Tolman-Oppenheimer-Volkoff equation as generalized Kolmogorov system

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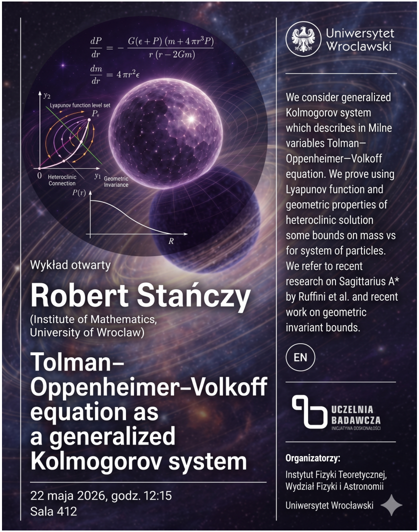
Będlewo, XII PoToR 2026

Tolman–Oppenheimer–Volkoff



Kolmogorov system and Lyapunov function

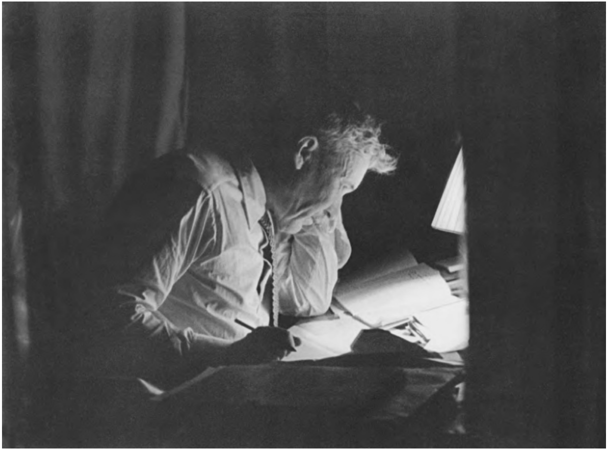
Assume $G(w,z) = H(w,z) = 0$ and Kolmogorov local assumptions $H_x(x,z) \leq 0$ and $G_y(w,y) \geq 0$ for the system

$$\begin{aligned} x' &= g(x)G(x,y), \\ y' &= h(y)H(x,y). \end{aligned}$$

Moreover, assume global assumptions $H_y \leq 0$ and $G_x \leq 0$ while the functions $-H(x,z)/g(x)$ and $G(w,y)/h(y)$ are nondecreasing with primitives $\mathcal{H}$ and $\mathcal{G}$, resp. We have a Lyapunov function

$$L = \mathcal{H} + \mathcal{G} + C.$$

Kolmogorov legacy



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Kolmogorov heritage

During his childhood, Andreï Nikolaievitch Kolmogorov found biology very interesting. In fact, in the book *Kolmogorov in Perspective*, one can read that he made the following comment about himself as a schoolboy:

"I was one of the best in my class in mathematics, but my real scientific passions were, first of all biology, and then russian history" ([Kol00], p. 5)

He kept these centers of interest for the rest of his life. Thus, in 1940, Kolmogorov dared to confront the feared Lysenko by defending Mendel's laws – a very dangerous move to make in the middle of Stalin's regime. And, according to V. I. Arnold, the last research done by Kolmogorov [KB67], published in 1967, was motivated by biological ideas about the structure of the brain ([Kol00], p. 94).

In fact, Kolmogorov made only a few isolated contributions to biomathematics; but they all demonstrate, as one would expect, a remarkable originality. In particular, the short note [Kol36] about the predator-prey equation is a model of perspicacity and has had great influence on the deterministic theory of population dynamics. It is one of the rare articles that Kolmogorov published in Italian, doubtlessly in honor of the mathematician Vito Volterra who inaugurated what would later be called *The Golden Age of biomathematics* [SZ78]. Kolmogorov's note represents a qualitative jump in the theory of predator-prey systems.

