Separability of the motion of spinning test particles in curved space-time

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Five stops along the way

01 **Motivation**
LISA, EMRIs, and precision physics at order $\mathcal{O}(S)$

02 **Setup**
MPD equations and the centroid ambiguity

03 **Canonical formalism**
Body-fixed and worldline-frame variables; Tulczyjew–Dixon via Dirac brackets

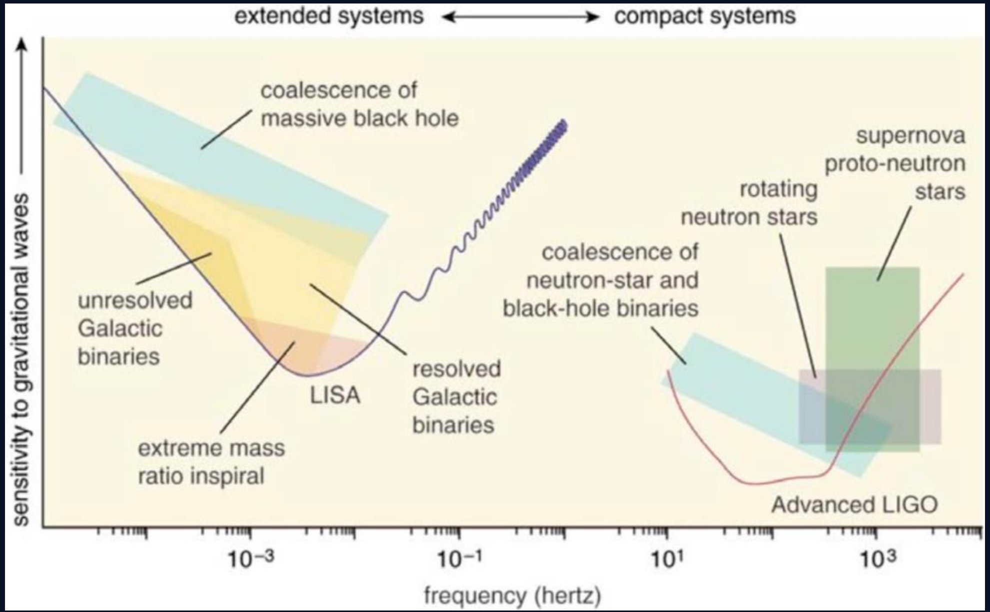
04 **Main result**
Separable parallel transport $\Longrightarrow$ separable spinning-particle motion

05 **Examples**
Kerr & Marck, Kerr-NUT-(A)dS, pp-wave, FLRW

One sentence — if parallel transport separates in the same coordinates as geodesics, the entire linear-in-spin Hamilton-Jacobi equation separates.

LISA — Laser Interferometer Space Antenna

- ESA / NASA L3 mission, launch window mid-2030s.
- Three-spacecraft constellation in heliocentric orbit, arms ≈ 2.5×10^6 km.
- Frequency band 10^{-4} Hz $\lesssim f \lesssim 10^{-1}$ Hz set by orbital dynamics + laser-noise floor.

