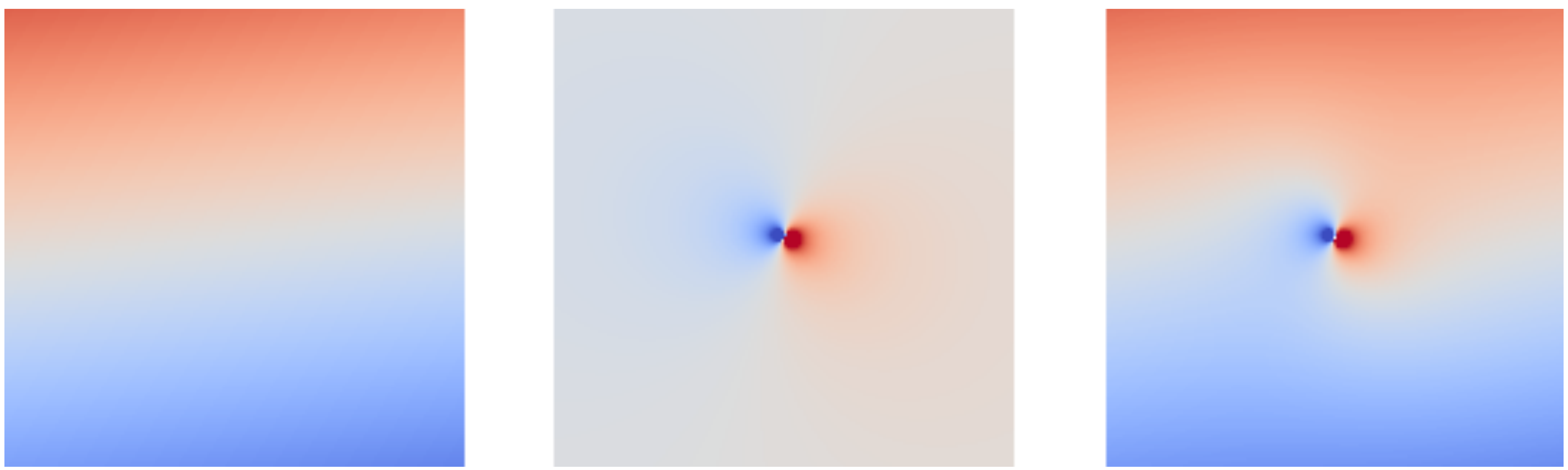
$$\mathscr{W}_{\mathbf{A}} = \frac{D_{SO}}{D_{SL}} \left(-\mathscr{T}(u_L) + \Sigma(a_S, a_L)\,\mathscr{T}(u_S)\right)$$

combination of u_L and u_S, no u_O

Main result: frequency variation

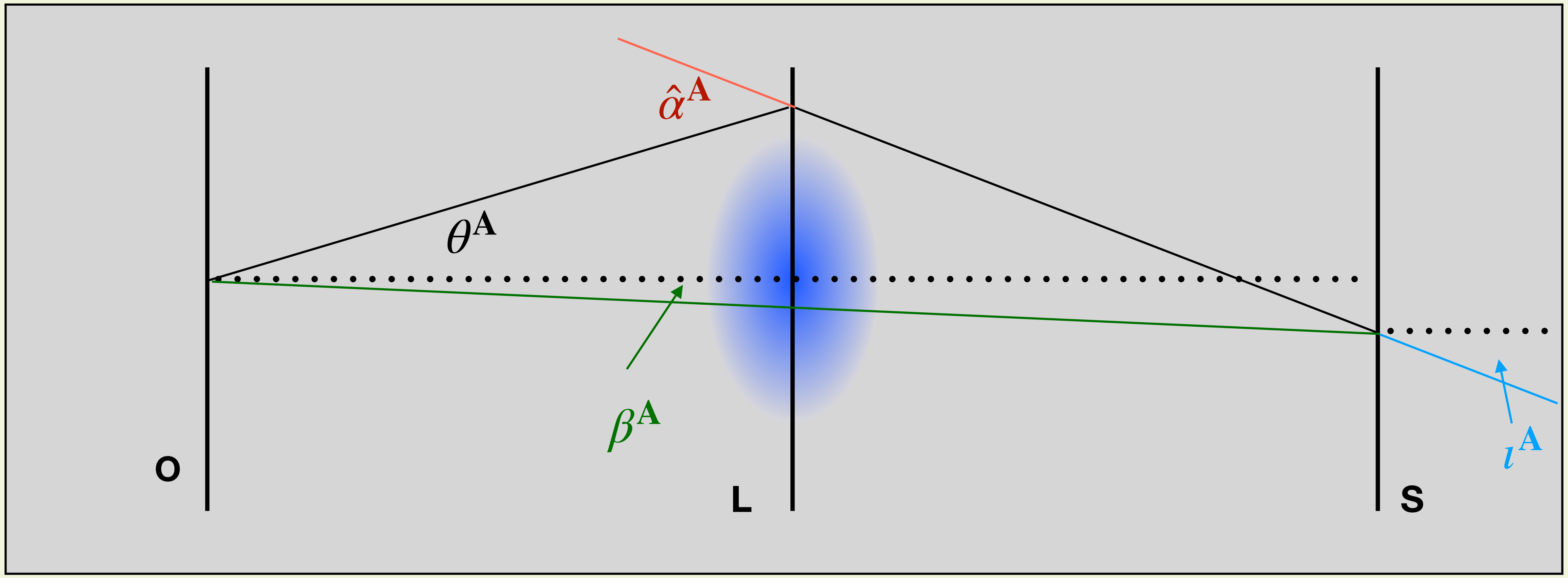
- Equivalent formulation in terms of $\theta^{\mathbf{A}}, \alpha^{\mathbf{A}}$

$$\frac{\Delta z}{1+z} = \frac{l_O \cdot u_O}{l_O \cdot U_O} \left(\theta^{\mathbf{A}} \mathscr{U}_{\mathbf{A}} - \alpha^{\mathbf{A}} \mathscr{W}_{\mathbf{A}}\right)$$



$\theta_{\mathbf{A}} \mathscr{U}^{\mathbf{A}}$ $-\alpha_{\mathbf{A}} \mathscr{W}^{\mathbf{A}}$ sum