Kolmogorov and population dynamics by Karl Sigmund (Vienna)

Lyapunov function proof

Proof. Multiply by $\mathcal{H}' = -H(x,z)/g(x)$ and $\mathcal{G}' = G(w,y)/h(y)$

$$x' = g(x)G(x,y),$$
$$y' = h(y)H(x,y).$$

Add the above and for $L = \mathcal{H} + \mathcal{G} + C$ obtain

$$L' = (-G(x,y) + G(w,y))H(x,z) + G(w,y)(-H(x,z) + H(x,y))\,.$$

Hence

$$L' = G_x(w-x)(H(x,z) - H(w,z)) + (G(w,y) - G(w,z))H_y(y-z)\,.$$

Thus we get the claim $L' = -G_x H_x (w-x)^2 + G_y H_y (y-z)^2 \le 0\,.$

Astrophysical nonrelativistic model

For nonrelativistic astrophysical model

$$x' = y - x = G,$$
$$y' = y(2 - x) = yH,$$

we get Lyapunov function

$$L = (x-2)^2 + 2y - 4\log y + C\,.$$

Smoluchowski–Poisson eq. for a cloud of self-gravitating particles

$$\varrho_t = \nabla \cdot \Big(P'(\varrho\theta^{-d/2})(\theta P'(\varrho\theta^{-d/2})\nabla\varrho + \varrho\nabla(\Delta)^{-1}\varrho) \Big)$$

with equation of state binding the density ϱ and the temperature θ with the pressure p via relation

$$p = \theta^{d/2+1} P(\varrho\theta^{-d/2}) \, .$$

As a special, easiest case one can consider $P = \mathrm{Id}$ yielding $p = \theta\varrho$. If we look for radial, stationary solutions in integrated density variables, we end up with the system of ODE's

$$x' = y - x, y' = (2 - x)y$$

and look for the heteroclinic trajectory linking $(0, 0)$ with $(2, 2)$ to find the density profile ϱ related to x for given mass $M = m(R)$ and radius of the system R.

Form Łódź to Wrocław, Rzewuski-Krzywicki-Biler-Stańczy

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Mathematical Methods in the Applied Sciences

Article

Radially symmetric solutions of the Poisson–Boltzmann equation

A. Krzywicki, T. Nadzieja

First published: May/June 1989 | https://doi.org/10.1002/mma.1670110307 |

Stańczy-Przeradzki-Nowak-Włodarski-Mazur-Banach-Steinhaus-Hilbert

Astrophysical relativistic model

In relativistic astrophysics we model mass distribution by

$$\begin{aligned} x' &= y - x = G, \\ y'(1-x) &= y(2-3x) - y^2 = yH(1-x), \end{aligned}$$

we get Lyapunov function centered at $(1/2, 1/2)$

$$2L = -6x - 3\log(1-x) + 2y - \log y + C\,.$$

Tolman–Oppenheimer–Volkoff

- Tolman–Oppenheimer–Volkoff eq.

$$\frac{dP}{dr} = -\frac{G(\varrho(r)c^2 + P(r))(m(r)c^2 + 4\pi r^3 P(r))}{rc^2(rc^2 - 2Gm(r))}$$

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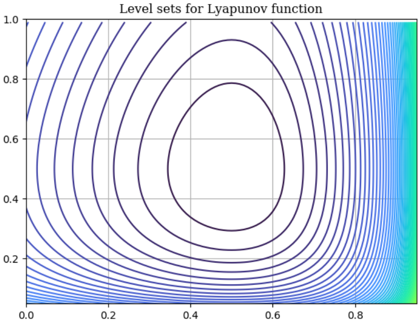
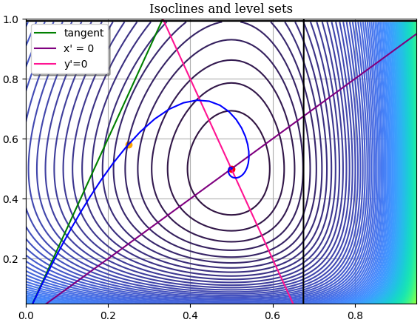
- Milne: $x' = y - x$, $y' = 2y - y(x + y)/(1 - x)$ if $P = c^2 \varrho$
- Lyapunov function

