Einstein constraints and differential forms

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Introduction

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$\Longrightarrow$ there is a local $SO(3)$-gauge freedom.

♠ This freedom can be used to reduce the order of the constraints—one can remove all the second order derivatives appearing in the scalar constraint imposed on (θ^I, r_J).

A Hamiltonian formulation of TEGR

- The phase space:

$$(\xi^I, \theta^J, r_K, \zeta_L), \quad I, J, K, L \in \{1, 2, 3\},$$

—for each value of the index ξ^I is a zero-form, θ^J a one-form, r_K a two-form, ζ_L a three-form on a 3-dim. manifold Σ.

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♠ r_J is the momentum conjugate to θ^J, ζ_I is the momentum conjugate to ξ^I.

♠ (θ^I) constitute a coframe on Σ such that the spatial metric reads

$$q = \Big(\delta_{IJ} - \frac{\xi_I \xi_J}{1 + \delta^{KL} \xi_K \xi_L}\Big) \theta^I \otimes \theta^J.$$

Constraints and Hamiltonian

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♠ There are six primary constraints which generate Lorentz (gauge) transformations on the phase space.

♠ The secondary constraints are a vector and a scalar constraints.

♠ All the constraints are of the first class and the Hamiltonian is a sum of the constraints smeared with appropriate fields.

The constraints

♠ The constraints (in a smeared form):

$$B(a) = \int_\Sigma a \wedge \Big(-\frac{\xi_I q_{JK}}{(\xi^0)^2} \theta^I \wedge *(d\xi^J \wedge \theta^K) + q_{IJ}\theta^I \wedge *d\theta^J + \\ + \vec{\theta}^I \lrcorner \zeta_I + \xi^I r_I \Big),$$

$$R(b) = \int_\Sigma b \wedge (\theta^I \wedge *r_I + q_{IJ} d\xi^I \wedge \theta^J),$$

$$V(\vec{N}) = \int_\Sigma d\xi^I \wedge \vec{N} \lrcorner \zeta_I - d\theta^I \wedge (\vec{N} \lrcorner r_I) - (\vec{N} \lrcorner \theta^I) \wedge dr_I$$

Here $*$ is the Hodge "star" given by the spatial metric q, $\vec{\theta}^I$ is a vector field obtained from θ^I by raising up its index by the metric dual to q, the indices $I, J, K, \ldots$ are raised up (lowered) by δ^{IJ} (δ_{IJ}), and $\xi^0 \equiv \sqrt{1 + \xi^K \xi_K}$.

The constraints

$$
\begin{aligned}
S(N) = \int_\Sigma N\Big(&\frac{1}{2}(r_I \wedge \theta^J) \wedge *(r_J \wedge \theta^I) - \frac{1}{4}(r_I \wedge \theta^I) \wedge *(r_J \wedge \theta^J) - \\
&- d(\vec{\theta}^J \lrcorner \zeta_J) - \xi^I \wedge dr_I + \frac{1}{4(\xi^0)^4} d(\xi_I \theta^I) \wedge \xi_J \theta^J \wedge *(d(\xi_K \theta^K) \wedge \xi_L \theta^L) + \\
&+ \frac{1}{2(\xi^0)^2} d(\xi_I \theta^I) \wedge \xi_J \theta^J \wedge *(d\theta_K \wedge \theta^K) - \\
&- \frac{1}{(\xi^0)^2}(q_{IJ} d\xi^J \wedge \theta^I + \xi_I d\theta^I) \wedge \theta^K \wedge *(d\theta_K \wedge \xi_L \theta^L) + \\
&+ \frac{1}{2} d\theta_I \wedge \theta^J \wedge *(d\theta_J \wedge \theta^I) - \frac{1}{4} d\theta_I \wedge \theta^I \wedge *(d\theta_J \wedge \theta^J)\Big),
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- Is it possible to zero (ξ^I) by gauge transformations?