Main result: viewing direction



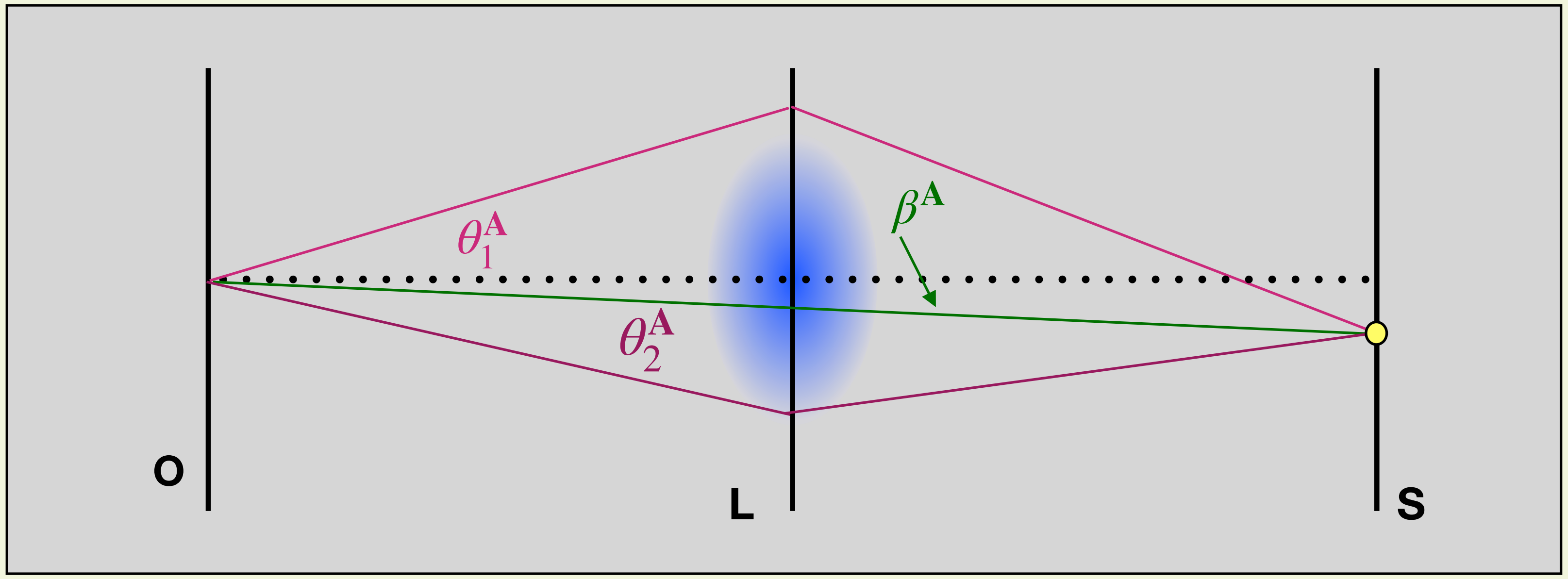
$$\iota^{\mathbf{A}} = \nu \left(-\frac{a_S}{a_L} \frac{D_{LO}}{D_{SL}} \theta^{\mathbf{A}} + \Sigma(a_S, a_L) \frac{D_{SO}}{D_{SL}} \beta^{\mathbf{A}} \right)$$

$$\nu = \frac{l_S \cdot U_S}{l_S \cdot u_S} \frac{l_O \cdot u_O}{l_O \cdot U_O}$$

aberration effect pre-factor

Dependence on distances

- Frequency difference between two images of a single source, point mass lens

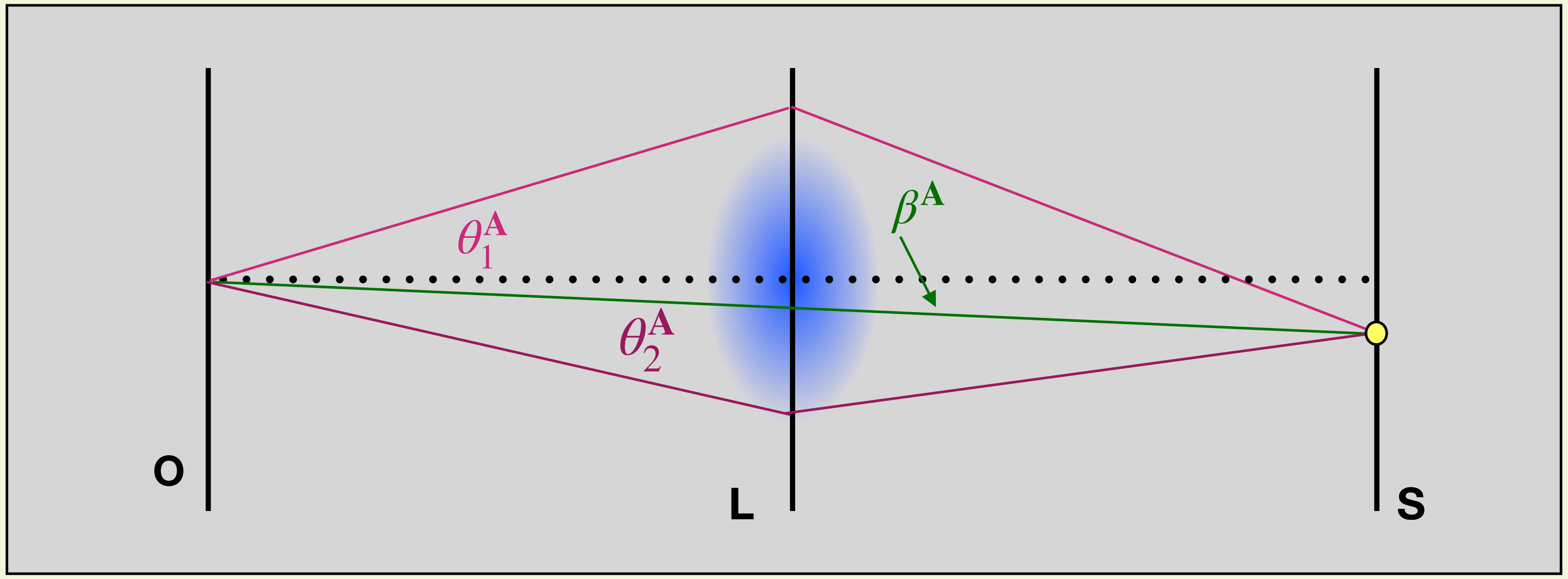


$$\beta_1^{\mathbf{A}} = \beta_2^{\mathbf{A}} \equiv \beta^{\mathbf{A}}$$

$$\Delta\zeta_1 - \Delta\zeta_2 = \left(\theta_1^{\mathbf{A}} - \theta_2^{\mathbf{A}}\right) \mathscr{V}_{\mathbf{A}} \left(\frac{l_O \cdot u_O}{l_O \cdot U_O}\right) \qquad \mathscr{V}_{\mathbf{A}} = -\mathscr{T}(u_O)_{\mathbf{A}} + \frac{D_{SO}}{D_{SL}}\mathscr{T}(u_L)_{\mathbf{A}} + \frac{a_S}{a_L}\frac{D_{LO}}{D_{SL}}\mathscr{T}(u_S)_{\mathbf{A}}$$

