Hidden symmetries and boundary data for asymptotically Kerr–de Sitter spacetimes

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What and Why?

Kerr-de Sitter spacetime: a solution of the vacuum Einstein field equations

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 0, \quad \Lambda > 0,$$

describing a rotating black hole.

Main properties:

- depends on three parameters: cosmological constant Λ, mass m and angular momentum J,
- admits two Killing fields,

$$\nabla_{(a} K_{b)} = 0.$$

What and Why?

Conserved quantities for geodesic motion:

- energy E,
- axial angular momentum L_ϕ,
- Carter constant C.

Carter constant and hidden symmetry: Kerr-de Sitter metric also admits a Killing tensor K,

$$\nabla_{(a}K_{bc)} = 0$$

such that $C := K_{ab}p^a p^b$, where p is the four-momentum.

What and Why?

Origin of the hidden symmetry:

closed Conformal Killing-Yano (cCKY) 2-form Q,

$$\nabla_a Q_{bc} + \tfrac{2}{3} g_{a[b} \nabla^d Q_{c]d} = 0.$$

Killing tensor can be obtained by squaring $\star Q$.

Theorem[1]

The most general solution of the vacuum Einstein field equations with cosmological constant that admits a non-degenerate cCKY tensor is the Kerr–NUT–(anti-)de Sitter metric.

(Not closed) Conformal Killing-Yano (CKY) equation

$$\nabla_a Q_{bc} - \nabla_{[a} Q_{bc]} + \tfrac{2}{3} g_{a[b} \nabla^d Q_{c]d} = 0.$$

CKY equation is conformally invariant under

$$g \longmapsto \Omega^2 g, \quad Q \longmapsto \Omega^3 Q$$

[1] P. Krtouš, V. P. Frolov, and D. Kubizňák, Hidden symmetries of higher dimensional black holes and uniqueness of the Kerr-NUT-(A)dS spacetime, Phys. Rev. D **78**, 2008.

What and Why?

Expected late-time behavior of black holes

Dynamical black holes settle down to a stationary final state described by the Kerr-de Sitter solution

$$\Updownarrow$$

Generic black hole is an asymptotically Kerr-de Sitter spacetime

Question 1

What necessary conditions must the spacetime satisfy to be asymptotically Kerr–de Sitter?

Question 1.1

Local characterization of Kerr-de Sitter spacetimes?

Setup

Closed conformal Killing-Yano (cCKY) tensor

Kerr-de Sitter metric admits a cCKY 2-form $\tilde{Q}$ satisfying

$$\tilde{\nabla}_a \tilde{Q}_{bc} + \tfrac{2}{3}\tilde{g}_{a[b}\tilde{\nabla}^d \tilde{Q}_{c]d} = 0$$

Asymptotically (local) Kerr-de Sitter spacetime

Let M be a compactification of a spacetime satisfying:

- conformal boundary Σ of M has the same local intrinsic conformal geometry as $\Sigma_{\text{Kerr-de Sitter}}$
- there exists an asymptotic solution of the cCKY equation Q such that
$$Q \stackrel{\Sigma}{=} Q_{\text{Kerr-de Sitter}}$$
where $\stackrel{\Sigma}{=}$ denotes equality on Σ.
- $\Lambda > 0$ and stress-energy tensor $\tilde{T}$ vanishes on Σ

Setup

What obstructs a spacetime from being locally asymptotically Kerr-de Sitter?

Compactification of a spacetime $(\tilde{M}, \tilde{g})$**:** $(M, \mathbf{g}, \sigma)$, where

- M is a compact manifold with boundary Σ
- $\mathbf{g} := [g]$ is a conformal class of metrics
- $\sigma := [\rho]$ is a conformal class of defining functions of the boundary Σ

Then

$$\tilde{g} = \sigma^{-2}\mathbf{g}$$

solves the Einstein field equations.

Setup

cCKY equations corresponding to the physical metric $\widetilde{g}$:

$$\widetilde{\nabla}_a \widetilde{Q}_{bc} - \widetilde{\nabla}_{[a} \widetilde{Q}_{bc]} + \tfrac{2}{3} \widetilde{g}_{a[b} \widetilde{\nabla}^d \widetilde{Q}_{c]d} = 0, \qquad \widetilde{\nabla}_{[a} \widetilde{Q}_{bc]} = 0.$$

Strategy

- Assume that the cCKY equations hold only asymptotically
- Write the system in terms of the unphysical conformal class of metrics and the conformal class of defining functions
- Couple it to the Einstein field equations with stress-energy tensor that vanishes on the conformal boundary
- Look at constraints for solving the Einstein-cCKY system

Conformal approach to Einstein-cCKY system

Let

$$\widetilde{T}_{ab} = \sigma\tau_{ab},$$

where τ_{ab} extends smoothly to Σ.