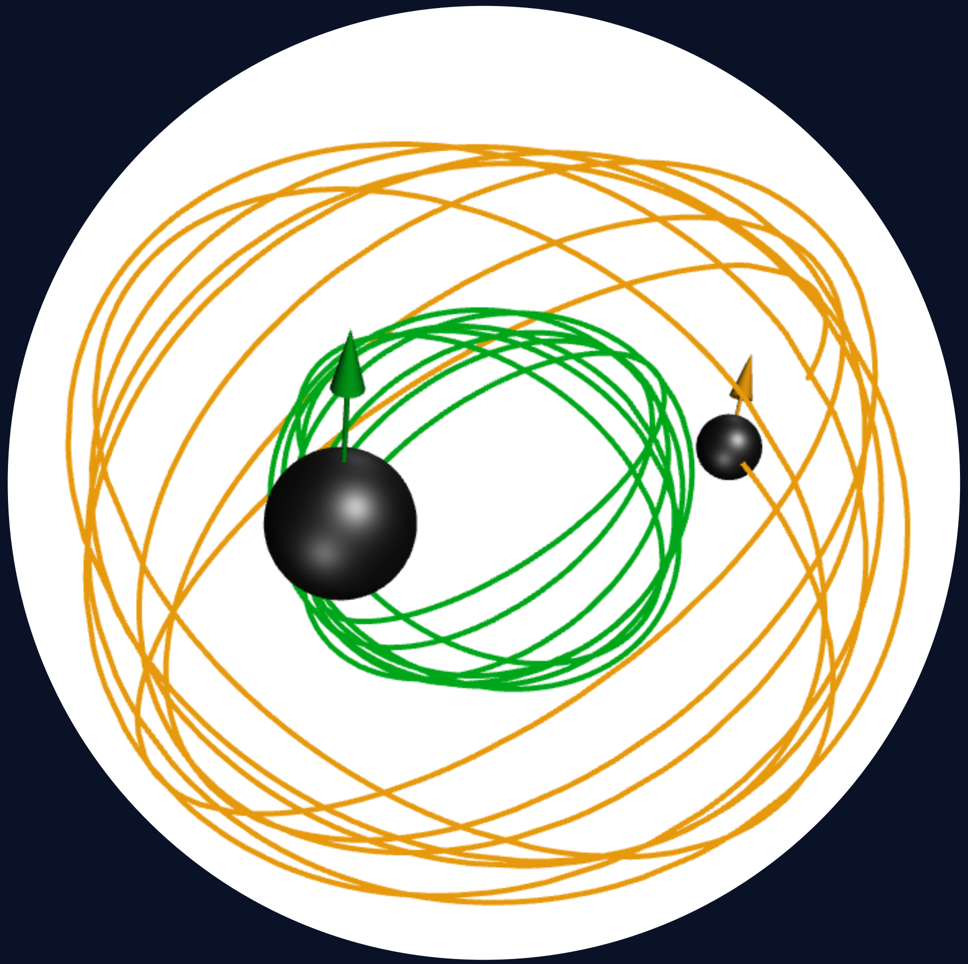


- The targetable mass range is fixed by the innermost stable circular orbit (ISCO):

$$f_{\mathrm{ISCO}} \approx \frac{2.2\ \mathrm{kHz}}{M/M_\odot} \quad \Rightarrow \quad 2 \times 10^4\, M_\odot \lesssim M_{\mathrm{LISA}} \lesssim 2 \times 10^7\, M_\odot$$

Extreme mass ratio inspirals (EMRIs)

- Stellar-mass compact secondary $\mathcal{M}$ ~ 1 – 50 M⊙.
- Massive central black hole M ~ 10^5 – 10^7 M⊙.
- Mass ratio $\varepsilon = \mathcal{M} / M$ ~ 10^{-7} – 10^{-4}.
- Formation: „loss cone“ N-body scattering; tidal capture; Hills mechanism; AGN-disc capture & migration.
- ~ 10^5 strong-field cycles near the ISCO per event.
- *Strong-field relativistic modelling problem, strong-field relativity precision test*