Dependence on distances

- Frequency difference between two images of a single source, point mass lens



$$\beta_1^{\mathbf{A}} = \beta_2^{\mathbf{A}} \equiv \beta^{\mathbf{A}}$$

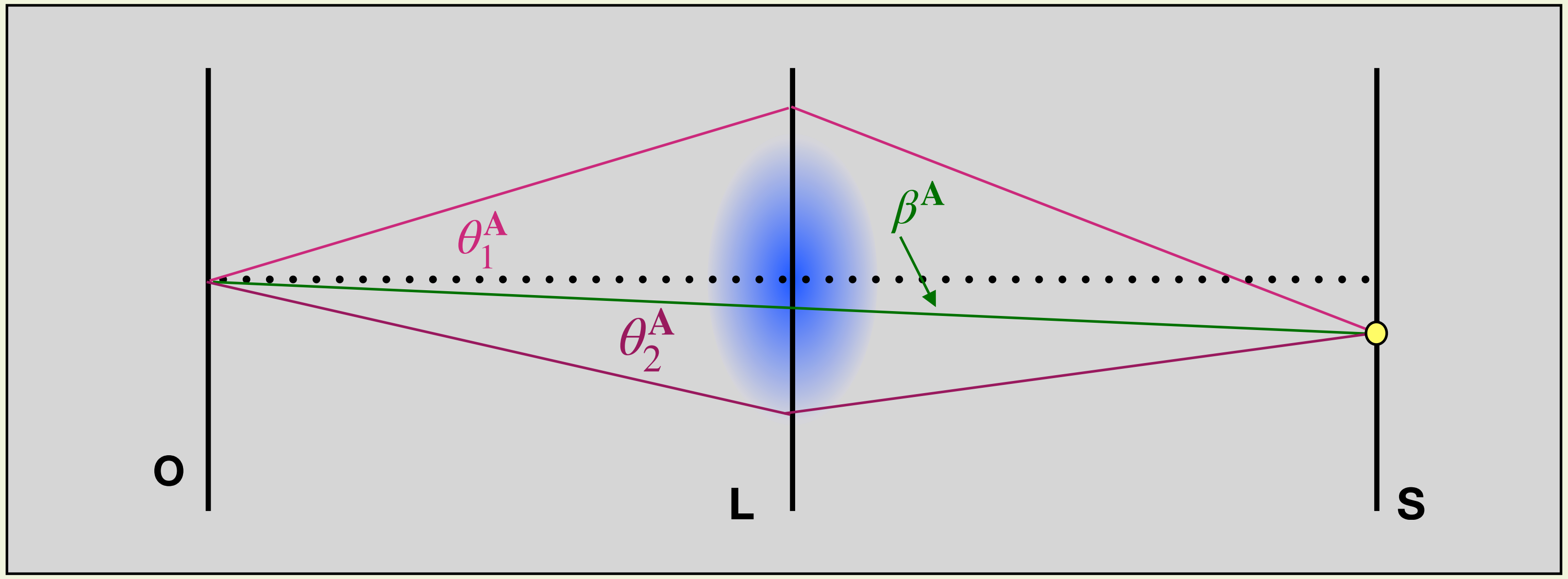
$$\Delta\zeta_1 - \Delta\zeta_2 = \underbrace{\left(\theta_1^{\mathbf{A}} - \theta_2^{\mathbf{A}}\right)}_{\approx 2\theta_E} \mathscr{V}_{\mathbf{A}} \left(\frac{l_O \cdot u_O}{l_O \cdot U_O}\right)$$

$$\mathscr{V}_{\mathbf{A}} = -\mathscr{T}(u_O)_{\mathbf{A}} + \frac{D_{SO}}{D_{SL}}\mathscr{T}(u_L)_{\mathbf{A}} + \frac{a_S}{a_L}\frac{D_{LO}}{D_{SL}}\mathscr{T}(u_S)_{\mathbf{A}}$$

Einstein radius $\theta_E = \sqrt{\dfrac{4GM\,D_{SL}}{D_{SO}\,D_{LO}}}$

Dependence on distances

- Frequency difference between two images of a single source, point mass lens



$$\beta_1^{\mathbf{A}} = \beta_2^{\mathbf{A}} \equiv \beta^{\mathbf{A}}$$

$$\Delta\zeta_1 - \Delta\zeta_2 = \underbrace{\left(\theta_1^{\mathbf{A}} - \theta_2^{\mathbf{A}}\right)}_{\approx 2\theta_E} \mathscr{V}_{\mathbf{A}} \underbrace{\left(\frac{l_O \cdot u_O}{l_O \cdot U_O}\right)}_{\approx 1}$$

$$\mathscr{V}_{\mathbf{A}} = -\mathscr{T}(u_O)_{\mathbf{A}} + \frac{D_{SO}}{D_{SL}} \mathscr{T}(u_L)_{\mathbf{A}} + \frac{a_S}{a_L} \frac{D_{LO}}{D_{SL}} \mathscr{T}(u_S)_{\mathbf{A}}$$

Einstein radius $\theta_E = \sqrt{\dfrac{4GM\, D_{SL}}{D_{SO}\, D_{LO}}}$

Dependence on distances

- General expression $$\Delta\zeta = 4\sqrt{\frac{GM}{c^2 D_{eff}}}\,\frac{v_{eff}}{c}$$

$$c\Delta\zeta = 830\,\mathrm{cm\,s^{-1}}\left(\frac{M}{M_\odot}\right)^{1/2}\left(\frac{D_{eff}}{1\,\mathrm{kpc}}\right)^{-1/2}\frac{v_{eff}}{c}$$

$$c\Delta\zeta = 0.23\,\mathrm{cm\,s^{-1}}\left(\frac{M}{M_\odot}\right)^{1/2}\left(\frac{D_{eff}}{1\,\mathrm{kpc}}\right)^{-1/2}\frac{v_{eff}}{100\,\mathrm{km\,s^{-1}}}$$

- Here $$v_{eff} = \left| C_S\,\vec{v}_S^{tr} + C_L\,\vec{v}_L^{tr} + C_O\,\vec{v}_O^{tr} \right|$$

$$D_{eff} = \frac{D_{LO}\,D_{SO}}{D_{SL}}$$

Can this effect be large?

Particular cases

$$\Delta\zeta = 4\sqrt{\frac{GM}{c^2 D_{eff}}}\frac{v_{eff}}{c} \qquad c\Delta\zeta = 0.23\,\mathrm{cm\,s^{-1}}\left(\frac{M}{M_\odot}\right)^{1/2}\left(\frac{D_{eff}}{1\,\mathrm{kpc}}\right)^{-1/2}\frac{v_{eff}}{100\,\mathrm{km\,s^{-1}}}$$

- Massive lens / galaxy cluster

$$c\,\Delta\zeta \cong 2\theta_E\,v_{eff}$$

$$c\Delta\zeta \cong 970\,\mathrm{cm\,s^{-1}}\left(\frac{\theta_E}{10''}\right)\left(\frac{v_{eff}}{100\,\mathrm{km\,s^{-1}}}\right)$$

Particular cases

$$\Delta\zeta = 4\sqrt{\frac{GM}{c^2 D_{eff}}}\,\frac{v_{eff}}{c}$$

$$c\Delta\zeta = 0.23\,\mathrm{cm\,s^{-1}}\left(\frac{M}{M_\odot}\right)^{1/2}\left(\frac{D_{eff}}{1\,\mathrm{kpc}}\right)^{-1/2}\frac{v_{eff}}{100\,\mathrm{km\,s^{-1}}}$$