$$2L = 2y - \log(y) - 6x - 3\log(1 - x) + C$$

Lyapunov function level sets

Lyapunov function centered at $(1/2, 1/2)$ and singular as x approaches 1

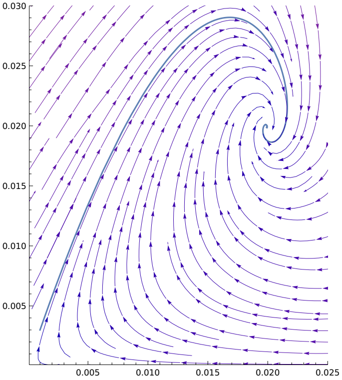


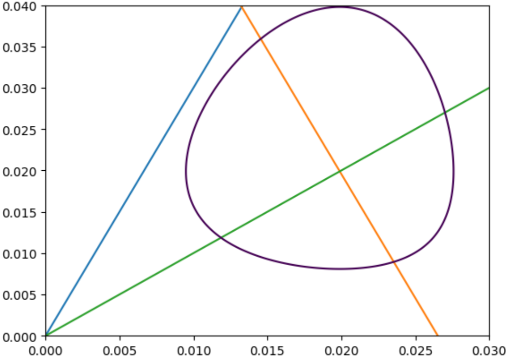


Bounds for heteroclinic connection following from Lyapunov function levels

Heteroclinic connection and geometric invariance

- Heteroclinic trajectory, unstable, iso Y'=0, iso X'=0, Lyapunov





Derivation of a priori bounds for rescaled mass

Define for $y' = 0$ the non-increasing isocline s by

$$H(s(y), y) = 0$$

and assume that it intersects tangent to unstable manifold at $Y > 0$. Then the upper bound for rescaled mass x can be estimated by

$$X = \mathcal{H}^{-1}(G(Y))$$

defined by

$$\mathcal{H}(X) + \mathcal{G}(z) = \mathcal{H}(w) + \mathcal{G}(Y)\,,$$

where (w, z) corresponds to stationary solution and

$$g(x)\mathcal{H}'(x) = -H(x, z)\,,\ h(y)\mathcal{G}'(y) = G(w, y)\,.$$

Nonrelativistic case

In the case $a(x) = 2 - x$ and $b(x) = 0$ we obtain the upper bound X for the x variable of the heteroclinic trajectory joining $(0,0)$ and $(2,2)$ as

$$X = \mathcal{H}|_{[2,\infty)}^{-1}(4 - 2\log 3)) \ ,$$

where $2\mathcal{H}(x) = (x-2)^2, x \geq 2$, whence $\mathcal{H}|_{[2,\infty)}^{-1}(y) = 2 + \sqrt{2y}, y \geq 0$. Thus

$$X = 2 + 2\sqrt{2 - \log 3}\,.$$

Relativistic case

In the case $a(x) = \frac{2-3x}{1-x}$ and $b(x) = \frac{1}{1-x}$ we have the upper bound for the x variable of heteroclinic trajectory joining $(0,0)$ and $(1/2, 1/2)$ as

$$2\mathcal{H}(X) = \log z - 2z + 2(a(0)+1)w - \log((a(0)+1)w)\,,$$

where the function $\mathcal{H}$ is defined as
$2\mathcal{H}(x) = -6x - 3\log(1-x) + 3 - 3\log 2, 1/2 \le x < 1$. Thus

$$2X = 2 + W(-2\exp(-4/3 - 2/3\,\log 2)) = 2 + W\left(-2^{1/3}e^{-4/3}\right)\,.$$

where W is Lambert function, productlog, i.e. inverse of $(\cdot)\exp(\cdot)$ and

