Relativistic figures of equilibrium in the Wald magnetosphere

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Based on joint work with **Patryk Mach**, **Audrey Trova**, and **Bakhtinur Juraev**
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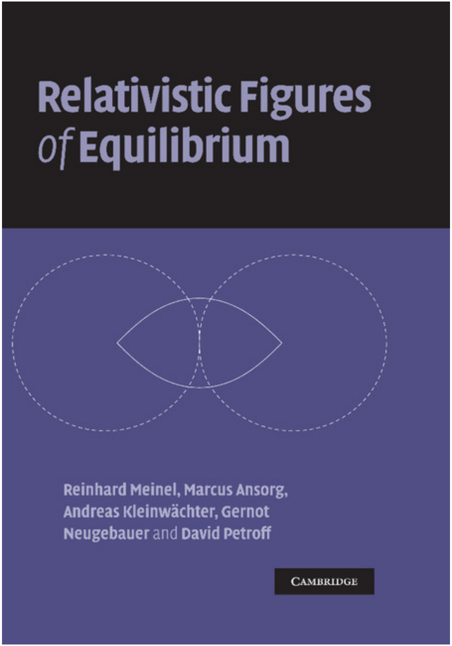
Relativistic Figures
of Equilibrium
Reinhard Meinel, Marcus Ansorg,
Andreas Kleinwächter, Gernot
Neugebauer and David Petroff
CAMBRIDGE

Outline of the talk

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2. Generalization to non-vacuum spacetimes: consistency conditions

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3. Energy-momentum conservation
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5. AKM code and numerical results

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$$F_{\alpha\beta} = \nabla_\alpha \xi_\beta - \nabla_\beta \xi_\alpha = -2\nabla_\beta \xi_\alpha \implies \nabla^\mu F_{\alpha\mu} = -2\nabla^\mu \nabla_\mu \xi_\alpha$$

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- At the same time, from the relation $(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)\xi_\gamma = R_{\gamma\mu\alpha\beta}\xi^\mu$ involving the Riemann tensor, one can deduce that

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$$\nabla^\mu F_{\alpha\mu} = 4\pi j_\alpha$$

- Combining the above relations, we find

$$\boxed{j_\alpha = \frac{1}{2\pi} R_{\alpha\mu}\xi^\mu}$$

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- Asymptotically, it corresponds to a uniform magnetic field of strength $2b$
- It serves as a toy model of black holes magnetospheres

Generalization to non-vacuum spacetimes

- We consider stationary, axisymmetric and circular spacetimes, and work in cylindrical coordinates $(t, \rho, \zeta, \varphi)$:

$$\mathrm{d}s^2 = g_{tt}\mathrm{d}t^2 + 2g_{t\varphi}\mathrm{d}t\mathrm{d}\varphi + g_{\rho\rho}\mathrm{d}\rho^2 + g_{\zeta\zeta}\mathrm{d}\zeta^2 + g_{\varphi\varphi}\mathrm{d}\varphi^2,$$

where $g_{\alpha\beta} = g_{\alpha\beta}(\rho, \zeta)$

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  where $g_{\alpha\beta} = g_{\alpha\beta}(\rho, \zeta)$
- The Killing vectors are $k^\alpha = (1, 0, 0, 0)$, $\eta^\alpha = (0, 0, 0, 1)$
- As the gravitational source, we take a rotating perfect fluid with the energy-momentum tensor

$$T^{\mathrm{pf}}_{\alpha\beta} := (\varepsilon + p)u_\alpha u_\beta + p g_{\alpha\beta}$$

  and $u^\rho = u^\zeta = 0$

Electric 4-current

- The Einstein equation reads

$$R_{\alpha\beta} = 8\pi\left(T^{\rm pf}_{\alpha\beta} - \frac{1}{2}T^{\rm pf}g_{\alpha\beta}\right) = 8\pi\left[(\varepsilon + p)u_\alpha u_\beta + \frac{1}{2}(\varepsilon - p)g_{\alpha\beta}\right],$$

where we have used $T^{\rm pf} := \left(T^{\rm pf}\right)^\mu{}_\mu = -\varepsilon + 3p$

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- Recalling that $j_\alpha = (2\pi)^{-1} R_{\alpha\mu} \xi^\mu$, for $\xi_\alpha = a k_\alpha + b \eta_\alpha$ we get

$$\boxed{\begin{aligned} j_\alpha &= 4 \left[(\varepsilon + p) u_\mu \xi^\mu u_\alpha + \frac{1}{2} (\varepsilon - p) \xi_\alpha \right] \\ &= 4 \left[(\varepsilon + p)(a u_t + b u_\varphi) u_\alpha + \frac{1}{2} (\varepsilon - p)(a k_\alpha + b \eta_\alpha) \right] \end{aligned}}$$