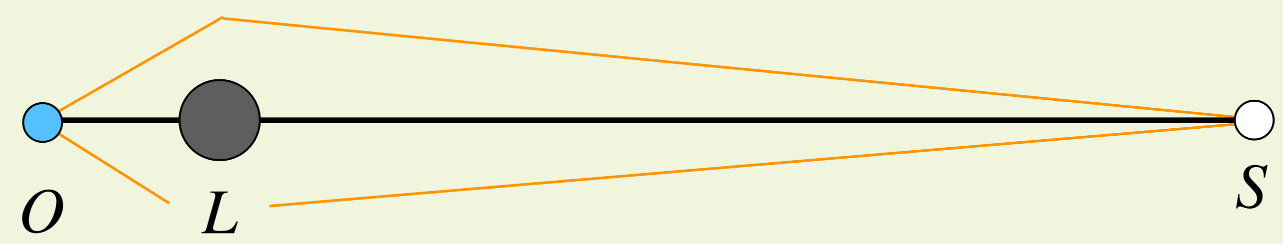
Particular cases

$$\Delta\zeta = 4\sqrt{\frac{GM}{c^2 D_{eff}}}\frac{v_{eff}}{c} \qquad c\Delta\zeta = 0.23\,\mathrm{cm\,s^{-1}}\left(\frac{M}{M_\odot}\right)^{1/2}\left(\frac{D_{eff}}{1\,\mathrm{kpc}}\right)^{-1/2}\frac{v_{eff}}{100\,\mathrm{km\,s^{-1}}}$$

- microlensing $D_{LO} \ll D_{SO}$

$$D_{eff} \cong D_{LO}$$

$$v_{eff} = \left|\vec{v}_L^{tr} - \vec{v}_O^{tr}\right|$$



GW: [Biesiada et al 2021, Suyampraksam et al 2025, …], EM: [Rahvar 2020]

Particular cases

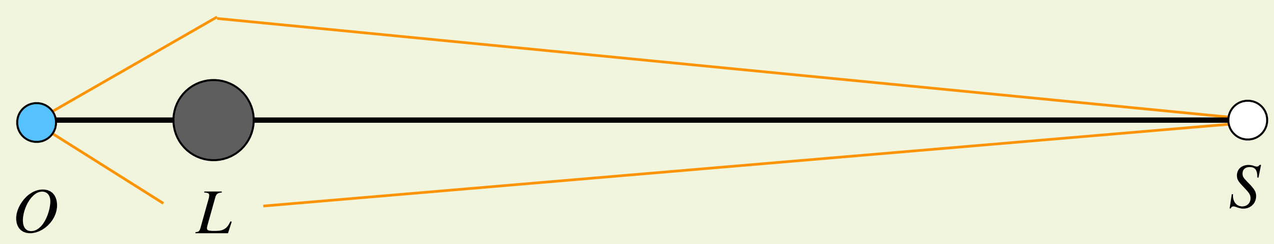
$$\Delta\zeta = 4\sqrt{\frac{GM}{c^2 D_{eff}}}\,\frac{v_{eff}}{c}$$

$$c\Delta\zeta = 0.23\,\mathrm{cm\,s^{-1}}\left(\frac{M}{M_\odot}\right)^{1/2}\left(\frac{D_{eff}}{1\,\mathrm{kpc}}\right)^{-1/2}\frac{v_{eff}}{100\,\mathrm{km\,s^{-1}}}$$

- microlensing $D_{LO} \ll D_{SO}$

$$D_{eff} \cong D_{LO}$$

$$v_{eff} = \left|\vec{v}_L^{tr} - \vec{v}_O^{tr}\right|$$

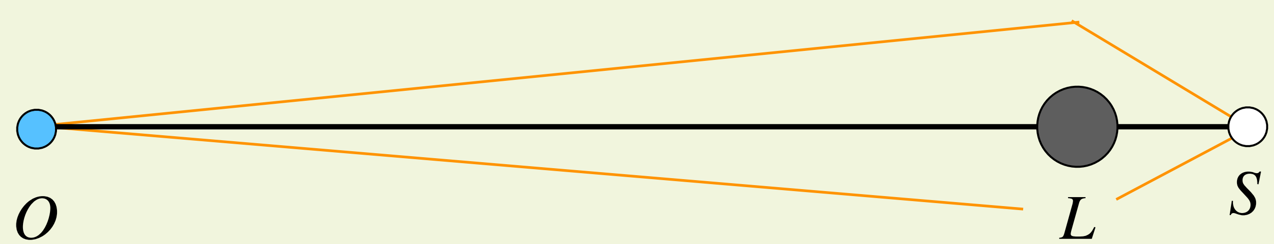


GW: [Biesiada et al 2021, Suyampraksam et al 2025, …], EM: [Rahvar 2020]

- lens and source close to each other $D_{SL} \ll D_{SO}$

$$D_{eff} \cong D_{SL}$$

$$v_{eff} = \left|\vec{v}_L^{tr} - \vec{v}_S^{tr}\right|$$



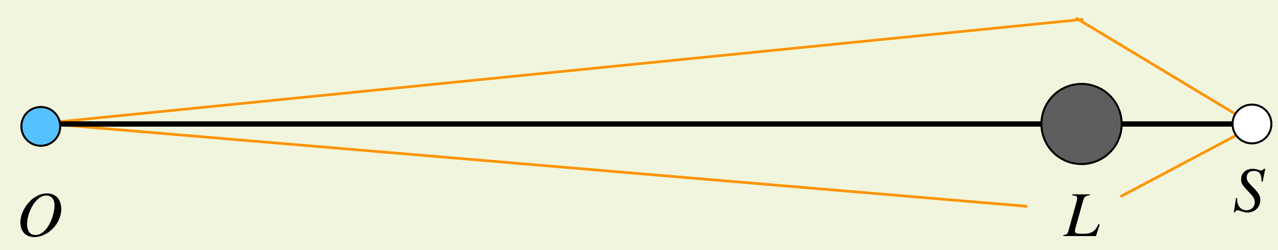
Particular cases

$$\Delta\zeta = 4\sqrt{\frac{GM}{c^2 D_{eff}}}\frac{v_{eff}}{c} \qquad c\Delta\zeta = 0.23\,\mathrm{cm\,s^{-1}}\left(\frac{M}{M_\odot}\right)^{1/2}\left(\frac{D_{eff}}{1\,\mathrm{kpc}}\right)^{-1/2}\frac{v_{eff}}{100\,\mathrm{km\,s^{-1}}}$$