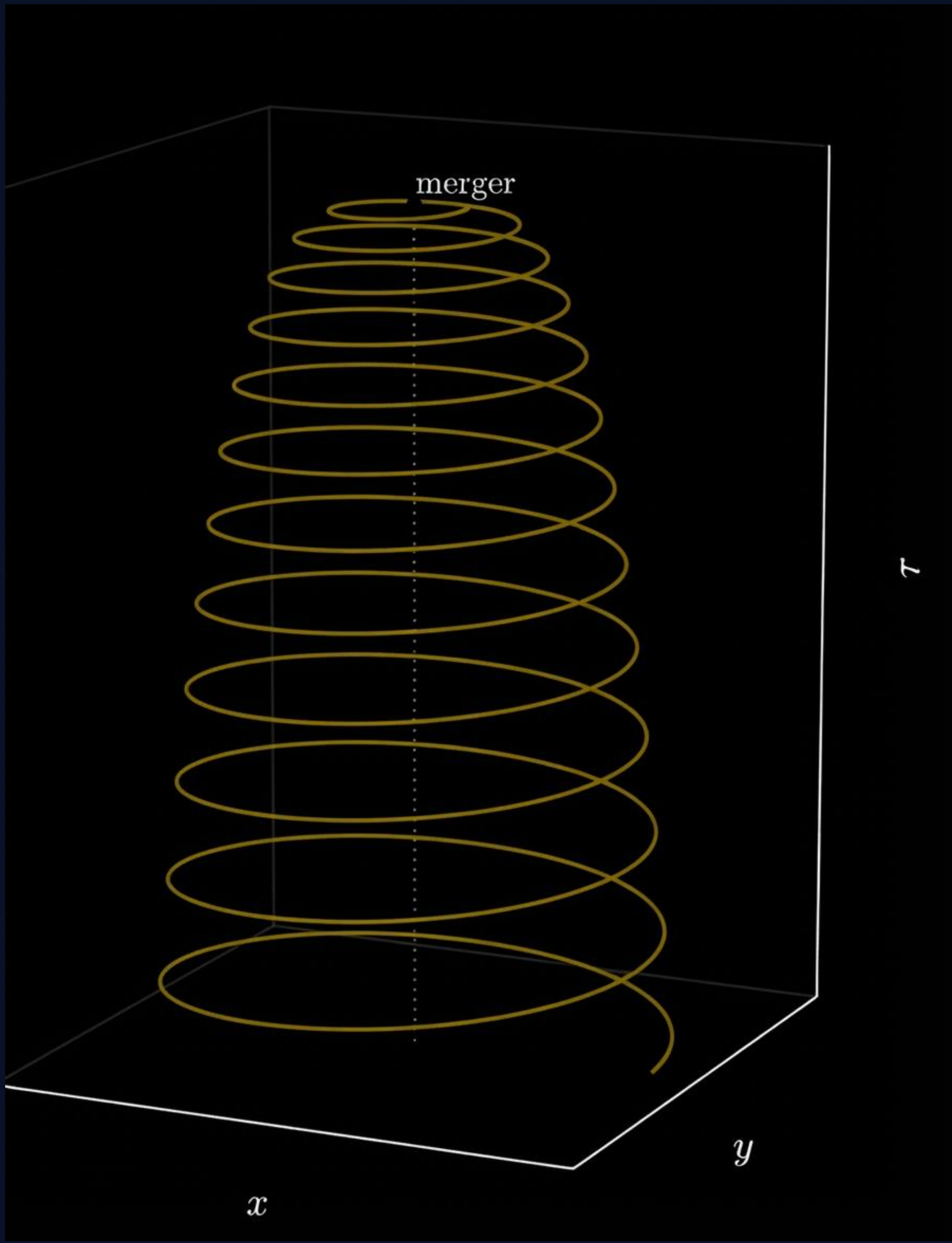


Two-timescale expansion — counting orders in ε

- Orbital timescale $T_{orb} \sim M$; radiation-reaction $T_{rr} \sim M / \varepsilon$.
- Action–angle expansion separates the fast (orbital) and slow (inspiral) dynamics:

$$\dot{J}_a = \varepsilon\, F_a^{(1)}(J,\varphi) + \varepsilon^2\, F_a^{(2)}(J,\varphi) + \cdots$$
$$\dot{\varphi}_i = \omega_i(J) + \varepsilon\, g_i^{(1)}(J,\varphi) + \cdots$$

Phase scales as ε^{-1} (adiabatic) + ε^{0} (first post-adiabatic, 1PA) + ε^{1} (2PA) + …



Spin of secondary enters at first post-adiabatic order

- For a maximally spinning compact secondary $|S| \leq \mathcal{M}^2$.
- Dimensionless spin ratio is therefore one mass ratio below unity:
- $\mathcal{O}(S)$ enters the inspiral PHASE at first post-adiabatic order (1PA) — the same order as CONSERVATIVE gravitational self-force (GSF), one beyond the adiabatic dissipative GSF.
- 1PA accuracy is the LISA waveform requirement; omitting $\mathcal{O}(S)$ systematically biases the inferred primary mass and spin.