$$X < 0.69342\,.$$

Predator-prey model application

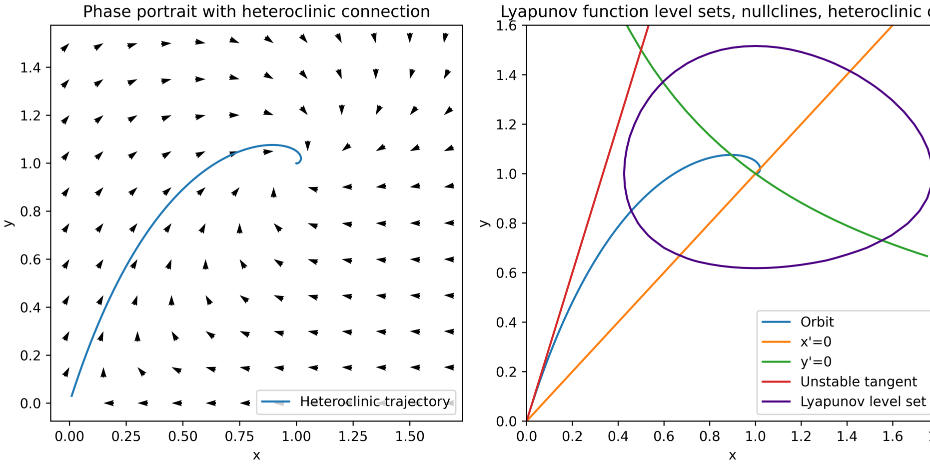


Figure: Typical phase portrait demonstrating the intersection of the isoclines at the equilibrium (w, z). Isocline $y' = 0$ intersects unstable tangent to heteroclinic.

Crash course in ODE's

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- integrability of the inverse of RHS at 0
- not Lipschitz regularity is all that matters

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  Synge–Jüttner and Fermi-Dirac $\rho \sim p$

  Michie–King and RAR $\rho \sim p^{5/7}$
- $p(R) = 0$ implies unique zero solution for S-J, F-D, not for M-K, RAR

Theorem (Krasnosielskiĭ Cone Expansion)

Let X *be a real Banach space, let* $K \subset X$ *be a cone, and let* $0 < \beta_1 < \beta_2$*.*
Define $\Omega_{\beta_i} := \{u \in X : \|u\| < \beta_i\}, \qquad i = 1, 2.$ *Suppose that*

$$T : K \cap \left(\overline{\Omega}_{\beta_2} \setminus \Omega_{\beta_1}\right) \longrightarrow K$$

is completely continuous and satisfies

$$\|Tu\| \le \|u\| = \beta_1, \qquad u \in K \cap \partial\Omega_{\beta_1},$$

and

$$\|Tu\| \ge \|u\| = \beta_2, \qquad u \in K \cap \partial\Omega_{\beta_2}.$$

Then T *possesses a fixed point* $Tu_* = u_*, \quad u_* \in K \cap \left(\overline{\Omega}_{\beta_2} \setminus \Omega_{\beta_1}\right).$
In particular, $\beta_1 \le \|u_*\| \le \beta_2$*, and hence* $u_* \neq 0$*.*

Let

$$(\mathcal{H}u)(r) := \int_r^R F_u(s)\, ds, \tag{1}$$

where

$$F_u(s) := \frac{m_u(s) + 4\pi s^3 p(u(s))}{s[s - 2m_u(s)]}. \tag{2}$$

The associated density operator is

$$(\mathcal{T}u)(r) := \rho((\mathcal{H}u)(r)). \tag{3}$$

A fixed point

$$u = \mathcal{T}u$$

determines the enthalpy profile

$$h = \rho^{-1}(u) = \mathcal{H}u.$$

Fix $0 < a < 1$ and $\mu > 0$. Define a cone

$$\mathcal{C}_{a,\mu} := \left\{ u \in X : \begin{array}{c} u \geq 0, \quad u \text{ is nonincreasing}, \\ \displaystyle\int_0^r s^2 u(s)\, ds \leq r^3 u(r), \quad 0 < r \leq aR, \\ m_u(aR) \geq \mu R^3 \|u\|_\infty \end{array} \right\}.$$