- lens and source close to each other $D_{SL} \ll D_{SO}$

$$D_{eff} \cong D_{SL}$$

$$v_{eff} = \left|\vec{v}_L^{tr} - \vec{v}_S^{tr}\right|$$



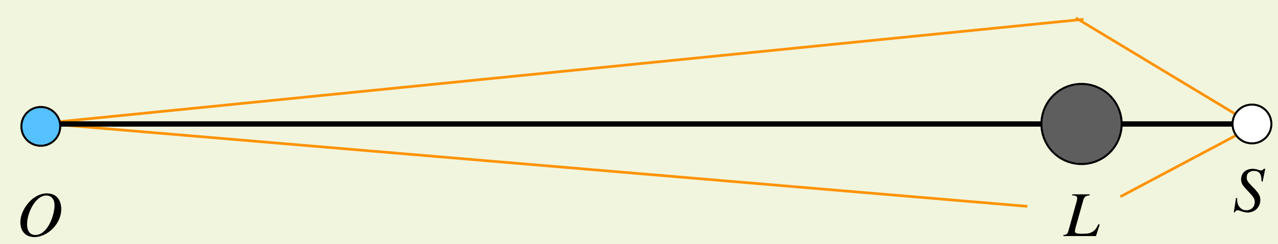
Particular cases

$$\Delta\zeta = 4\sqrt{\frac{GM}{c^2 D_{eff}}}\frac{v_{eff}}{c} \qquad c\Delta\zeta = 0.23\,\mathrm{cm\,s^{-1}}\left(\frac{M}{M_\odot}\right)^{1/2}\left(\frac{D_{eff}}{1\,\mathrm{kpc}}\right)^{-1/2}\frac{v_{eff}}{100\,\mathrm{km\,s^{-1}}}$$

- lens and source close to each other $D_{SL} \ll D_{SO}$

$$D_{eff} \cong D_{SL}$$

$$v_{eff} = \left|\vec{v}_L^{tr} - \vec{v}_S^{tr}\right|$$

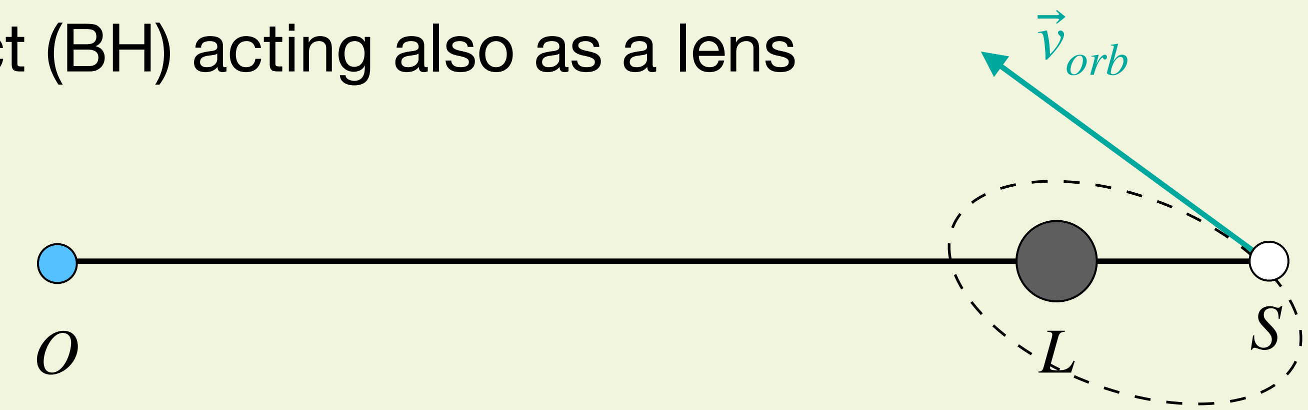


- self-lensing

source orbiting a massive object (BH) acting also as a lens

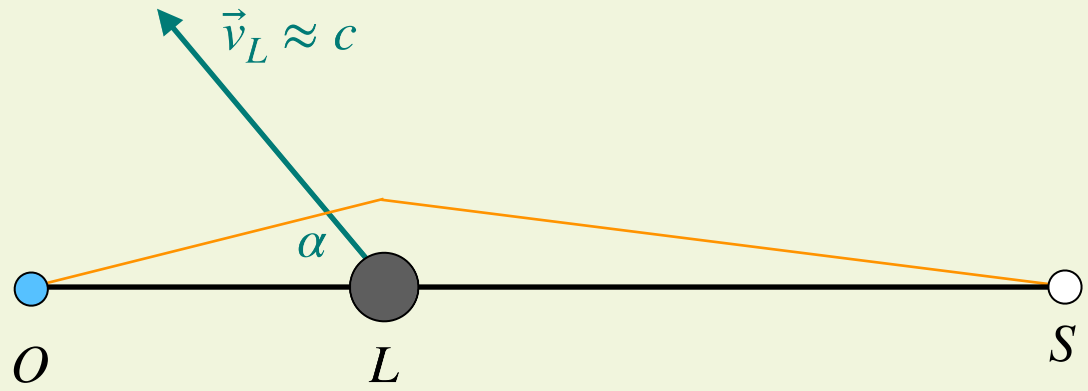
$$D_{eff} \cong R_{orb}$$

$$v_{eff} \cong v_{orb}$$



in GW context: [Ubach 2026, Rahvar et al 2011, Samsing et al 2024…]