$$\frac{S}{\mathcal{M}\, M} \sim \frac{\mathcal{M}}{M} = \varepsilon$$

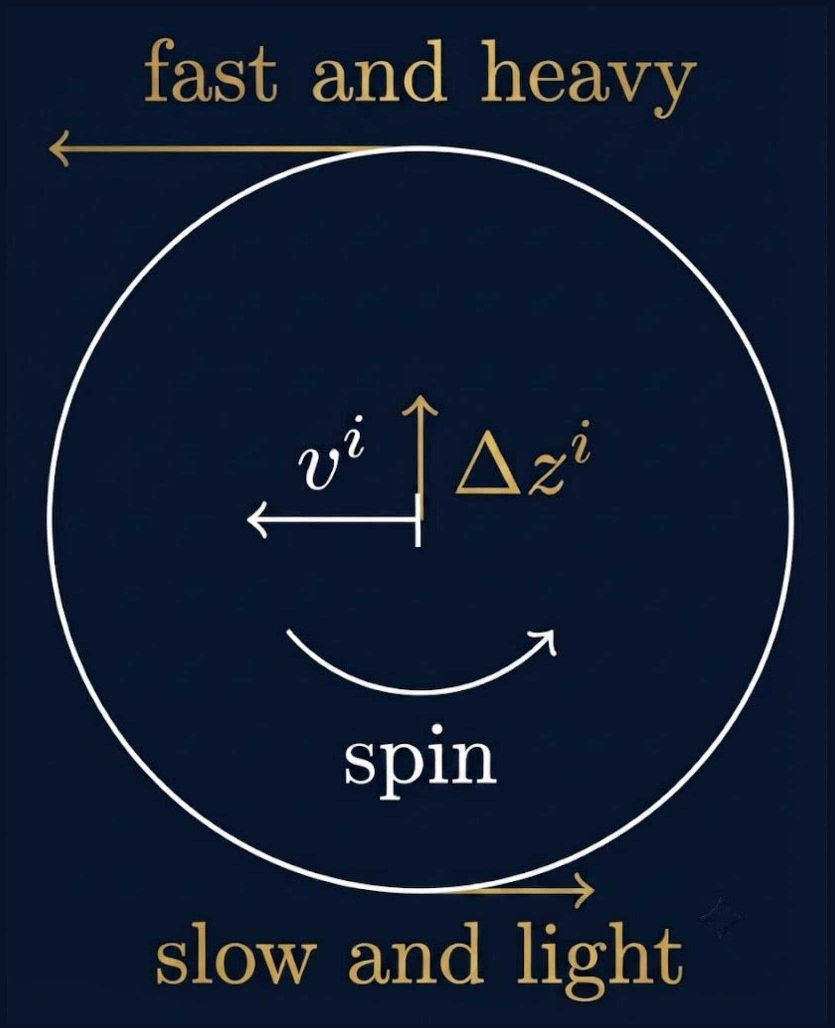
MPD equations — spin-curvature force, parallel-transported spin

- Four-momentum P^μ ; antisymmetric spin tensor $S^{\mu\nu}$. The *Mathisson–Papapetrou–Dixon* (MPD) system: 10 equations, 13 unknowns.
- Tulczyjew–Dixon (TD) spin supplementary condition $S^{\mu\nu} P_\nu = 0$ closes the system in the body rest frame.
- To linear order in spin $u^\mu \parallel P^\mu \implies$ spin is parallel-transported along (nearly) geodesics.

$$\frac{DP^\mu}{d\tau} = -\frac{1}{2} R^\mu{}_{\nu\alpha\beta}\, u^\nu S^{\alpha\beta}\,,$$
$$\frac{DS^{\mu\nu}}{d\tau} = P^\mu u^\nu - u^\mu P^\nu \;=\; \mathcal{O}(S^2)\,.$$

The position of a relativistic body is ambiguous

- Center of mass is FRAME-DEPENDENT in relativity — different observers deduce different center of mass
- A spin supplementary condition (SSC) selects worldline; physical predictions should be independent.
- (Pryce–)Newton–Wigner theorem (flat space-time): if we require a CANONICAL Poisson-bracket structure on (x, p, S), the position operator is essentially UNIQUE — *the Newton–Wigner centroid*.
- Tulczyjew–Dixon condition $S^{\mu\nu} P_\nu = 0$ is manifestly covariant, body-rest centroid, no preferred frame imposed.
- Changing SSC ⇔ worldline shift $\delta x^\mu_\perp$ orthogonal to P^μ at $\mathcal{O}(S)$



Adapted from Steinhoff 1501.04951

Effective action with a body-fixed frame (Bailey & Israel, Hanson & Regge,...)

