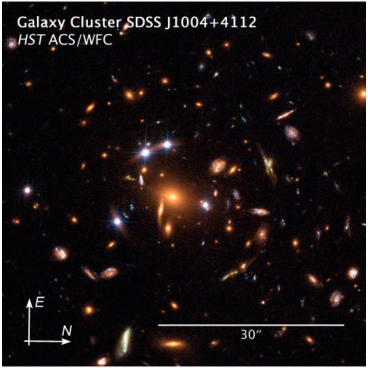
Gauge dependence of the gravitational lensing.

Szymon Sikora
Astronomical Observatory, Jagiellonian University

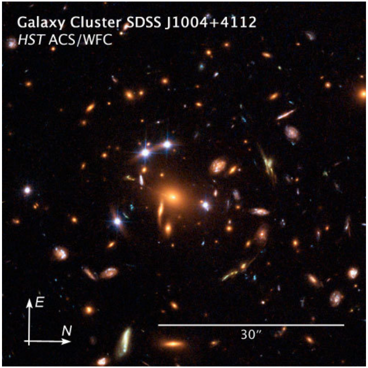
.

in collaboration with:
Krzysztof Głód, Alicja Pucek, Jan Ostrowski

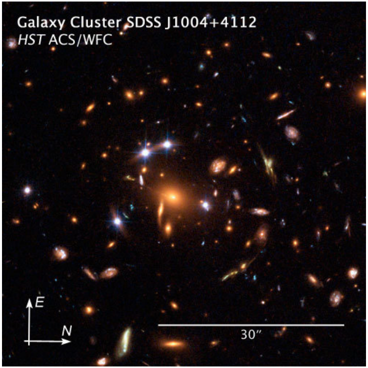
6 August 2026



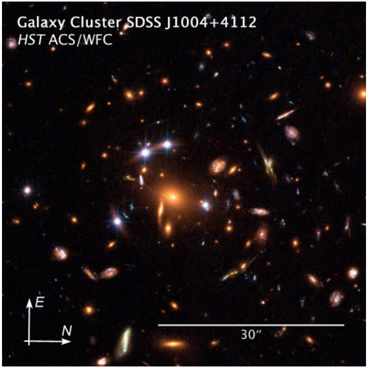
In the context of the gravitational lensing the linear perturbation theory is considered.
The metric: $g_{mn} = g^{(0)}_{mn} + \varepsilon\, g^{(1)}_{mn}$.
The gauge transformation: $\widetilde{x}^n = x^n - \varepsilon\, \xi^n$.



In the standard approach the longitudinal gauge is adopted.
$ds^2 = -(1+2\psi)dt^2 + (1-2\psi)A(t)^2\,\hat{g}_{ij}\,dx^i\,dx^j$.



The aim of the gravitational lensing is to derive the lens density distribution $\rho(\vec{x})$ on basis of the observed images.
The energy density ρ is not a gauge invariant quantity.



There was a discussion whether all possible inhomogeneities can be described within linear perturbation theory in the longitudinal gauge. See e.g.: Kolb et al. [Phys Rev D. 103002, (2008)], Van Acoleyen, [JCAP,10,028, (2008)]

The model setup. The background.

The metric: $g_{mn} = g^{(0)}_{mn} + \varepsilon\, g^{(1)}_{mn}$.

The background line element: $ds^2 = -B^2(t)\, dt^2 + A^2(t)\, \widehat{g}_{ij}\, dx^i\, dx^j$

The matter four-velocity: $u_n = u^{(0)}_n + \varepsilon\, u^{(1)}_n$, $u^a u_a = -1$.

$u^{(0)}_n = (-B, 0, 0, 0)$

The background model is dust: $\partial_t \left(\frac{A(\partial_t A)^2}{B^2} - \frac{\Lambda}{3} A^3 \right) = 0$

The first order perturbations.