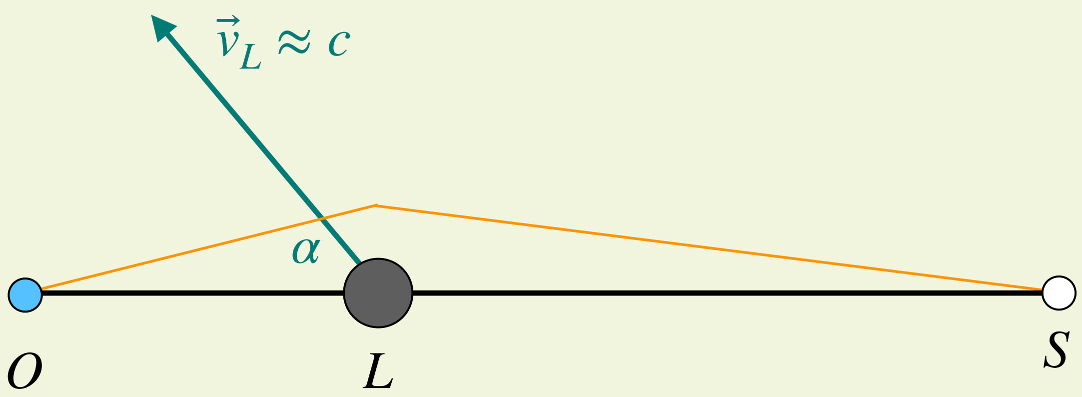
Relativistic lenses



$$\Delta\zeta = \sqrt{\frac{4GM\,D_{SO}}{D_{LO}\,D_{SL}}}\, f(v_L, \alpha)$$

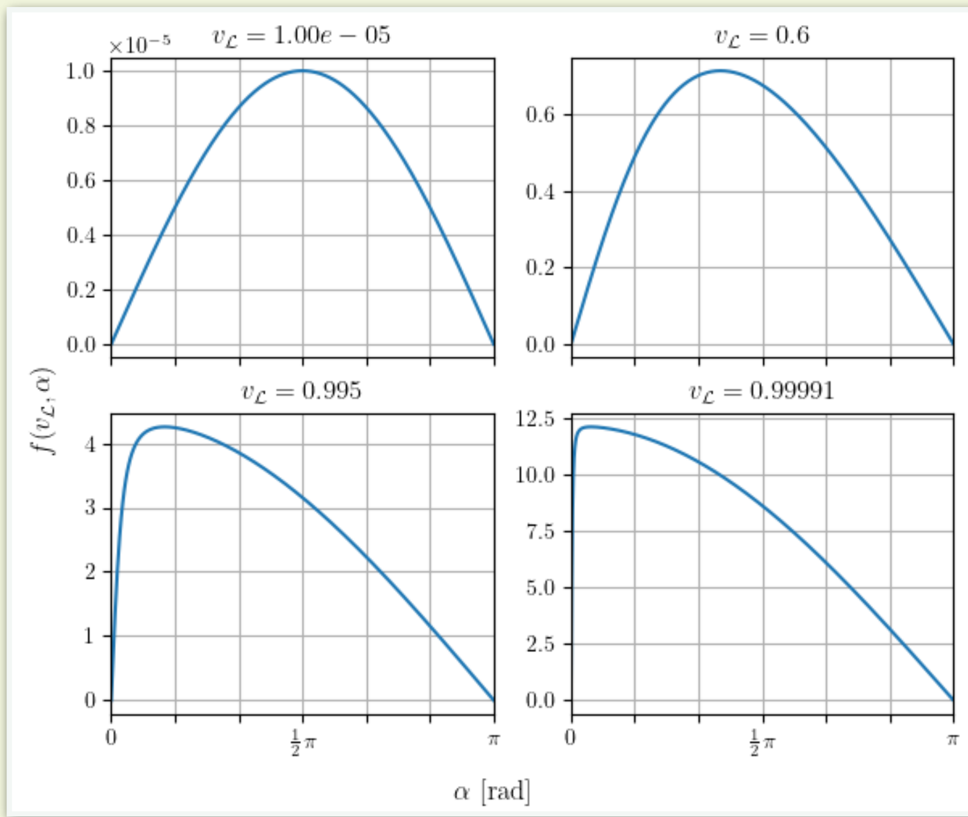
$$f(v_L, \alpha) = \frac{\gamma_L^{1/2}\,\sin\alpha\, v_L}{(1 - v_L\,\cos\alpha)^{1/2}}$$

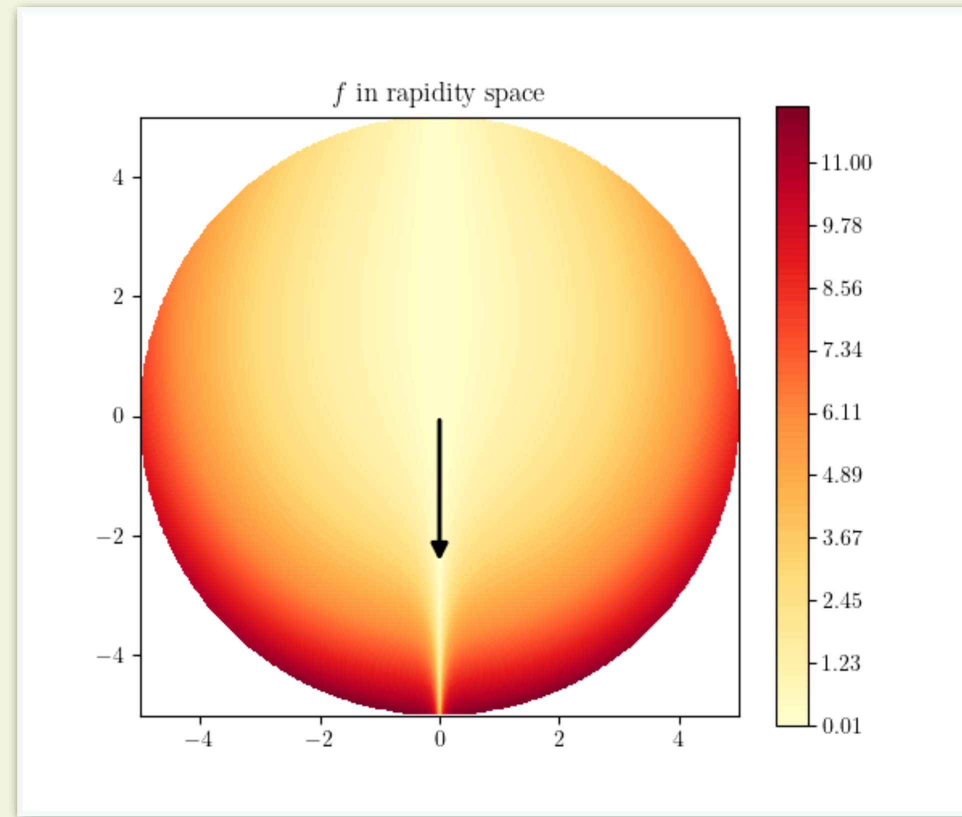
Relativistic lenses



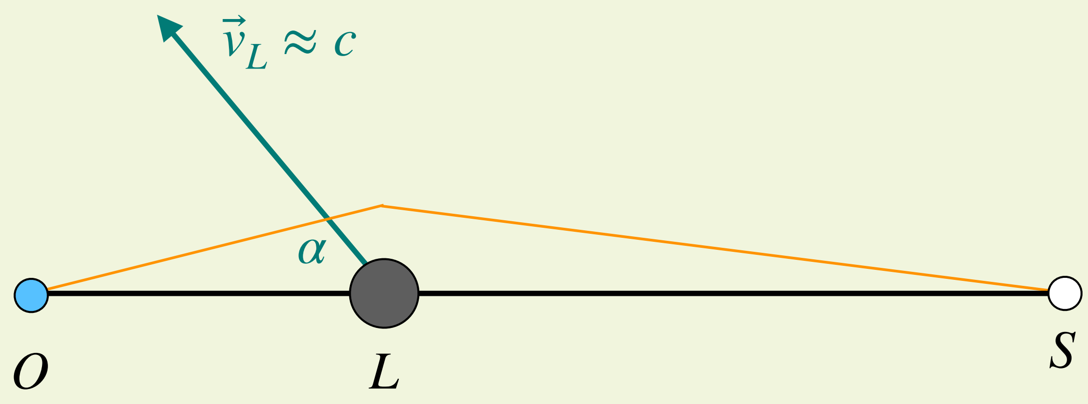
$$\Delta\zeta = \sqrt{\frac{4GM\,D_{SO}}{D_{LO}\,D_{SL}}}\; f(v_L, \alpha)$$

$$f(v_L, \alpha) = \frac{\gamma_L^{1/2}\,\sin\alpha\, v_L}{(1 - v_L\,\cos\alpha)^{1/2}}$$