- Body-fixed Lorentz frame $\Lambda^{A}{}_{\hat{A}}$ attached to the rigid body — carries the body's internal axes through the curved space-time.
- Spherical symmetry of the body $\Longrightarrow$ Lagrangian can depend only on Ω^{AB}.

$$S = \int \left[p_\mu \, \dot{x}^\mu + \frac{1}{2} S_{AB} \, \Omega^{AB} - H(p, x, S) \right] d\tau \,,$$

$$\Omega^{AB} \equiv \Lambda^{A}{}_{\hat{A}} \frac{d\Lambda^{B\hat{A}}}{d\tau} \,, \qquad S^{AB} = 2 \frac{\partial L}{\partial \Omega^{AB}} = 2\Pi_{A}{}^{\hat{A}} \Lambda_{B\hat{A}} \,.$$

Canonical conjugates $x^\mu, p_\mu, ; \Lambda^{B}{}_{\hat{A}}, \Pi_{B}{}^{\hat{A}}$

Warm-up — global Lorentz rotation of the tetrad

- Type-II generating function for a constant Lorentz transform $L^{A'}{}_{B}$ with $L^T \eta L = \eta$.
- Yields the expected tetrad-change rule, with NO coordinate shift.
- Spin transforms as a rank-2 tensor.

$$G = x^\mu p'_\mu + \Lambda^B{}_{\hat{A}} L^{B'}{}_B \Pi_{B'}{}^{\hat{A}},$$

$$p_\mu = p'_\mu,$$

$$\Pi_B{}^{\hat{A}} = L^{B'}{}_B \Pi_{B'}{}^{\hat{A}},$$

$$\Lambda^{B'}{}_{\hat{A}} = L^{B'}{}_B \Lambda^B{}_{\hat{A}},$$

$$S_{A'B'} = L_{A'}{}^A L_{B'}{}^B S_{AB}.$$

Background-frame variables — phase-space–dependent tetrad

- Take $L^{B'}{}_{B}(x, p')$ with $L^{T}\,\eta\, L = \eta$ at every phase-space point.
- Same generating function now produces BOTH a momentum shift AND a coordinate shift.
- The coordinate shift δx^{μ} *can* be related the SSC centroid shift — canonically generated.

$$p_\mu = p'_\mu \; + \frac{1}{2} S^{AB}\, L_{B'A}\, \frac{\partial L^{B'}{}_{B}}{\partial x^\mu}\;,$$

$$x'^\mu = x^\mu \; + \frac{1}{2}\, S^{AB}\, L_{B'A}\, \frac{\partial L^{B'}{}_{B}}{\partial p'_\mu}\;.$$

Worldline-frame variables — the new construction

- Take a momentum-dependent tetrad $e^{\mu}{}_{A}(x, p)$ directly — skip the intermediate background tetrad.
- Two connection-like phase-space objects emerge: $\omega_{\mu AB}$ and $\alpha^{\mu}{}_{AB}$:

$$p'_\mu = P_\mu + \frac{1}{2}\omega_{\mu AB}\, S^{AB}\,, \qquad \omega_{\mu AB} \equiv e^{\nu}_{A;\mu}\, e_{B\nu}\,,$$

$$x'^{\mu} = x^{\mu} - \frac{1}{2}\alpha^{\mu}{}_{AB}\, S^{AB}\,, \qquad \alpha^{\mu}{}_{AB} \equiv \frac{\partial e_{A\nu}}{\partial p_\mu}\, e^{\nu}_{B}\,.$$

Result: a single phase-space covariant derivative $\mathcal{D}_\nu$ encapsulates both the connection and the momentum-derivative pieces.