Normalization: $\Lambda = 3$.

Conformal Einstein system coupled to σ

$$\nabla_a\nabla_b\sigma + \sigma P_{ab} - \tfrac{1}{4}\mathbf{g}_{ab}(\Delta\sigma + P_c{}^c\sigma) = \tfrac{1}{2}\sigma^2\mathring{\tau}_{ab}\,,$$
$$(\nabla^a\sigma)(\nabla_a\sigma) - \tfrac{\sigma}{2}(\Delta\sigma + P_a{}^a\sigma) + 1 = \tfrac{1}{12}\sigma^3\tau\,,$$

where $P_{ab} := R_{ab} - \frac{1}{4}Rg_{ab}$ and $\mathring{}$ denotes a trace-free part.

cCKY system

$$\widetilde{\nabla}_a \widetilde{Q}_{bc} - \widetilde{\nabla}_{[a} \widetilde{Q}_{bc]} + \tfrac{2}{3} \widetilde{g}_{a[b} \widetilde{\nabla}^d \widetilde{Q}_{c]d} = \mathcal{O}(\sigma),$$
$$\widetilde{\nabla}_{[a} \widetilde{Q}_{bc]} = \mathcal{O}(\sigma),$$

Conformal version coupled to σ:

$$\nabla_a Q_{bc} - \nabla_{[a} Q_{bc]} + \tfrac{2}{3} \mathbf{g}_{a[b} \nabla^d Q_{c]d} = \mathcal{O}(\sigma^4),$$
$$\sigma \nabla_{[a} Q_{bc]} - 3(\nabla_{[a}\sigma) Q_{bc]} = \mathcal{O}(\sigma^5).$$

Constraints arising from Einstein-cCKY system

Conformally covariant Einstein-cCKY system coupled to σ

To be solved for σ and Q:

$$\begin{array}{r}\text{asymptotic}\\ \text{vacuum Einstein}\end{array}\begin{cases}\nabla_a\nabla_b\sigma+\sigma P_{ab}-\frac{1}{4}\mathbf{g}_{ab}(\Delta\sigma+P_c{}^c\sigma)=\frac{1}{2}\sigma^2\mathring{\tau}_{ab},\\ (\nabla^a\sigma)(\nabla_a\sigma)-\frac{\sigma}{2}(\Delta\sigma+P_a{}^a\sigma)+1=\frac{1}{12}\sigma^3\tau\end{cases}$$

$$\begin{array}{r}\text{asymptotic}\\ \text{cCKY}\end{array}\begin{cases}\nabla_a Q_{bc}-\nabla_{[a}Q_{bc]}+\frac{2}{3}\mathbf{g}_{a[b}\nabla^d Q_{c]d}=\mathcal{O}(\sigma^4),\\ \sigma\nabla_{[a}Q_{bc]}-3(\nabla_{[a}\sigma)Q_{bc]}=\mathcal{O}(\sigma^5)\end{cases}$$

subject to Kerr-de Sitter boundary conditions at Σ.

Schematically, each equation has the following form:

$$\mathcal{E}_{a..}(\sigma,Q)=\mathcal{O}(\sigma^k),\quad k\in\mathbb{N}.$$

Constraints on Σ: Apply a conformally covariant normal derivative operator of normal order $<k$, evaluate on Σ:

$$(\nabla^i_{\hat{n}}+\dots)\circ\mathcal{E}_{a..}(\sigma,Q)\stackrel{\Sigma}{=}0,\qquad i<k,$$

where $\hat{n}$ is the unit normal vector to Σ.

Conformal fundamental forms of Σ

Asymptotic vacuum Einstein system:

$$\nabla_a \nabla_b \sigma + \sigma P_{ab} - \tfrac{1}{4}\mathbf{g}_{ab}(\Delta\sigma + P_c{}^c\sigma) = \tfrac{1}{2}\sigma^2 \mathring{\tau}_{ab},$$
$$(\nabla^a\sigma)(\nabla_a\sigma) - \tfrac{\sigma}{2}(\Delta\sigma + P_a{}^a\sigma) + 1 = \tfrac{1}{12}\sigma^3\tau$$

Constraints on the conformal boundary Σ

Subcritical conformal fundamental forms of Σ vanish:

- trace-free extrinsic curvature: $\mathring{\mathrm{II}}_{ab}$,
- electric part of the Weyl tensor: $W_{\hat{n}a\hat{n}b}$.