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- Note that we treat Wald's magnetosphere as a (gravitationally) test field: it is coupled to the fluid via the Maxwell equation and energy-momentum conservation, but not Einstein equation

Relativistic Ohm's law

- We can orthogonally decompose the electric 4-current as

$$j_\alpha =: \rho_\mathrm{e} u_\alpha + J_\alpha, \qquad u_\mu J^\mu = 0,$$

where $\rho_\mathrm{e} = -u_\mu j^\mu$ and J_α are, respectively, the electric charge and current densities in a comoving frame

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- ***Relativistic Ohm's law*** states that $J_\alpha = \sigma E_\alpha$, where E_α denotes the electric field measured by a comoving observer,

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- In standard ideal magnetohydrodynamics, $E_\alpha = 0$ and $\sigma = \infty$ (perfect conductor model)

Consistency conditions

- Under the assumption of relativistic Ohm's law, one can show that $J_\alpha = 0$, i.e.

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- Combining the above expression for j_α with the previous one, we come to $(\varepsilon - p)(a u^\varphi - b u^t) = 0$; hence, either $\varepsilon = p$ (ultra-stiff equation of state), or
$$\boxed{\Omega := \frac{u^\varphi}{u^t} = \frac{b}{a} = \mathrm{const}} \qquad \text{(rigid rotation);}$$
for the remainder of the talk, we focus on the latter case

Energy-momentum conservation

- The energy-momentum conservation equation reads

$$\nabla^\mu \left(T^{\mathrm{pf}}_{\alpha\mu} + T^{\mathrm{em}}_{\alpha\mu}\right) = 0,$$

where $T^{\mathrm{em}}_{\alpha\beta} := \frac{1}{4\pi}\left(F_{\mu\alpha}F^{\mu}{}_{\beta} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}g_{\alpha\beta}\right)$ is the electromagnetic energy-momentum tensor

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- After some calculations, we can write this condition as

$$\partial_{\alpha}p - (\varepsilon + p)\partial_{\alpha}\ln u^{t} - 2a^{2}(\varepsilon + 3p)\partial_{\alpha}(u^{t})^{-2} = 0 \qquad (*)$$

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- Then $(*)$ becomes

$$\boxed{h' - \frac{h}{u^t} + \frac{4a^2}{(u^t)^3}\left(h + \frac{2p}{\rho_{\mathrm{b}}}\right) = 0}$$

with $h' := \mathrm{d}h/\mathrm{d}u^t$

Example 1: Constant energy density

- If $\varepsilon = \mathrm{const}$ in the fluid region, we have $\mathrm{d}\ln h = \mathrm{d}p/(\varepsilon + p) = \mathrm{d}\ln(\varepsilon + p)$, so that $h \propto \varepsilon + p$, or equivalently $\rho_{\mathrm{b}} = \mathrm{const}$

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- Choosing $\rho_{\mathrm{b}} = \varepsilon$, we get $p/\rho_{\mathrm{b}} = h - 1$, which leads to the equation

$$h' - \frac{h}{u^t} + \frac{4a^2}{(u^t)^3}(3h - 2) = 0$$

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- The solution is

$$\boxed{h = -\frac{1}{18a}\left[\sqrt{6\pi}u^t \operatorname{erf}(U) e^{U^2} - 12a\right] + C u^t e^{U^2}}$$

with

$$\boxed{U := \frac{\sqrt{6}a}{u^t}}$$

Example 2: Polytrope

- The polytropic equation of state reads $\varepsilon(\rho_{\rm b}) = \rho_{\rm b} + K\rho_{\rm b}^{\Gamma}/(\Gamma - 1)$, where K and Γ are constants, so we find

$$h = \frac{\mathrm{d}\varepsilon}{\mathrm{d}\rho_{\rm b}} = 1 + \frac{\Gamma K}{\Gamma - 1}\rho_{\rm b}^{\Gamma-1}, \qquad p = -\varepsilon + \rho_{\rm b} h = K\rho_{\rm b}^{\Gamma}$$

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- The solution is

$$\boxed{h = -\frac{(\Gamma - 1)\left[\sqrt{2\pi\Gamma}u^t \operatorname{erf}(U)e^{U^2} - 4a\sqrt{3\Gamma - 2}\right]}{2a(3\Gamma - 2)^{3/2}} + Cu^t e^{U^2}}$$

with

$$\boxed{U := \sqrt{\frac{2(3\Gamma - 2)}{\Gamma}\frac{a}{u^t}}}$$

Example 3: MIT bag model

- In the MIT bag model one postulates $\varepsilon(\rho_{\mathrm{b}}) = A\rho_{\mathrm{b}}^{4/3} + B$, where $A := (9/4)\pi^{2/3}\hbar m_{\mathrm{b}}^{-4/3}$, and B is called the bag constant

