12th POTOR conference, Bedlewo, August 2-7

Varying cosmological constants,diffusion and black hole's evaporation law

Zbigniew Haba
Institute of Theoretical Physics, University of Wroclaw

Plan of the talk

- Bianchi identities
- Relativistic diffusive gas
- Integrated Bianchi identities and black hole's evaporation

Bianchi

The Bianchi identity leads to the equation

$$\nabla_\mu \tilde{T}^\mu_\nu = 0. \tag{1}$$

If we can define a fluid four-velocity u^μ then

$$\tilde{T}^{\mu\nu} = 8\pi G T^{\mu\nu} + \Sigma^{\mu\nu} - g^{\mu\nu}\Lambda, \tag{2}$$

where (ideal fluid)

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + g^{\mu\nu}P. \tag{3}$$

ρ has the meaning of the energy-density and P is the pressure. The Bianchi identity (1) gives an equation for the "fluids"

$$8\pi\partial_\mu G T^\mu_\nu + 8\pi G \nabla_\mu T^\mu_\nu - \partial_\nu \Lambda + \nabla_\mu \Sigma^\mu_\nu = 0. \tag{4}$$

Assume the metric

$$dl^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{00} dt^2 + \gamma_{jk} dx^j dx^k = -|g_{00}| dt^2 + \gamma_{jk} dx^j dx^k. \tag{5}$$

The model of the gas is simple in FLRW metric

$$dl^2 = -dt^2 + a^2\Big((1-kr^2)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\Big). \tag{6}$$

Then Bianchi is

$$8\pi\partial_t(G\rho) + 24\pi GH(\rho + P) + \partial_t\Lambda = 0 \tag{7}$$

This looks like a local form of $dE + pdV$

Non-conserved energy-momentum:relativistic diffusion

We have the phase space distribution Ω for a microscopic (statistical) description. Then,

$$T^{\mu\nu} = \sqrt{g} \int d\mathbf{p} p_0^{-1} p^\mu p^\nu \Omega, \tag{8}$$

where p_0 is determined from the mass-shell condition $p_\mu p^\mu = -m^2$
We can also define the conserved particle current ($\nabla^x_\mu N^\mu = 0$)

$$N^\mu = \sqrt{g} \int d\mathbf{p} p_0^{-1} p^\mu \Omega. \tag{9}$$

The diffusion equation is

$$(p^\mu \partial^x_\mu - \Gamma^k_{\mu\nu} p^\mu p^\nu \partial^p_k)\Omega = \kappa^2 \triangle_H \Omega, \tag{10}$$

where κ^2 is the diffusion constant.

The operator $\triangle_H$ is the Laplace-Beltrami operator on the mass hyperboloid $\mathcal{H}_+$

$$\triangle_H = \frac{1}{\sqrt{h}} \partial_j^p h^{jk} \sqrt{h} \partial_k^p, \tag{11}$$

where

$$h^{jk} = m^2 \gamma^{jk} + p^j p^k, \tag{12}$$

$\partial_j^p = \frac{\partial}{\partial p^j}$ and $h = \det(h_{jk}) = m^{-4} \det(\gamma_{jk}) \omega_g^{-2}$ is the determinant of the metric h_{jk}, $\omega_g = \sqrt{m^2 + \gamma_{jk} p^j p^k}$
If $\kappa \neq 0$ then the energy-momentum is not conserved but

$$\nabla_\mu^x T^{\mu\nu} = 3\kappa^2 N^\nu. \tag{13}$$

The equation for ρ from Bianchi can be solved if $P = w\rho$.

$$\rho(t) = 3na^{-3(1+w)}(T_0 + \kappa^2 \int_{t_0}^{t} a(s)^{3w} ds) \tag{14}$$

From thermodynamics this equation defines the temperature

$$T = a^{-3w}(T_0 + \kappa^2 A). \tag{15}$$

where

$$A = \int_{t_0}^{t} a(s)^{3w} ds$$

For an exponential expansion $a = \exp(Ht)$ we have

$$T = \exp(-3wHt)\Big(T_0 - \frac{\kappa^2}{3wH}\exp(3wHt_0)\Big) + \frac{\kappa^2}{3wH}$$

For a power-law expansion $a = t^{\alpha}$ we get

$$T = (T_0 - \tfrac{\kappa^2}{1+3w\alpha} t_0^{1+3w\alpha}) t^{-3w\alpha} + \tfrac{\kappa^2}{1+3w\alpha} t.$$