- monotonicity $\frac{d}{dr}\left(\frac{m_u(r)}{r}\right) \geq 0$.
- Harnack - no concentration at 0 - cone expansion

Theorem

Assume the following conditions:

- *the equation-of-state asymptotics assumptions* $p \sim \rho^{7/5}$ *at* 0
- *the constants* a, μ, q, β_2 *satisfy*
  $0 < a < 1, \qquad 0 < q < 1, \qquad 0 < \beta_2 \leq \beta_{\max};$
- *the derived constants* ε_* *and* γ_* *satisfy* $\varepsilon_* < 1, \qquad \gamma_* \geq \frac{1}{3};$
- *the mass-spread constant satisfies*

$$0 < \mu \leq \frac{4\pi}{3}\gamma_* a^3;$$

- *the outer expansion condition* $\beta_2 > \beta_{\mathrm{exp}}(R)$ *holds.*

Then there exists $\beta_1 \in (0, \beta_2)$ *and a nonzero fixed point of the density operator* $\mathcal{T}$

$$u_* \in \mathcal{C}_{a,\mu} \cap \left(\overline{\Omega}_{\beta_2} \setminus \Omega_{\beta_1}\right)$$

The associated enthalpy

$$h_* = \rho^{-1}(u_*) = \mathcal{H}u_*$$

satisfies

$$h_*(R) = 0, \qquad h_*(0) > 0,$$

and, together with

$$m_*(r) = 4\pi \int_0^r s^2 u_*(s)\, ds,$$

solves the TOV system

$$m_*'(r) = 4\pi r^2 \rho(h_*(r)),$$
$$h_*'(r) = -\frac{m_*(r) + 4\pi r^3 P(h_*(r))}{r[r - 2m_*(r)]}.$$

Moreover,

$$\frac{d}{dr}\left(\frac{m_*(r)}{r}\right) \geq 0, \qquad 0 < r \leq aR,$$

$$m_*(aR) \geq \mu R^3 \|u_*\|_\infty, \frac{2m_*(r)}{r} \leq q < 1, \qquad 0 < r \leq R.$$

Asymptotics and properties of Michie-King model

Note that as $r \sim 0$ then for $c = 2\pi(\rho(0) + p(0))(p(0) + \rho(0)/3)$ we have

$$p(r) \sim p(0) - cr^2$$

while

$$p(r) \sim C(R - r)^{7/2}$$

for $C = (2RM\chi/7(R - 2M))^{7/2}$ with $\rho(p) \sim \chi p^{5/7}$ at 0.

