Some inequalities among curvature invariants

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Some inequalities among curvature invariants
Eur. Phys. J. C 86, 205 (2026).

Decomposition of the Riemann tensor R_{abcd} in D-dimensions $(D \geq 3)$ into the Weyl tensor C_{abcd} and the Ricci tensor R_{ab}

$$\begin{aligned} R_{abcd} =& C_{abcd} + \frac{1}{D-2}(g_{ac}R_{bd} - g_{ad}R_{bc} + g_{bd}R_{ac} - g_{bc}R_{ad}) \\ &- \frac{R}{(D-1)(D-2)}(g_{ac}g_{bd} - g_{ad}g_{bc}) \end{aligned}$$

We define the nth Ricci invariant as

$$R_n = R_{a_1 b_1} R_{a_2 b_2} \cdots R_{a_n b_n} g^{b_1 a_2} g^{b_2 a_3} \cdots g^{b_{n-1} a_n} g^{b_n a_1}$$

$R_1 = R^a{}_a$, $R_2 = R_{ab}R^{ab}$ ...
We denote the Kretschmann scalar by

$$K = R_{abcd}R^{abcd}$$

and the square of the Weyl tensor by

$$I_1 = C_{abcd}C^{abcd}$$

In 4D –> 14 algebraically independent curvature invariants.
Polynomial relations between invariants are called *syzygies*.

J. Dragašević, I. Moslavac, I. Smolić *Weighing the curvature invariants*
Eur. Phys. J. C 85, 818 (2025)
It was proved that if the metric has

- static
$$ds^2 = -f(x^m)\,dt^2 + h_{ij}(x^m)\,dx^i\,dx^j$$
- generalized FLRW
$$ds^2 = -dt^2 + a(t)^2 h_{ij}(x^m)\,dx^i\,dx^j$$
- generalized Bianchi type I form
$$ds^2 = -dt^2 + h_{ij}(t)\,dx^i\,dx^j$$

then the following inequalities hold

- $R_1^2 \leq DR_2$
- $2R_2 \leq (D-1)K$

Let (M,g) be a D-dimensional Lorentzian manifold and P be a symmetric rank-2 tensor : $\forall p \in M$, $P^a_{\ b} : T_pM \rightarrow T_pM$.
The eigenvalue problem for $P^a_{\ b}$ leads to the Segre classification.

	Segre	Plebański
A1	$[111,1]$	$[S_1 - S_2 - S_3 - T]_{(1111)}$
	$[11(1,1)]$	$[S_1 - S_2 - 2T]_{(111)}$
	$[(11)1,1]$	$[2S_1 - S_2 - T]_{(111)}$
	$[(11)(1,1)]$	$[2S - 2T]_{(11)}$
	$[1(11,1)]$	$[S - 3T]_{(11)}$
	$[(111),1]$	$[3S - T]_{(11)}$
	$[(111,1)]$	$[2S - 2T]_{(11)}$
A2	$[11,Z\bar{Z}]$	$[S_1 - S_2 - Z - \bar{Z}]_{(1111)}$
	$[(11),Z\bar{Z}]$	$[2S_1 - Z - \bar{Z}]_{(111)}$
A3	$[11,2]$	$[S_1 - S_2 - 2N]_{(112)}$
	$[1(1,2)]$	$[S - 3N]_{(12)}$
	$[(11),2]$	$[2S - 2N]_{(12)}$
	$[(11,2)]$	$[4N]_{(2)}$
B	$[1,3]$	$[S - 3N]_{(13)}$
	$[(1,3)]$	$[4N]_{(3)}$

The canonical form of P for a given Segre class in D dimensions is

$$\begin{aligned}
A1: \quad & P_{ab} = \sum_{i=1}^{D-1} \lambda_i \, x_a^{(i)} x_b^{(i)} - \lambda_D u_a u_b \ , \\
A2: \quad & P_{ab} = \sum_{i=1}^{D-2} \lambda_i \, x_a^{(i)} x_b^{(i)} + \lambda_{D-1} k_{(a} l_{b)} + \lambda_D (k_a k_b - l_a l_b) \ , \\
A3: \quad & P_{ab} = \sum_{i=1}^{D-2} \lambda_i \, x_a^{(i)} x_b^{(i)} - 2\lambda_{D-1} k_{(a} l_{b)} \pm k_a k_b \ , \\
B: \quad & P_{ab} = \sum_{i=1}^{D-2} \lambda_i \, x_a^{(i)} x_b^{(i)} + \lambda_{D-1} k_{(a} l_{b)} + k_{(a} x_{b)} \ ,
\end{aligned} \tag{1}$$

where $(x^{(1)}, \ldots, x^{(D-1)}, u)$ is an orthonormal tetrad with u timelike, $(x^{(1)}, \ldots, x^{(D-2)}, k, l)$ is a real partially null tetrad and $\lambda_i \in \mathbb{R}$ -eigenvalues or their linear combinations.