Varying cosmological constants
if $\Lambda = const$ then

$$G \simeq (T_0 + \kappa^2 A)^{-1}$$

In general, in terms of entropy S

$$G \simeq \exp(-S)$$

If $G = const$

$$\Lambda = \Lambda_0 - K\kappa^2 \int_{t_0}^{t} a^{-3}$$

as in unimodular gravity
Solutions of Einstein equations(for large time):
$a(t) \simeq \exp(\Lambda_0 t)$ if $\Lambda_0 > 0$
and
$a(t) \simeq Kt$ if $\Lambda_0 = 0$
(coasting cosmology)

Thermodynamics in an expanding universe

First law of thermodynamics

$$TdS = dE + pdV = \frac{\partial \rho}{\partial T} VdT + (p+\rho)dV = T\left(\frac{\partial S}{\partial T} dT + \frac{\partial S}{\partial V} dV\right). \tag{16}$$

Assume that in the diffusive system the entropy is the sum

$$S = S_{st} + S_{diss}, \tag{17}$$

where S_{st} is the constant fixed first for $\kappa = 0$ and then required to be constant for $\kappa \neq 0$

For the relativistic diffusion

$$T\partial_t S_{diss} = V(\partial_t \rho) + V^{-1}\partial_t V \rho(1+w) = V3\kappa^2 n a^{-3} \\ = 3\kappa^2 n V_0 \tag{18}$$

Then

$$S_{diss} = 3\kappa^2 n V_0 \int_{t_0}^{t} \frac{ds}{T(s)} + C = 3nV_0 \ln(T_0 + \kappa^2 A) + C, \tag{19}$$

Statistical physics:solutions of the diffusion equation

The solution with the Jüttner initial distribution in de Sitter space is

$$\Omega_t = \frac{n}{8\pi}(\kappa^2 H^{-1} a)^{-3} \exp(-\sqrt{a^2\mathbf{p}^2 + m^2} H\kappa^{-2}). \tag{20}$$

The solutions of massless equation

$$\Omega_t = \frac{n}{8\pi}(T_0 + \kappa^2 A)^{-3} \exp\Big(-\frac{a^2|\mathbf{p}|}{T_0 + \kappa^2 A}\Big), \tag{21}$$

where

$$A(t) = \int_{t_0}^{t} a(s)ds. \tag{22}$$

We consider a covariant form of first law of thermodynamics. We introduce the inverse temperature β as a four-vector $\beta^\mu = \beta u^\mu$ The entropy density σ^μ as the four-vector

$$\sigma^\mu = \sigma u^\mu. \tag{23}$$

We express the first law of thermodynamics of a moving fluid as

$$\beta^\nu \nabla_\mu T^\mu_\nu = -\nabla_\mu \sigma^\mu. \tag{24}$$

For a diffusing system of particles having a phase space distribution Ω we can calculate the statistical physics (microscopic) entropy σ^μ

$$\sigma^\mu = -\sqrt{g} \int d\mathbf{p} p_0^{-1} p^\mu \Omega \ln \Omega = \sigma u^\mu . \tag{25}$$

In the comoving frame $\sigma = \sigma^0$

$$\sigma = -\sqrt{g} \int d\mathbf{p} \Omega \ln \Omega = 3na^{-3} \ln(T_0 + \kappa^2 A) - a^{-3}(n \ln n - 3n). \tag{26}$$

n- particle density
time $\simeq \exp S$

Integrated Bianchi identity

$$8\pi\partial_t G\rho + 8\pi G(\partial_t\rho + 3H(\rho + P)) + \partial_t\Lambda = 0. \tag{27}$$

After integration

$$\begin{aligned}&\int d\mathbf{x}\sqrt{\gamma}(\partial_t G\rho + G\partial_t\rho) + G\int d\mathbf{x}(\rho + P)\partial_t\sqrt{\gamma}\\ &= -\frac{1}{8\pi}\int d\mathbf{x}\sqrt{\gamma}\partial_t\Lambda.\end{aligned} \tag{28}$$

We can identify here the first law of thermodynamics

$$dE + PdV = dQ = TdS. \tag{29}$$

When we define the mass as

$$M = \int d\mathbf{x}\sqrt{\gamma}\rho \tag{30}$$

then

$$M\partial_t G + G(\partial_t M + \int d\mathbf{x} P \partial_t \sqrt{\gamma}) = -\frac{1}{8\pi}\int d\mathbf{x}\sqrt{\gamma}\partial_t \Lambda \tag{31}$$

In terms of the entropy

$$M\partial_t G + GT\partial_t S = -\frac{1}{8\pi}\int d\mathbf{x}\sqrt{\gamma}\partial_t \Lambda. \tag{32}$$