Unstable manifold trajectory for Fermi–Dirac distribution

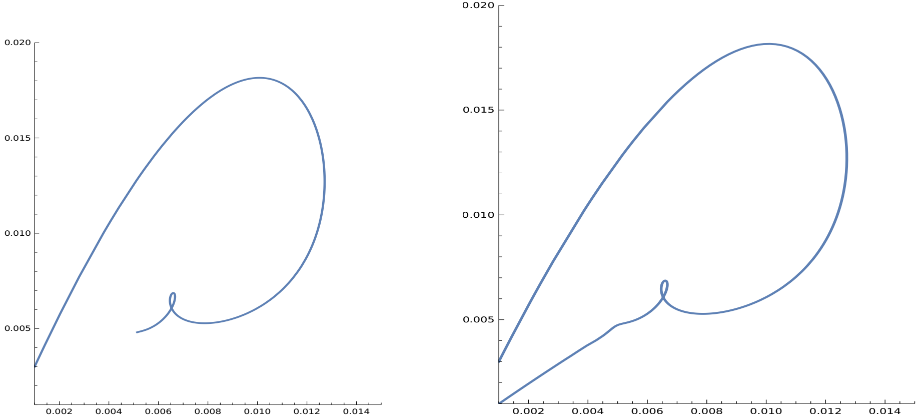

Figure: *The orbit starting from* $(0,0)$ *along unstable manifold*

Unstable manifold trajectory for Michie–King distribution

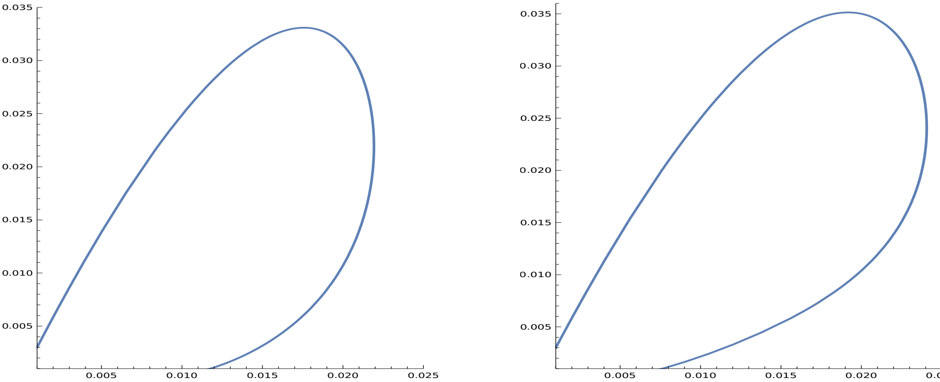

Figure: *The orbit from* $(0, 0)$ *along unstable manifold for* $a = 0.2, 0.1$ *resp.*

Mass vs radius limits

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$$2GM/Rc^2 < 1$$

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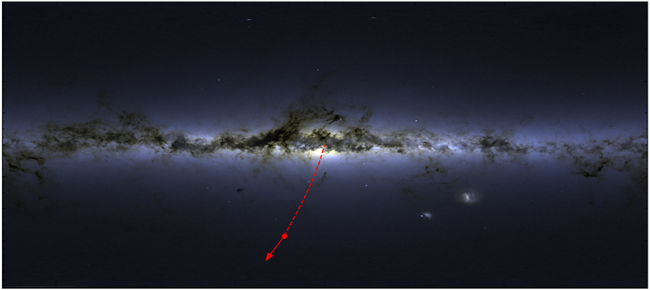
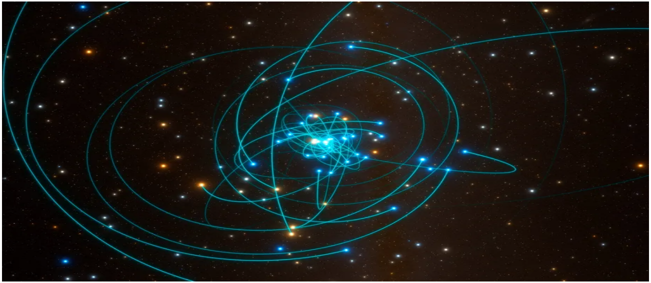
- 0.55 numeric Chavanis, Bors, Stańczy

Koposov - S5 HVS1; Calçada - ESO S2





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Acknowledgements

I would like to thank Pierre-Heinri Chavanis and Shahar Hod for valuable comments, Dorota Bors for cooperation. Thank you for your attention!

Linear equation of state $P = \kappa c^2 \rho$ with mass bound X

We can depict the non-monotone dependence of X on κ with the pictures.