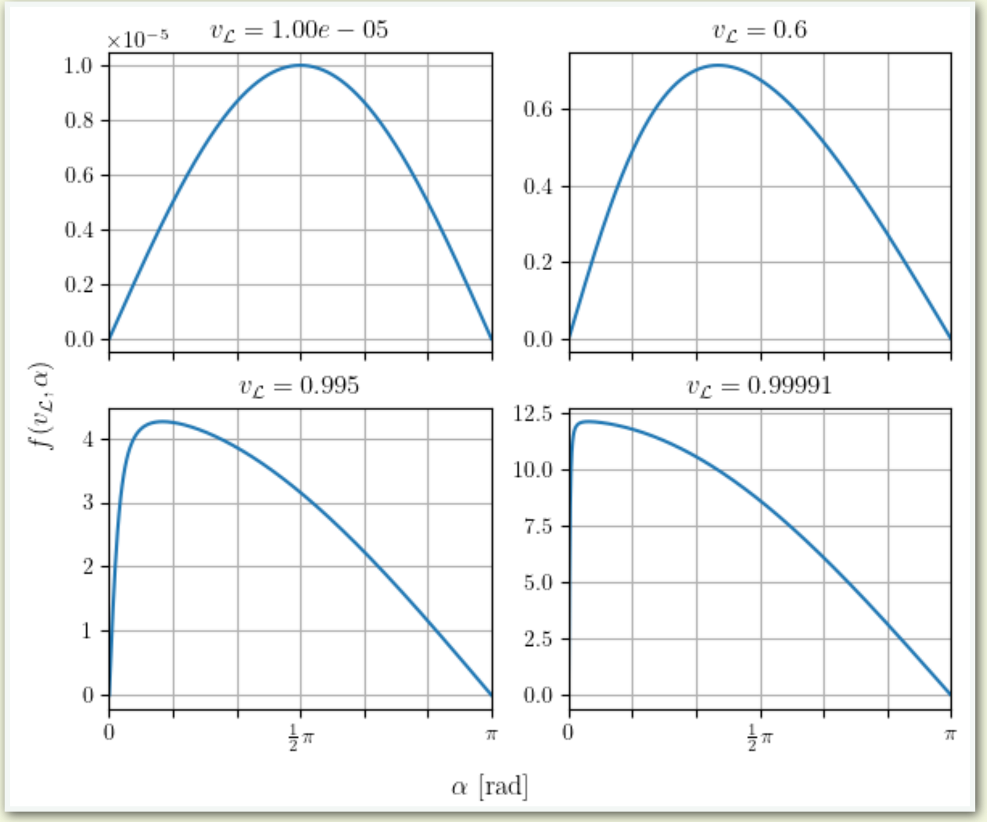


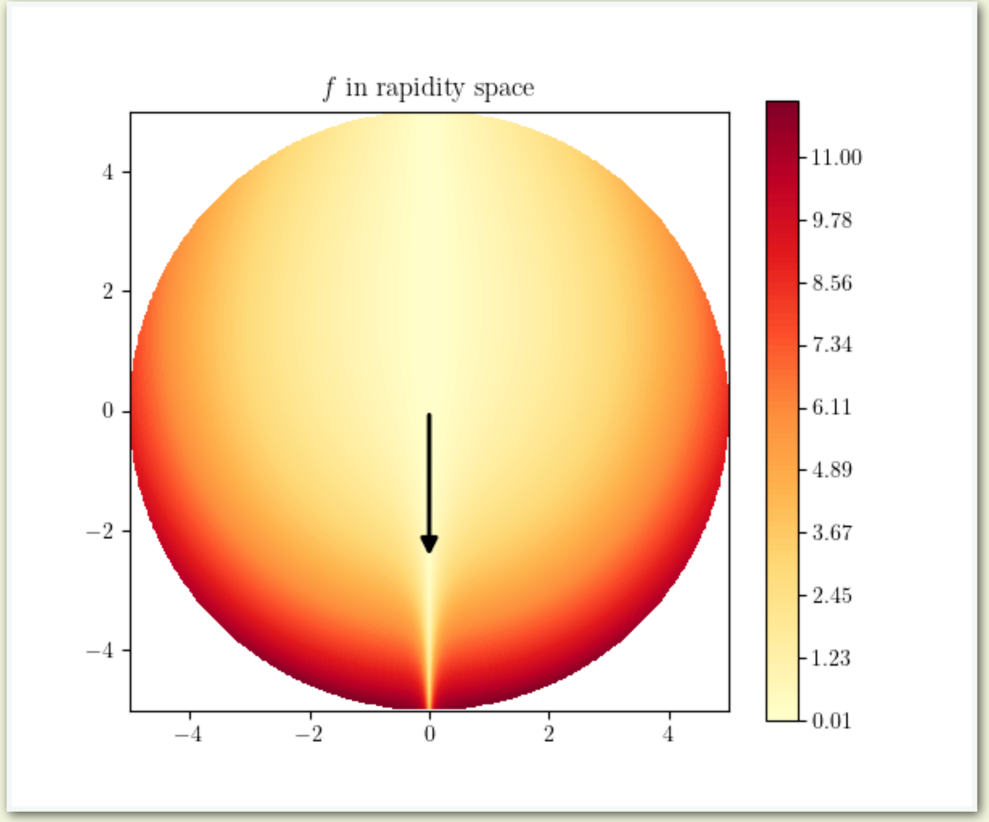
Relativistic lenses



$$\Delta\zeta = \sqrt{\frac{4GM\,D_{SO}}{D_{LO}\,D_{SL}}}\,f(v_L,\alpha)$$

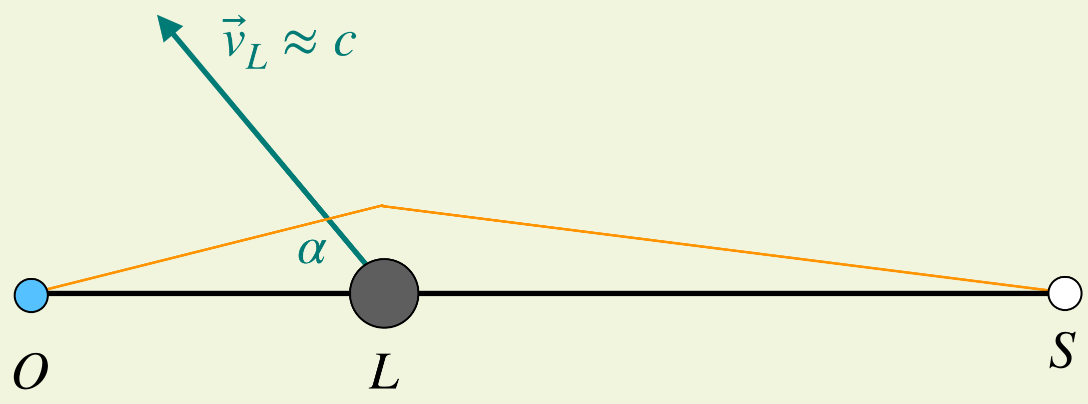
$$f(v_L,\alpha) = \frac{\gamma_L^{1/2}\,\sin\alpha\,v_L}{(1 - v_L\,\cos\alpha)^{1/2}}$$