Lemma (I)

Let P' be related to a symmetric rank-2 tensor P at $p \in M$ by

$$P' = \alpha P + \beta g \, ,$$

with α, $\beta \in \mathbb{R}$ and $\alpha \neq 0$. Then P' is of the same Segre type as P.

We define the P_n invariant as

$$P_n = P_{a_1 b_1} P_{a_2 b_2} \cdots P_{a_n b_n} g^{b_1 a_2} g^{b_2 a_3} \cdots g^{b_{n-1} a_n} g^{b_n a_1}$$

$P_1 = P^a{}_a$, $P_2 = P_{ab} P^{ab}$...

Theorem (I)

Let (M, g) be a D-dimensional spacetime, $D \geq 2$, and P be a rank-2 symmetric tensor of Segre type A1, A3 or B. Then

$$P_s^{2m} \leq D^{2m-s} P_{2m}^s \, , \tag{2}$$

for every $s, m \in \mathbb{N}$ such that $1 \leq s < 2m$

Proof: Generalized mean inequality

$$\left(\sum_{i=1}^{D} x_i^s \right)^{2m} \leq D^{2m-s} \left(\sum_{i=1}^{D} x_i^{2m} \right)^s$$

Theorem (II)

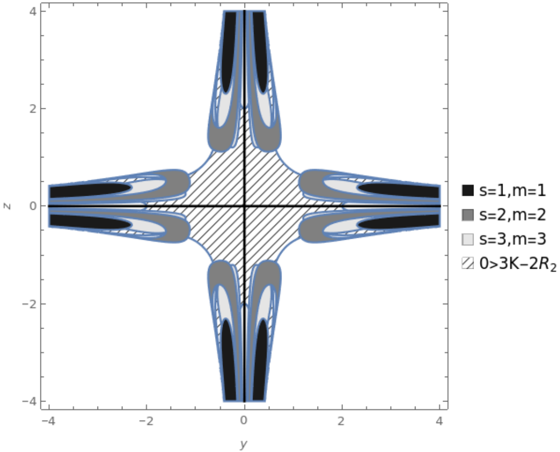
Let (M, g) be a D-dimensional spacetime, $D \geq 2$, of Segre class A1, A3 or B. Let I_1 denote the square of the Weyl tensor and let K be a Kretschmann scalar. If $I_1 \geq 0$, then

$$2R_2 \leq (D-1)K\,. \tag{3}$$

Proof:

- $D = 2$, $K = 2R_2 = R_1^2$
- $D \geq 3$, we use Theorem I for $s = m = 1$ and the syzygy :

$$K = I_1 + \frac{4}{D-2}R_2 - \frac{2}{(D-1)(D-2)}R_1^2$$



Schmidt metric: $ds^2 = -dt^2 + 2\, y\, z\, dt dx + dx^2 + dy^2 + dz^2$

- $A1$ $[1(11,1)]$ $([S - 3\,T]_{(11)})$ — Theorem I hold $(yz = 0)$
- $A2$ $[11, Z\bar{Z}]$ $([S_1 - S_2 - Z - \bar{Z}]_{(1111)})$

Consequences:

- If any of the inequalities from the Theorem I is violated for the Ricci tensor, then the spacetime is unphysical.
- If the sth Ricci invariant blows-up so will the invariants of higher $2m$th orders for every $1 \leq s < 2m$. Under additional assumptions this also applies to the Kretschmann scalar via Theorem II.
- A. Bokulić, T. Jurić, I. Smolić
  Conundrum of regular black holes with nonlinear electromagnetic fields
  Phys. Rev. D 113, 024044 (2026)
- The algebraic restrictions on physical geometries distinguish general relativity from other theories.

Theorem I:

$$P_s^{2m} \leq D^{2m-s} P_{2m}^s$$

Theorem II ($I_1 \geq 0$):

$$2R_2 \leq (D-1)K$$

J. Dragašević, I. Moslavac, I. Smolić *Weighing the curvature invariants* Eur. Phys. J. C 85, 818 (2025):

- $R_1^2 \leq DR_2$
- $2R_2 \leq (D-1)K$

Thank you for listening!

S. J. Szybka, Y. Kravetska, KN, *Some inequalities among curvature invariants* Eur. Phys. J. C 86, 205 (2026).