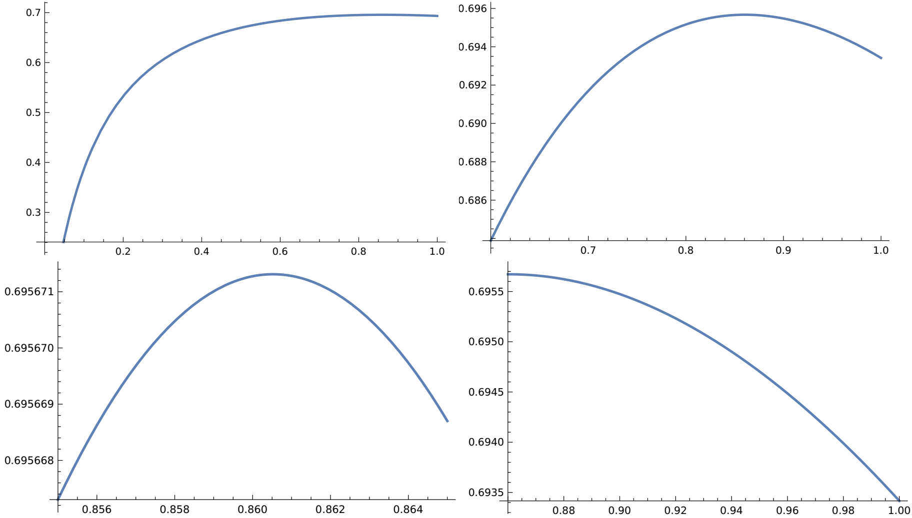

Figure: Dependence of the bound X of x of the heteroclinic trajectory on κ.

In the case $a(x) = 2 - \frac{1+\kappa}{2\kappa}\frac{x}{1-x} = \frac{4\kappa-(1+5\kappa)x}{2\kappa(1-x)}$ with $\kappa \in (0, 1]$ and $b(x) = \frac{1+\kappa}{2}\frac{1}{1-x}$ we have

$$2\kappa(1-x)(a(x) - xb(x)) = 4\kappa - x(\kappa^2 + 6\kappa + 1)$$

thus

$$z = \frac{4\kappa}{(\kappa+1)^2 + 4\kappa} .$$

Moreover, we have

$$zB(x) - A(x) = -\frac{1+5\kappa}{2\kappa}x - \frac{1+\kappa}{2\kappa}\frac{4\kappa+(1+\kappa)^3}{4\kappa+(1+\kappa)^2}\log(1-x) + D_\kappa ,$$

where D_κ is chosen, so that $A(z) = zB(z)$, in fact $A(z) = 0$ and $B(z) = 0$, i.e.

$$D_\kappa = \frac{1+5\kappa}{2\kappa}z + \frac{1+\kappa}{2\kappa}\frac{4\kappa+(1+\kappa)^3}{4\kappa+(1+\kappa)^2}\log(1-z) ,$$

whence

$$D_\kappa = \frac{2(1+5\kappa)}{4\kappa+(1+\kappa)^2} + \frac{1+\kappa}{2\kappa}\frac{4\kappa+(1+\kappa)^3}{4\kappa+(1+\kappa)^2}\log\left(\frac{(1+\kappa)^2}{4\kappa+(1+\kappa)^2}\right) .$$

Furthermore, solving

$$2\kappa(1-x)(a(x)-x(a(0)+1)b(x)) = 4\kappa - x(3\kappa^2+8\kappa+1) = 0$$

gives

$$w = 4\kappa/(3\kappa^2+8\kappa+1)\,.$$

Hence, the upper bound for the x variable of the heteroclinic trajectory joining $(0,0)$ and (z,z) follows and is equal to

$$X = 1 + W\left(-\alpha_\kappa \exp\left(-\alpha_\kappa - (E_\kappa - D_\kappa)\frac{2\kappa}{1+\kappa}\frac{4\kappa+(1+\kappa)^2}{4\kappa+(1+\kappa)^3}\right)\right)/\alpha_\kappa$$

with

$$\alpha_\kappa = \frac{1+5\kappa}{1+\kappa}\frac{4\kappa+(1+\kappa)^2}{4\kappa+(1+\kappa)^3},$$

where $E_\kappa = (a(0)+1)w - z - z\log((a(0)+1)w/z)$ and W is Lambert function or productlog, i.e. inverse of $(\cdot)\exp(\cdot)$. In fact

$$E_\kappa = \frac{12\kappa}{3\kappa^2+8\kappa+1} - \frac{4\kappa}{4\kappa+(1+\kappa)^2} - \frac{4\kappa}{4\kappa+(1+\kappa)^2}\log\left(\frac{3\kappa^2+18\kappa+3}{3\kappa^2+8\kappa+1}\right).$$