Fourth conformal fundamental form $\mathring{\mathrm{IV}}$ satisfies,

$$\overline{\nabla}^b \mathring{\mathrm{IV}}_{ab} \stackrel{\Sigma}{=} \lim_{\sigma\to 0}\left(\sigma^{-1}\tau^{\top}_{\hat{n}a}\right) - \tfrac{1}{3}\overline{\nabla}_a\tau,$$

where $\top$ is the projection onto Σ, and is otherwise undetermined.

Boundary data for vacuum conformal Einstein system:

$$\overline{\mathbf{g}} \quad \text{and} \quad \mathring{\mathrm{IV}}$$

Constraints arising from Einstein-cCKY system

Asymptotic cCKY equations

$$\nabla_a Q_{bc} - \nabla_{[a} Q_{bc]} + \tfrac{2}{3}\mathbf{g}_{a[b}\nabla^d Q_{c]d} = \mathcal{O}(\sigma^4),$$
$$\sigma\nabla_{[a} Q_{bc]} - 3(\nabla_{[a}\sigma) Q_{bc]} = \mathcal{O}(\sigma^5).$$

At the second normal order,

$$2\mathring{\mathrm{IV}}_{a[b} Q^{\top}_{|\hat{n}|c]} + \overline{\mathbf{g}}_{a[b}\mathring{\mathrm{IV}}_{c]}{}^{d} Q^{\top}_{\hat{n}d} \stackrel{\Sigma}{=} 0,$$

Algebraic constraint on the boundary data $\mathring{\mathrm{IV}}$.

Determining $\mathring{\mathrm{IV}}$

Given

$$Q \overset{\Sigma}{=} Q_{\text{Kerr-de Sitter}}$$

we can solve the algebraic constraint on $\mathring{\mathrm{IV}}$ to obtain

$$\mathring{\mathrm{IV}}_{ab} = F \mathring{A}^{\mathrm{KdS}}_{ab},$$

where $\mathring{A}^{\mathrm{KdS}}$ is a trace-free tensor determined by the geometry of Σ and F is a function constrained by

$$\overline{\nabla}^b \mathring{\mathrm{IV}}_{ab} \overset{\Sigma}{=} \lim_{\sigma \to 0} \left(\sigma^{-1} \tau_{\hat{n}a} \right) - \tfrac{1}{3} \overline{\nabla}_a \tau.$$

Vacuum solution. If $\tau_{ab} = 0$, then F is constant,

$$F = 2m \left(1 + \frac{J^2}{m^2} \right)^{-2}.$$

Quasi-local charge [2]

$$C_\xi[N] := \frac{1}{8\pi}\int_N \left(\mathring{\mathrm{IV}}_{ab} + \tfrac{1}{3}\tau\overline{g}_{ab}\right)\xi^a\hat{r}^b \mathrm{d}N\,,$$

where ξ is a conformal Killing vector, N a closed surface of Σ and $\hat{r}$ is the unit normal to N in Σ.

Boyer-Lindquist chart (t, r, θ, ϕ)**:** Charges corresponding to ∂_t and ∂_ϕ evaluated on $t = \mathrm{const.}$ spheres S_t can be written as

$$\text{quasi-local mass} := C_{\partial_t}[S_t] = \tfrac{1}{8\pi}\int_{S^2}\left(F + \tfrac{\tau}{3}\left(1 + \tfrac{J^2}{m^2}\right)^{-2}\right)\mathrm{d}S^2,$$

$$\text{quasi-local angular momentum} := C_{\partial_\phi}[S_t] = -\tfrac{3J}{16m\pi}\int_{S^2} F\sin^2\theta\,\mathrm{d}S^2,$$

where dS^2 is the volume element of the unit sphere.

[2] A. Ashtekar, B. Bonga and A. Kesavan, Asymptotics with a positive cosmological constant: I. Basic framework, Class. Quant. Grav. 32, 025004, 2015.

Takeaways

Summary:

- In asymptotically locally Kerr–de Sitter spacetimes, the asymptotic cCKY equations impose an algebraic constraint on the boundary-data tensor $\mathring{\mathrm{IV}}$,
- This constraint reduces $\mathring{\mathrm{IV}}$ to a single scalar function that enters the definitions of the quasi-local mass and angular momentum.

Based on:

Samuel Blitz and JK, *Asymptotia of Kerr-de Sitter Black Holes*, Class. Quantum Grav. 42 20LT02, 2025.

Thank you!