Gauge $\xi^I = 0$

♠ A Lorentzian boost of $(\xi_0^I, \theta_0^J, r_K^0, \zeta_L^0)$: a curve

$$\lambda \mapsto (\xi^I(\lambda), \theta^J(\lambda), r_K(\lambda), \zeta_L(\lambda))$$

being the solution of

$$\frac{d\xi^I}{d\lambda} = \{\xi^I, B(a(\lambda))\} = \vec{\theta}^I \lrcorner a(\lambda), \qquad \frac{d\zeta_L}{d\lambda} = \{\zeta_L, B(a(\lambda))\},$$
$$\frac{d\theta^J}{d\lambda} = \{\theta^J, B(a(\lambda))\} = \xi^J a(\lambda), \qquad \frac{dr_K}{d\lambda} = \{r_K, B(a(\lambda))\},$$

with the initial value $(\xi_0^I, \theta_0^J, r_K^0, \zeta_L^0,)$ at $\lambda = 0$.

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♠ *Yes!* $\Longrightarrow$ Each point $(\xi_0^I, \theta_0^J, r_K^0, \zeta_L^0,)$ can be gauge-transformed to $(\xi^I = 0, \theta^J, r_K, \zeta_L,)$.

Gauge $\xi^I = 0$

- The constraints (in the not smeared form):

$$\vec{\theta}^I \lrcorner \zeta_I + \theta_J \wedge *d\theta^J = 0,$$

$$\vec{\theta}^I \lrcorner r_I = 0,$$

$$dr^J + d\theta^I \wedge \vec{\theta}^J \lrcorner r_I = 0,$$

$$[*(r_I \wedge \theta^J)][*(r_J \wedge \theta^I)] - \frac{1}{2}[*(r_I \wedge \theta^I)]^2 +$$

$$+2*d(\theta_J \wedge *d\theta^J) + [*(d\theta_I \wedge \theta^J)][*(d\theta_J \wedge \theta^I)] - \frac{1}{2}[*(d\theta_I \wedge \theta^I)]^2 = 0.$$

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- (θ^I) is now an orthonormal coframe of q.

- This suggests to express the constraints in terms of the components of r_I and $d\theta^I$ in the coframe and decompose the components into *irreducible representations* of the $SO(3)$ group preserving the metric q.

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$$\zeta_I = \zeta_{IJKL}\,\theta^J \otimes \theta^K \otimes \theta^L, \qquad \epsilon = \epsilon_{JKL}\,\theta^J \otimes \theta^K \otimes \theta^L.$$

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- The decomposition of (r^I) and $(d\theta^J)$:

$$r^I{}_{JK} = \frac{1}{2} r^{IL}\epsilon_{LJK}, \quad r^{IL} := r^I{}_{JK}\epsilon^{JKL} = r^{[IL]} + \mathring{r}^{IL} + \frac{1}{3} r^K{}_K \delta^{IL},$$
$$\beta^I{}_{JK} = \frac{1}{2}\beta^{IL}\epsilon_{LJK}, \quad \beta^{IL} := \beta^I{}_{JK}\epsilon^{JKL} = \beta^{[IL]} + \mathring{\beta}^{IL} + \frac{1}{3}\beta^K{}_K\delta^{IL},$$
$$\beta^{[IL]} = \epsilon^{ILJ}\beta_J, \qquad \beta_I := \frac{1}{2}\epsilon_{IJK}\beta^{JK}.$$

The constraints

- In terms of the irreducible components:

$$\zeta_{IJKL} + \frac{1}{4}\beta_I \epsilon_{JKL} = 0,$$
$$r^{[IJ]} = 0,$$
$$\frac{2}{3} r^K{}_K{}^{,J} + 2\mathring{r}^{JM}{}_{,M} + 3\mathring{r}^{JM}\beta_M + \mathring{r}^{IM}\mathring{\beta}_{IN}\epsilon^{JN}{}_M = 0,$$
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♠ The rotation constraint forces the antisymmetric part of (r_I) to vanish.

The TEGR constraints vs the standard ones

♠ By analyzing the origin of the momenta (r_I) we get

$$r_{IJ} = 2(K_{IJ} - K^L{}_L \delta_{IJ}), \qquad K_{IJ} = \frac{1}{2} r_{IJ} - \frac{1}{4} r^K{}_K \delta_{IJ},$$

$$r^L{}_L = -4K^L{}_L, \qquad \mathring{r}_{IJ} = 2\mathring{K}_{IJ},$$

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♠ If R_q is the Ricci scalar of q, then

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♠ Direct calculations show that the TEGR constraints in the gauge $\xi^I = 0$ are equivalent to the standard vacuum Einstein constraints with zero cosmological constant.