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- Here we find

$$h = \frac{4}{3}A\rho_\mathrm{b}^{1/3}, \qquad p = \frac{1}{3}A\rho_\mathrm{b}^{4/3} - B = \frac{1}{3}(\varepsilon - 4B),$$

so that $p/\rho_\mathrm{b} = h/4 - B[4A/(3h)]^3$, and we get the equation

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- The solution is

$$\boxed{h = \frac{1}{3\sqrt{3}|a|}\left\{2304a^4A^3B + 384a^2A^3B(u^t)^2 + \left[32A^3B + 729Ca^4e^{12a^2/(u^t)^2}\right](u^t)^4\right\}^{1/4}}$$

AKM code

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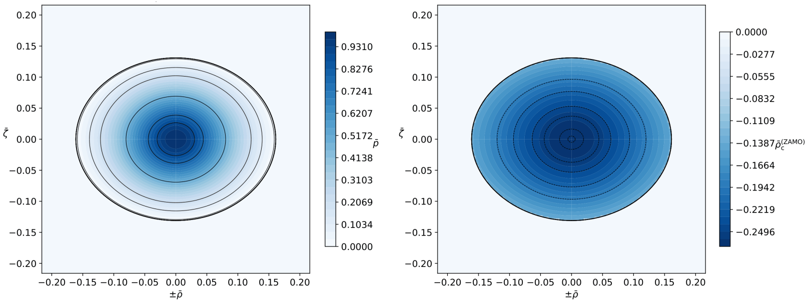
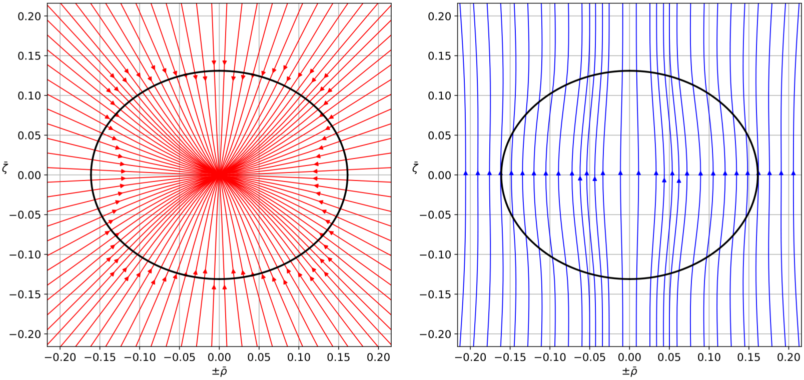
- The metric functions are expanded in Chebyshev polynomials in s, t, and a set of equations for the coefficients is solved via the Newton–Raphson method
- A series of solutions is produced by increasing the value of a (starting from $a = 0$), where each solution serves as the initial data for the next one; the values of two selected quantities can be prescribed

Numerical results (1)



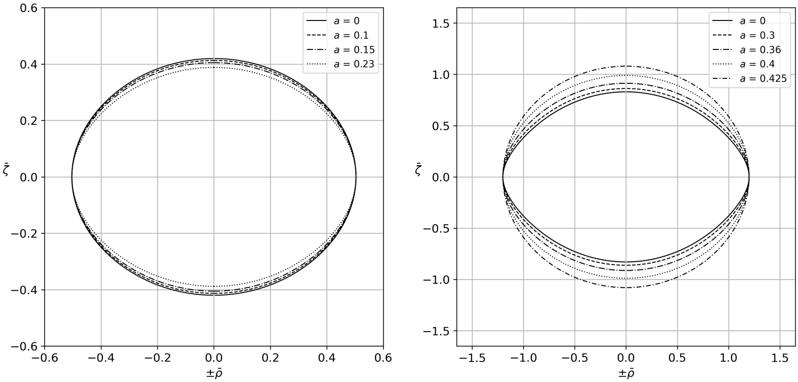
Pressure (left) and electric charge density (in the ZAMO frame; right) for a sample homogeneous configuration

Numerical results (2)



Electric (left) and magnetic (right) field lines in the ZAMO frame for the same homogeneous configuration

Numerical results (3)



Variation of the star surface with respect to a for two sample polytropic configurations (left: $\Gamma = 2$, right: $\Gamma = 5/3$)

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- For constant energy density or polytropic equation of state, the energy-momentum conservation equation is analytically integrable
- The shape of a star may change in different ways when the electromagnetic field is "added"
- Possible next steps: analysis of the solution space, more sophisticated equations of state, relaxation of Ohm's law

Main references

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