One compact phase-space covariant Hamiltonian

$$H = \frac{1}{2\mathcal{M}} g^{\mu\nu}(x'^{\alpha})\, p'_\mu (p'_\nu - \mathcal{D}_\nu\, e^\kappa_A\, e_{B\kappa}\, S^{AB}) + \mathcal{O}(S^2)\,,$$

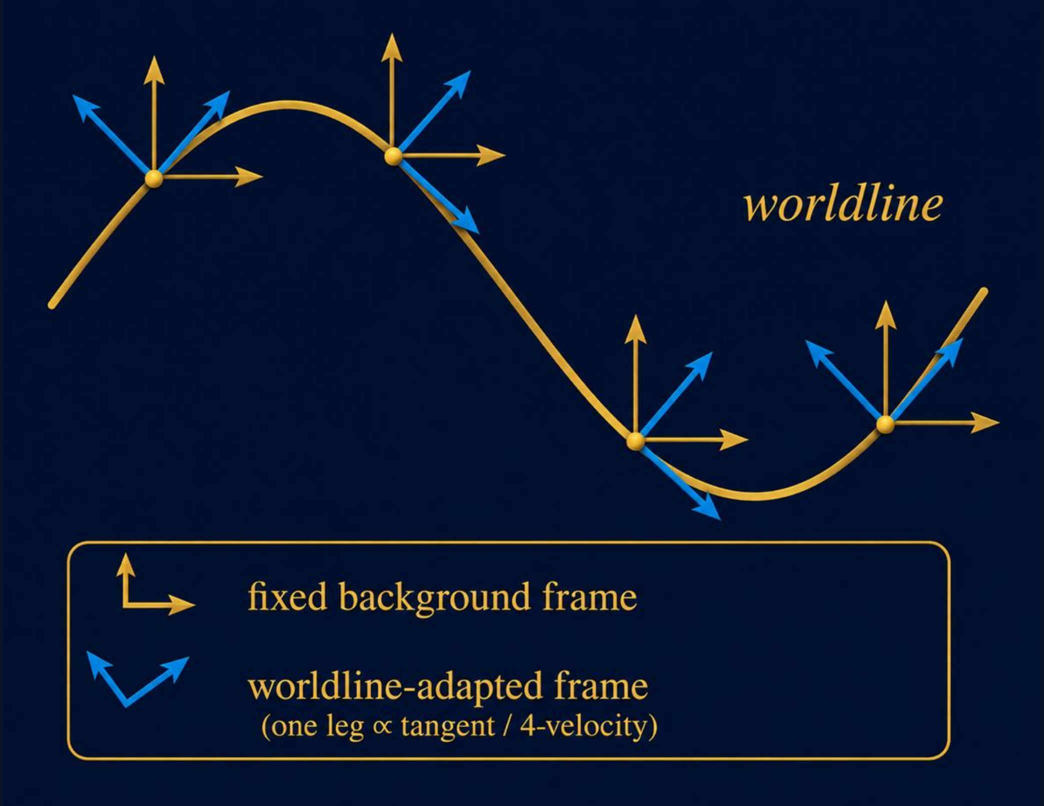
$$\mathcal{D}_\mu\, e^\kappa_A = e^\kappa_{A,\mu} + \Gamma^\kappa{}_{\mu\lambda}\, e^\lambda_A + \Gamma^\nu{}_{\mu\rho}\, p'_\nu\, \frac{\partial e^\kappa_A}{\partial p'_\rho}$$

Solving the Tulczyjew–Dixon SSC with Dirac brackets

- Choose $e^{\mu}{}_{0} = P^{\mu} / \sqrt{(-P_{\kappa} P^{\kappa})}$. In these tetrad indices the SSC is purely algebraic:

$$S^{0'B'} = 0 + \mathcal{O}(S^2)$$

- The $S^{0'B'}$ components have non-zero Poisson bracket only with each other ⟹ purely second-class ⟹ set to zero as an identity
- Surviving S^{IJ}, $I,J = 1,2,3$ — three physical d.o.f., $\mathfrak{so}(3)$ bracket.
- Comparison with Barausse et al. (2009) construction using Newton–Wigner SSC: Hamiltonians match at $\mathcal{O}(S)$ for „minimal“ choice of adapted tetrad.