If

$$\Gamma T dS = dM$$

or

$$\frac{1}{T} = \Gamma \frac{\partial S}{\partial M}$$

then

$$M\partial_t G + \Gamma G \partial_t M = -\tfrac{1}{8\pi} \int d\mathbf{x} \sqrt{\gamma} \partial_t \Lambda. \tag{33}$$

Usually, $\Gamma = 1$ (also for $P = 0$) then

$$\partial_t (MG) = -\tfrac{1}{8\pi} \int d\mathbf{x} \sqrt{\gamma} \partial_t \Lambda. \tag{34}$$

If we chose another integration with a time-independent factor v

$$\int dt\Big(\tilde{M}\partial_t G + G\partial_t(\tilde{M} + \int d\mathbf{x} v p \partial_t \sqrt{\gamma}\Big)$$
$$= -\frac{1}{8\pi}\int dt \int d\mathbf{x}\sqrt{\gamma} v \partial_t \Lambda$$

with $\tilde{M} = \int d\mathbf{x}\sqrt{\gamma} v\rho$. The only change would involve a change of measure $\sqrt{\gamma} \to v\sqrt{\gamma}$ in the definition of the mass and in entropy

$$S = \int d\mathbf{x}\sqrt{\gamma} v\sigma \tag{35}$$

Black hole radiation and entropy

Quantum radiation law (Stefan-Boltzmann-Planck) which follows from thermodynamics

A the area

$$\frac{dM}{dt} = -\lambda T^4 A, \tag{36}$$

The BH formula is

$$T_{BH} = \frac{\hbar}{8\pi GM} \tag{37}$$

and

$$S_{BH} = \frac{4\pi}{\hbar} M^2 G \tag{38}$$

If instead we use T_{BH} for varying G (with $\Gamma = 1$)

$$dS = \Gamma T_{BH}^{-1} dM \tag{39}$$

then the formula for the entropy is

$$S = \frac{8\pi}{\hbar} \int GMdM. \tag{40}$$

Black hole evaporation

$T \simeq (MG)^{-1}$ and $A \simeq (GM)^2$

$$\frac{dM}{dt} = -\hbar\alpha M^{-2} G^{-2}, \tag{41}$$

If $GM \simeq const$ then for $\Lambda = const$

$$M_t = M_0 - \hbar\alpha M_0^{-2} G_0^{-2}(t - t_0), \tag{42}$$

If $\Gamma = 1$ and Λ is time-dependent

$$MG = G_0 M_0 - \int_{t_0}^{t} \frac{ds}{8\pi} \int d\mathbf{x}\sqrt{\gamma}\partial_s\Lambda. \tag{43}$$

Then,

$$M = M_0 - \hbar\alpha \int_{t_0}^{t} d\tau \left(G_0 M_0 - \int_{t_0}^{\tau} \frac{ds}{8\pi} \int d\mathbf{x}\sqrt{\gamma}\partial_s\Lambda \right)^{-2} \tag{44}$$

For the BH entropy ($\Gamma = \frac{2}{3}$) Bianchi reads

$$\frac{3}{2} M \partial_t G + G \partial_t M + \frac{1}{8\pi} \int d\mathbf{x} \sqrt{\gamma} \partial_t \Lambda = 0. \tag{45}$$

The general option : with $\Lambda = const$ and $\Gamma \neq 0$, $\Gamma \neq \frac{3}{2}$)

$$M = M_0\Big(1-(3-2\Gamma)\hbar\alpha G_0^{-2}M_0^{-3}(t-t_0)\Big)^{\frac{1}{3-2\Gamma}} \equiv M_0 Z^{\frac{1}{3-2\Gamma}}, \quad (46)$$

$$G = G_0\Big(1-(3-2\Gamma)\hbar\alpha G_0^{-2}M_0^{-3}(t-t_0)\Big)^{-\frac{\Gamma}{3-2\Gamma}} \equiv G_0 Z^{-\frac{\Gamma}{3-2\Gamma}}. \quad (47)$$

Summarize: $\Lambda = const$

Γ	**T,L**	**time**
$0 < \Gamma < 1$	increasing to infinity	finite
$\Gamma = 1$	constant	all time
$1 < \Gamma < \frac{3}{2}$	decreasing to zero	finite
$\Gamma \geq \frac{3}{2}$	decreasing to zero	infinite

For $\Gamma \geq \frac{3}{2}$ the mass, the temperature and the luminosity tend to zero asymptotically in time.

Conclusion:if evaporation is treated with varying G and Λ (classical thermodynamics+classical Einstein equations) then we obtain different results then in Hawking's approach.