Reducing the order of the constraints

♠ Both the TEGR and the standard constraints form a system of PDE's of second order:

$$\mathring{r}^{IJ}\mathring{r}_{IJ} - \frac{1}{6}(r^K{}_K)^2 + 8\beta^I{}_{,I} + 6\beta^I\beta_I + \mathring{\beta}^{IJ}\mathring{\beta}_{IJ} - \frac{1}{6}(\beta^K{}_K)^2 = 0.$$

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♠ Is it possible to use the remaining gauge freedom to annihilate β_I without spoiling the gauge $\xi^I = 0$?

♠ A simpler question: does every Riemannian metric in the dimension 3 admit an orthonormal coframe (θ^I), for which $\beta_I = 0$?

Reducing the order of the constraints

♠ $\beta_I = 0$ iff any of the following conditions holds:

$$\begin{aligned} {\beta^J}_{IJ} &= 0, & \theta_I \wedge *d\theta^I &= 0, & \theta^J([\varepsilon_I, \varepsilon_J]) &= 0, \\ \mathcal{L}_{\varepsilon_I}\mathcal{E} &= 0, & \mathcal{L}_{\varepsilon_I}\epsilon &= 0, & \operatorname{div}_q \varepsilon_I &= 0, \\ d(\theta^I \wedge \theta^J) &= 0, & *\,d*\theta^I &= 0, & {\Gamma^J}_{IJ} &= 0. \end{aligned}$$

Here $\mathcal{E} = \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3$, and $({\Gamma^I}_{JK})$ are the LC connection coefficients in (θ^I).

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♠ The coframe (θ^I), which satisfies $*\, d * \theta^I = 0$, is said to be *coclosed*.

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Theorem (R. Bryant)

Let q be a real-analytic Riemannian metric defined on a 3-dimensional manifold. Then there exists locally an orthonormal coframe of the metric that is coclosed.

The first order system of constraints

♠ If a real-analytic metric q and an extrinsic curvature K satisfy the vacuum Einstein constraints, then the corresponding coclosed o.n. coframe (θ^I) and the momenta (r_J) satisfy

$$\beta_I = 0, \qquad \frac{2}{3} r^K{}_K{}^{,J} + 2\mathring{r}^{JM}{}_{,M} + \mathring{r}^{IM}\mathring{\beta}_{IN}\epsilon^{JN}{}_M = 0,$$

$$r^{[IJ]} = 0, \qquad \mathring{r}^{IJ}\mathring{r}_{IJ} - \frac{1}{6}(r^K{}_K)^2 + \mathring{\beta}^{IJ}\mathring{\beta}_{IJ} - \frac{1}{6}(\beta^K{}_K)^2 = 0.$$

and vice versa.

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and vice versa.

♠ Since $\beta^{[IJ]} = \epsilon^{IJK}\beta_K$ the equations (except the vector constr.)

$$\beta^{[IJ]} = 0,$$

$$r^{[IJ]} = 0, \qquad \mathring{r}^{IJ}\mathring{r}_{IJ} - \frac{1}{6}(r^K{}_K)^2 + \mathring{\beta}^{IJ}\mathring{\beta}_{IJ} - \frac{1}{6}(\beta^K{}_K)^2 = 0$$

are invariant w.r.t. the replacement $(r_I) \leftrightarrow (d\theta_I)$.

Other results

♠ A general (local) solution to the equation

$$*\, d * \theta^I = 0$$

($(\theta^I) \Longrightarrow$ a metric $q = \delta_{IJ}\, \theta^I \otimes \theta^J \Longrightarrow$ the Hodge "star" $*$). The solution is explicit up to six (one-dim.) integrals.

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♠ A method for generating new solutions of the constraints: if (θ^I) is coclosed and (θ^I, r_J) satisfy the constraints, then

$$(\theta^I, \lambda r_J + \kappa d\theta_J), \quad \lambda, \kappa \in \mathbb{R},$$

satisfy the vector and the rotation constraints. In some cases the scalar constraint can be easily solved by the appropriate choice of the constants λ and κ.

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Summary

♠ The vacuum Einstein constraints with zero cosmological constant were expressed in terms of differential forms, which include an orthonormal coframe of the spatial metric.

♠ If the spatial metric is real-analytic, then we can (locally) remove all second derivatives from the constraints requiring that the orthonormal coframe is coclosed.

References

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