The metric and velocity field perturbations:

$$g^{(1)}_{00} = -2B^2\mathfrak{a} \qquad g^{(1)}_{0i} = -A^2\,\widehat{\nabla}_i\,\mathfrak{b}$$

$$g^{(1)}_{ij} = 2A^2\left(\widehat{\nabla}_{(i}\widehat{\nabla}_{j)}\mathfrak{c} - \tfrac{1}{3}\Delta\mathfrak{c} + \mathfrak{d}\widehat{g}_{ij}\right) \qquad \Delta = \widehat{\nabla}^k\widehat{\nabla}_k.$$

$$u^{(1)}_0 = -B\mathfrak{a} \qquad u^{(1)}_i = -B\widehat{\nabla}_i\mathfrak{e}$$

The gauge transformations:

$$\mathfrak{a} = a + \tfrac{1}{B}\partial_t(Bf)$$
$$\mathfrak{b} = b - \partial_t g + \tfrac{B^2}{A^2}f$$
$$\mathfrak{c} = c + g$$
$$\mathfrak{d} = d + \tfrac{1}{3}\Delta g + \tfrac{\partial_t A}{A}f$$
$$\mathfrak{e} = e + f$$

The covariant model definition.

Nonconductive, inviscid and pressureless fluid:

$$q_n = -u^b h_n{}^a T_{ba} = 0, \tag{1}$$

$$\pi_{mn} = \left(h_m{}^b h_n{}^a - \frac{1}{3} h_{mn} h^{ba} \right) T_{ba} = 0, \tag{2}$$

$$p = \frac{1}{3} h^{ba} T_{ba} = 0, \tag{3}$$

The covariant model definition.

Nonconductive, inviscid and pressureless fluid:

$$\frac{A}{\partial_t A}\partial_t\left(d - \frac{1}{3}\Delta c\right) = \frac{AB}{\partial_t A}\partial_t\left(\frac{\partial_t A}{AB}\right)e + a, \tag{4}$$

$$\frac{1}{AB}\partial_t\left(\frac{A^3}{B}(b + \partial_t c)\right) = d - \frac{1}{3}\Delta c + a, \tag{5}$$

$$\partial_t\left(\frac{A^3}{B}\partial_t d\right) + \frac{1}{3}\Delta\left(\partial_t\left(\frac{A^3}{B}b\right) - AB\left(d - \frac{1}{3}\Delta c + a\right)\right) = \frac{AB}{\partial_t A}\partial_t\left(\frac{A(\partial_t A)^2}{B^2}a\right). \tag{6}$$

How to find the solution.

The equations (4-6) are satisfied when:

$$i = \frac{A^3}{B}\left(b + \partial_t c - \frac{B^2}{A\partial_t A}\left(d - \frac{1}{3}\Delta c\right)\right), \tag{7}$$

$$j = \frac{\partial_t A}{A} e - d + \frac{1}{3}\Delta c, \tag{8}$$

$$\partial_t i = j \left(\partial_t \left(\frac{A^2 B}{\partial_t A}\right) - AB\right), \tag{9}$$

$$\partial_t j = 0. \tag{10}$$

It is convenient to consider a new gauge function:

$$\mathfrak{h} = d - \frac{1}{3}\Delta c + \frac{\partial_t A}{A} f, \tag{11}$$

How to find the solution.

When $B = 1$: $\quad A(t) = \beta \sinh^{2/3}\left(\frac{\sqrt{3\Lambda}}{2}(t-\alpha)\right)$,

$$i = \mathcal{A}_3(t)\, C_j(r,\vartheta,\varphi) + C_i(r,\vartheta,\varphi)\,, \tag{12}$$

$$j = C_j(r,\vartheta,\varphi), \tag{13}$$

$$\mathcal{A}_3(t) = \frac{9\beta}{5\sqrt{3\Lambda}} \sinh^{5/3}\left(\frac{\sqrt{3\Lambda}}{2}(t-\alpha)\right) \times {}_2\mathrm{F}_1\left(\frac{5}{6}, \frac{3}{2}; \frac{11}{6}; -\sinh^2\left(\frac{\sqrt{3\Lambda}}{2}(t-\alpha)\right)\right)\,. \tag{14}$$

To summarize: The solution is described by the functions C_i, C_j, which guarantee that the model is a nonconductive, inviscid and pressureless fluid, and the two gauge functions $\mathfrak{c}$ and $\mathfrak{h}$.

How the gauge freedom is related to the gravitational lensing.