Canonical coordinates on the spin sector

- The $\mathfrak{so}(3)$ sector admits the classical canonical pair (π_ϕ, ϕ).
- One Casimir S^2 fixes the magnitude; one north-pole-style coordinate singularity at $\pi_\phi = S$.
- Similar constructions are possible in general dimensions.
- Together with (x'^μ, p_ν) this is a minimal canonical cover in any space-time.

$$S^{12} = \pi_\phi\,, \qquad S^{23} = \sqrt{S^2 - \pi_\phi^2}\,\sin\phi\,, \qquad S^{31} = \sqrt{S^2 - \pi_\phi^2}\,\cos\phi$$

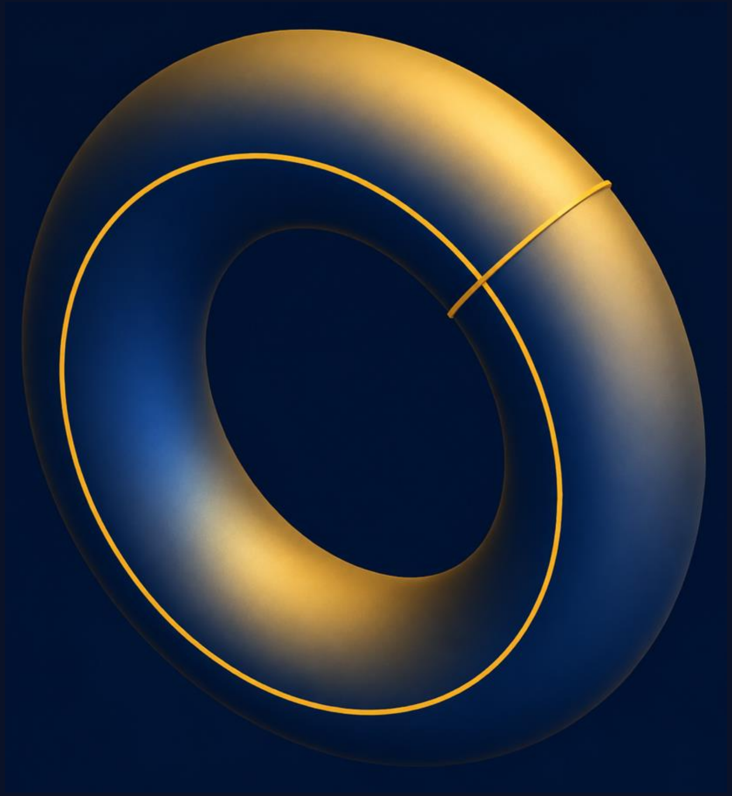
Parallel transport in an adapted tetrad — precession matrix

- Choose $e^{\mu}{}_{0} = u^{\mu}$ (geodesic D-velocity). Any parallel-transported V^{μ} obeys:
- Autoparallel geodesic $\Longrightarrow$ $\omega^{A}{}_{0} = \omega^{0}{}_{B} = 0$. The remaining $\omega^{I}{}_{J} \in \mathfrak{so}(D-1)$ is the precession matrix.
- Real, skew-symmetric $\Longrightarrow$ $\lfloor (D-1)/2 \rfloor$ pure-imaginary eigenvalue pairs (instantaneous precession 2-planes).

$$\frac{dV^{A}}{d\tau} = \omega^{A}{}_{B}\, V^{B}, \qquad \omega^{A}{}_{B} = -\, e^{A}_{\nu;\mu}\, e^{\nu}_{B}\, u^{\mu}.$$

Torus reducibility — when precession 2-planes are fixed

- If all $\omega^{I}{}_{J}(\tau)$ lie in a fixed Cartan subalgebra of $\mathfrak{so}(D-1)$ — i.e. pairwise commute — parallel transport reduces to commuting rotations in fixed 2-planes.
- Solution by quadrature, one precession angle per Cartan 2-plane:



$$\psi_{(2k-1,2k)}(\tau) = \int^{\tau} \omega^{2k-1}{}_{2k}(\tau')\, d\tau', \qquad k = 1, \ldots, \lfloor (D-1)/2 \rfloor$$

Separable geodesics and parallel transport

Definition: geodesic motion separable in $\hat{x}^\mu$ if separable solution to Hamilton-Jacobi equation

$$W_{(\mathrm{g}),\mu} W_{(\mathrm{g}),\nu} g^{\mu\nu} = -\mathcal{M}^{\,2}; \quad W_{(\mathrm{g})} = \sum_{\mu} w_{(\mu)}\left(\hat{x}^\mu; C_a\right)$$