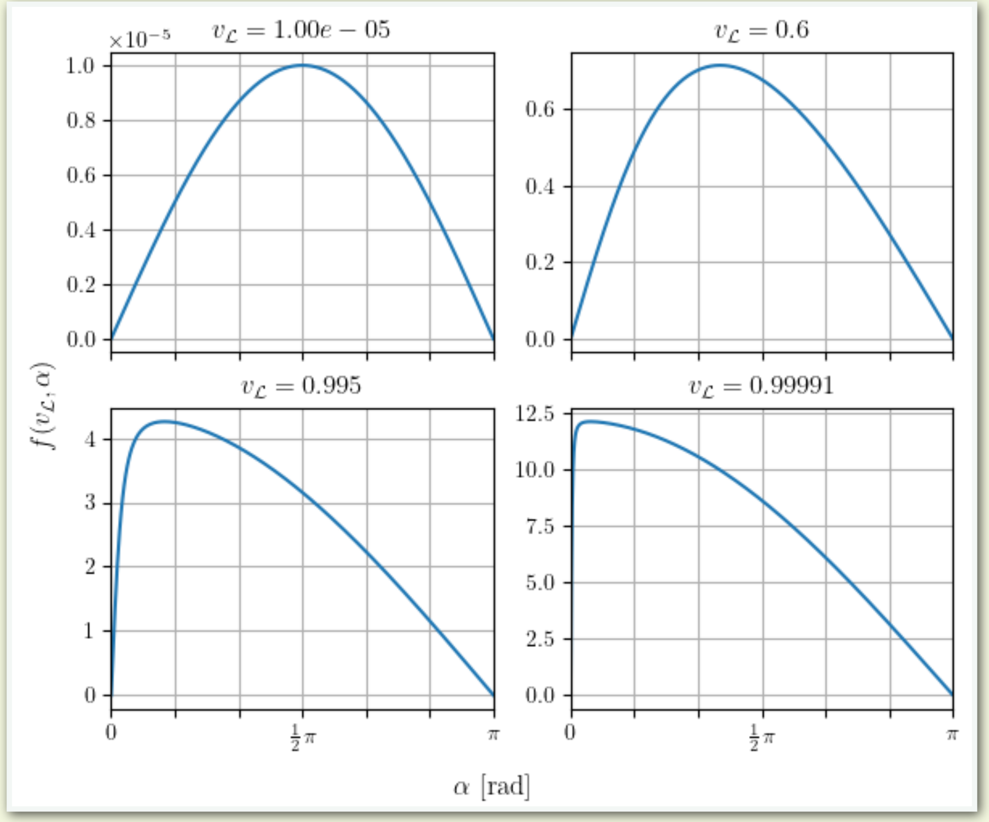
- different dependence on α for high velocities

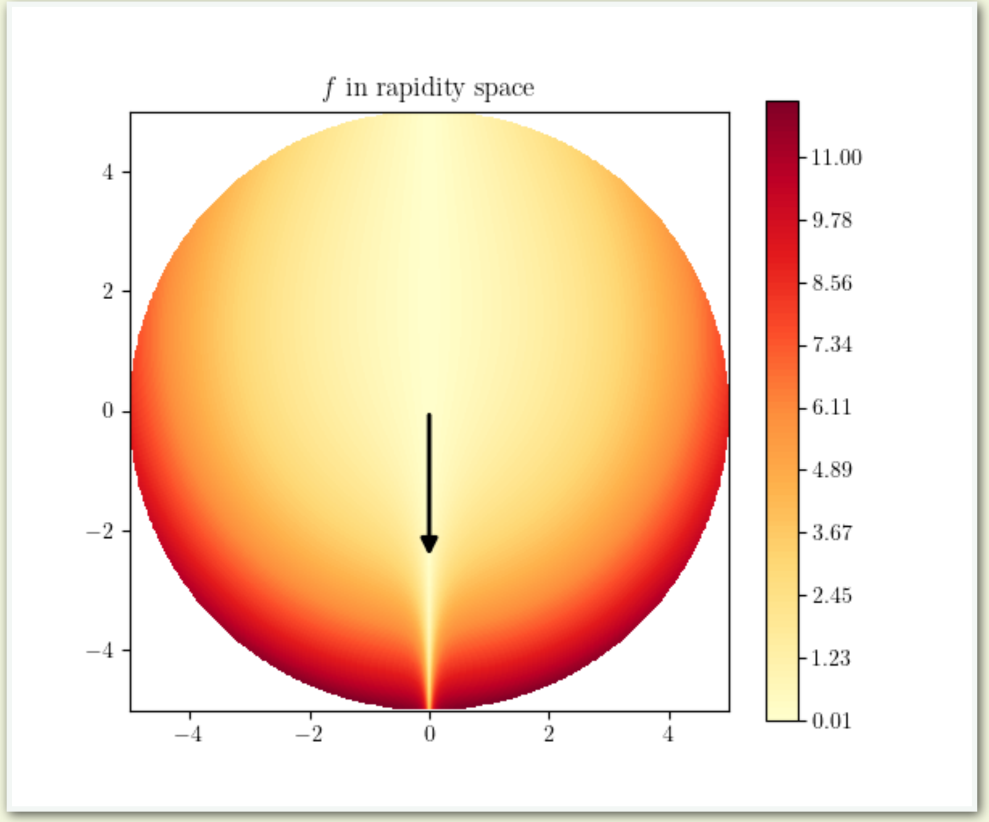
Relativistic lenses



$$\Delta\zeta = \sqrt{\frac{4GM\,D_{SO}}{D_{LO}\,D_{SL}}}\;f(v_L,\alpha)$$

$$f(v_L,\alpha) = \frac{\gamma_L^{1/2}\,\sin\alpha\,v_L}{(1 - v_L\,\cos\alpha)^{1/2}}$$





- different dependence on α for high velocities
- $\Delta\zeta$ can be as large as possible, but grows only like $\gamma_L^{1/2} = (1 - v_L^2/c^2)^{-1/4}$

Summary and takeaways

- General expression for the frequency effect of gravitational lensing (EM and GW)
- Velocities can be arbitrary large
- Cosmological effects included
  - It is **not** enough to substitute distances for angular diameter distances in FLRW case, like in the lensing equation
- Effect almost always very small
- Can be large if the lens is close to the source or close to the observer (microlensing)
- Self-lensing is the most promising astrophysical situation in which the effect can be observed (GW)

Acknowledgements:

Funded by the National Science Centre, Poland (NCN) under the Weave-UNISONO call (grant no. 2024/06/Y/ST2/00190), project *"Lensing of electromagnetic and gravitational waves"*.

National Science Centre
Poland

Relativistic lenses