General linear case

For linear pressure $p = \kappa\rho$. we have

$$\begin{cases} x'(s) = -\,x(s) + \, y(s)\,, \\ y'(s) = 2\,y(s) - \frac{1+\kappa}{2\kappa} \cdot \frac{y(s)(x(s)+\kappa y(s))}{1-x(s)}\,. \end{cases}$$

$$(x_\kappa, y_\kappa) = \left(\frac{4\kappa}{(1+\kappa)^2 + 4\kappa}, \frac{4\kappa}{(1+\kappa)^2 + 4\kappa}\right)$$

$$L = 2y - (5 + 1/\kappa)x - 2x_k \log\left(y(1-x)^{\delta_\kappa}\right) + C_\kappa$$

where the exponent δ_κ is defined by

$$8\kappa^2\delta_\kappa = (5\kappa + 1)(\kappa + 1)^2$$

while the constant $C_\kappa = (3 + 1/\kappa)x_\kappa + 2x_\kappa \log\left(x_\kappa(1-x_\kappa)^{\delta_\kappa}\right)$.

Appendix - Relativistic Equation of State for fermionic Michie–King distribution

- The energy–mass density

$$\rho = \frac{2m}{h^3} \int_0^\infty f_c(p) \left(1 + \frac{\varepsilon(p)}{mc^2}\right) d^3p$$

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$$f_c(p) = \left(\frac{1 - \exp((\varepsilon(p) - \varepsilon_c)/kT)}{\exp((\varepsilon(p) - \mu)/kT) + 1}\right)_+$$

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- the kinetic energy

$$\varepsilon(p) = \sqrt{c^2p^2 + m^2c^4} - mc^2 \,.$$

Appendix - relativistic Equation of State for fermionic Michie-King distribution - RAR

- $\rho = \frac{8\sqrt{2}\pi c^3 \kappa^{3/2} m^4}{h^3(1-\kappa y)^{3/2}} \int_0^y x^{1/2} \left(\frac{\kappa x}{1-\kappa y} + 1\right)^2 \left(\frac{\kappa x/2}{1-\kappa y} + 1\right)^{1/2} \frac{1-\exp(x-y)}{1+\chi \exp(x-y)} dx$

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- As $y \sim 0$ we have

$$\rho(y) \sim \frac{8\sqrt{2}\pi m \kappa^{3/2}}{1+\chi} \left(\frac{mc}{h}\right)^3 \int_0^y \sqrt{x}(y-x)dx = \rho_0 y^{5/2}$$

and $P(y) \sim p_0 y^{7/2}$ hence as $\rho \sim 0$ then $P(\rho) \sim p_0(\rho/\rho_0)^{7/5}$.

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  and $P(y) \sim p_0 y^{7/2}$ hence as $\rho \sim 0$ then $P(\rho) \sim p_0(\rho/\rho_0)^{7/5}$.
- as $y \sim 1/\kappa$ then $\rho(y) \sim \frac{\rho_\infty}{(1-\kappa y)^4}$ and

$$\rho_\infty = 8\pi c^3 m^4 h^{-3} \int_0^1 w^3 \frac{1-\exp((w-1)/\kappa)}{1+\chi \exp((w-1)/\kappa)} dw \,.$$

  while $P(y) \sim \frac{c^2 \rho_\infty}{3(1-\kappa y)^4}$. Thus as $\rho \sim \infty$ we have $P(\rho) \sim c^2\rho/3$.