Parallel transport separable along geodesics in in $\hat{x}^\mu$ if it separates the same way along geodesic congruence:

$$\psi_{(k)} = \sum_{\mu} \int \Psi_{(k|\mu)}(\hat{x}^\mu; C_a)\, \mathrm{d}\hat{x}^\mu$$

Hamilton-Jacobi solution correction

If a D-dimensional space-time admits a tetrad $e^{\mu}{}_{A}(x,p)$ *in which geodesic parallel transport is separable in some coordinates, then the canonical Hamiltonian for a spinning test particle in those coordinates — to linear order in spin — admits a SEPARATED Hamilton–Jacobi solution.*

$$W_{(1)} = \sum_{k} \sigma_{(k)} \left(\phi_{(k)} + \sum_{\mu} \int \Psi_{(k|\mu)}(\hat{x}^{\mu}; C_a)\, \mathrm{d}\hat{x}^{\mu} \right)$$

Separation constants $\sigma_{(k)}$ are constant values of momenta conjugate to spin sector coordinates $\phi_{(k)}$

Kerr-NUT-(A)dS spacetimes

Off-shell Carter metric $(\rho^2 = r^2 + (n + a\cos\theta)^2, \Delta_r = \Delta_r(r), \Delta_\theta = \Delta_\theta(\theta))$*:*

$$\mathrm{d}s^2 = -\frac{\Delta_r}{\rho^2}\left(\mathrm{d}t - \left(a\sin^2\theta + 2n(1-\cos\theta)\right)\mathrm{d}\varphi\right)^2 + \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left(a\,\mathrm{d}t - (r^2 + (a+n)^2)\,\mathrm{d}\varphi\right)^2 + \frac{\rho^2}{\Delta_r}\,\mathrm{d}r^2 + \frac{\rho^2}{\Delta_\theta}\,\mathrm{d}\theta^2$$

Killing–Yano tensor contractions with $p_\mu \rightarrow$ *Marck tetrad that leads to*

$$\omega_{12} = \frac{\sqrt{K}}{\rho^2}\left(\frac{E(r^2 + (a+n)^2) - aL}{K + r^2} + \frac{a(L - aE\sin^2\theta) - 2nEa(1-\cos\theta)}{K - (n + a\cos\theta)^2}\right)$$

Separation with Mino parameter $\mathrm{d}\lambda = \rho^2 d\tau$

Other examples

Gravitational plane wave in Rosen coordinates (β, γ, α functions of U):

$$\mathrm{d}s^2 = 2\,\mathrm{d}U\,\mathrm{d}V + e^{2\beta}\left[\,e^{2\gamma}(\cos\alpha\,\mathrm{d}x + \sin\alpha\,\mathrm{d}y)^2 + e^{-2\gamma}(-\sin\alpha\,\mathrm{d}x + \cos\alpha\,\mathrm{d}y)^2\,\right]$$

Covariantly constant null vector ℓ ⟹ single precession:

$$\omega_{12} = p_V \cosh(2\gamma)\,\frac{\mathrm{d}\alpha}{\mathrm{d}U}\,, \qquad \psi = \int \cosh(2\gamma)\,\mathrm{d}\alpha$$

FLRW (γ is the maximally symmetric spatial metric):

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\,\gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,,$$

$$\gamma_{ij}\,\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}\chi^2 + S_k(\chi)^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)$$

*We can directly construct a parallel-transported tetrad ⟹ spin **fully** decouples in these variables:*

$$\omega_{IJ} = 0 \quad \Rightarrow \quad W_{(1)} = \sigma\,\phi$$

THANK YOU · QUESTIONS WELCOME

Take it home.

- Worldline-frame canonical variables give a clean, covariant $\mathcal{O}(S)$ Hamiltonian — Tulczyjew–Dixon built-in via Dirac brackets.
- THEOREM. Separability of geodesic parallel transport in an adapted tetrad $\Longrightarrow$ separability of the linear-in-spin HJ equation.
- Examples: Kerr, any-D Kerr-NUT-(A)dS (general D in preparation), plane waves, FLRW.
- Payoff: globally regular closed-form spin corrections to EMRI orbits, action-angle formalism and related radiation-reaction balances — straight into LISA waveform modelling.