The spatial curvature scalar, energy density, its spatial gradient and expansion scalar:

$$\mathcal{R} = \varepsilon \frac{4}{A^2} \Delta j, \tag{15}$$

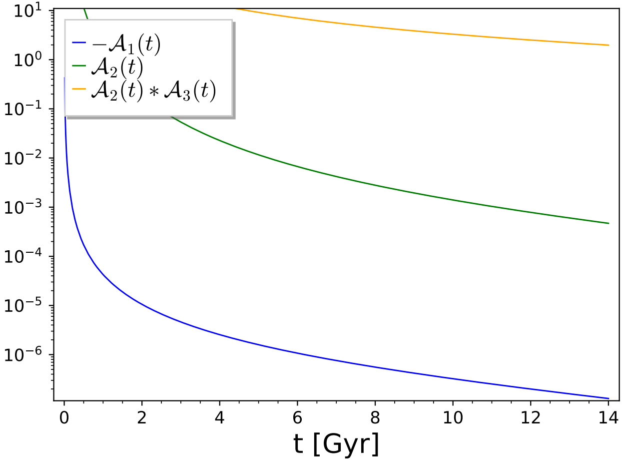
$$8\pi\rho = \frac{3(\partial_t A)^2}{A^2 B^2} - \Lambda + \varepsilon \Big(\mathcal{A}_1(t)\,(\mathfrak{h} + j) + \mathcal{A}_2(t)\,\Delta i \Big), \tag{16}$$

$$8\pi X_n = \varepsilon \frac{2\partial_t A}{A^4 B} (0, \partial_r \Delta i, \partial_\vartheta \Delta i, \partial_\varphi \Delta i), \tag{17}$$

$$\theta = 3\frac{\partial_t A}{AB} + \varepsilon \left[\frac{3}{2} \left(\frac{AB\,\Lambda}{\partial_t A} - \frac{3\,\partial_t A}{AB} \right) (h + j) + \frac{\triangle \partial_t c}{B} \right]. \tag{18}$$

$$\mathcal{A}_1(t) = -3 \bigg(\frac{3(\partial_t A)^2}{A^2 B^2} - \Lambda \bigg), \quad \mathcal{A}_2(t) = \frac{2\partial_t A}{A^4 B}. \tag{19}$$

Time dependence.



The family of gauges under consideration.

$$\Delta\psi = 4\pi\rho^{(1)} \quad \rho^{(1)}(r) = \frac{\rho_c}{(r/r_s)(1+(r/r_s))^2}$$

$$\mathfrak{h}(r) = (s-1)\frac{\partial_t A}{A^2} i = (s-1)\,\psi,$$

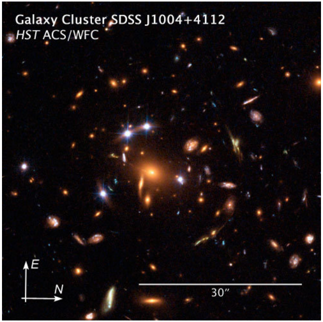
$$\partial_t \mathfrak{c} = \frac{B^2}{A\partial_t A}\mathfrak{h} + \frac{B}{A^3} i = \frac{s\,\psi}{A\partial_t A}\,.$$

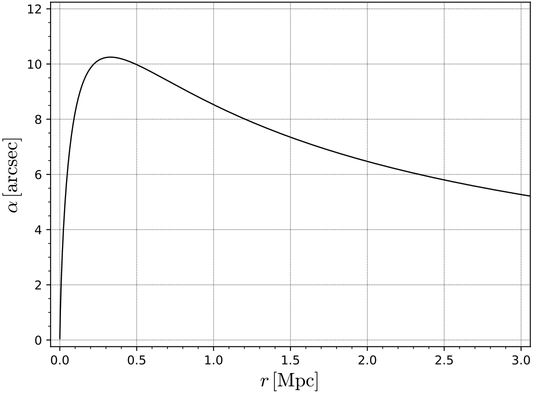
$$\mathfrak{c}(t,r) = \frac{s\,\psi}{\Lambda\beta^2}\left[\log\left(\frac{1+\chi}{\sqrt{\chi^2-\chi+1}}\right) + \sqrt{3}\,\mathrm{atan}\left(\frac{2\chi-1}{\sqrt{3}}\right)\right]\,,$$

where $\chi = \chi = \sinh^{2/3}(\sqrt{3\Lambda}t/2)$.

Work in progress...

$$\Delta\psi = 4\pi\rho^{(1)} \quad \rho^{(1)}(r) = \frac{\rho_c}{(r/r_s)(1+(r/r_s))^2}$$





Thank you for your attention!