Extremal (and Type D) horizons of the NUT type

PoToR Conference 2026

Maciej Ossowski
Institute of Theoretical Physics, Jagiellonian University

Based (mostly) on:
Phys. Rev. D 113, 044004 (2026) Eryk Buk, Jerzy Lewandowski, Denis Dobkowski-Ryłko, **MO**

Part I: Isolated horizon structure

Isolated horizons. Why?

The "proper" definition:

$$\underbrace{\mathcal{B}}_{\text{black hole region}} = \mathcal{M} \setminus J^-(\mathcal{I}^+), \qquad \underbrace{\mathcal{H}^+}_{\text{black hole event horizon}} = \partial\mathcal{B}$$

Isolated horizons. Why?

The "proper" definition:

$$\underbrace{\mathcal{B}}_{\text{black hole region}} = \mathcal{M} \setminus J^-(\mathcal{I}^+), \qquad \underbrace{\mathcal{H}^+}_{\text{black hole event horizon}} = \partial\mathcal{B}$$

- Problem: teleological nature - we know it is there only at the end
- Hawking: In stationary ST, satisfying EEs BH horizon is a Killing horizon

Isolated horizons. Why?

The "proper" definition:

$$\underbrace{\mathcal{B}}_{\text{black hole region}} = \mathcal{M} \setminus J^-(\mathcal{I}^+), \qquad \underbrace{\mathcal{H}^+}_{\text{black hole event horizon}} = \partial\mathcal{B}$$

- Problem: teleological nature - we know it is there only at the end
- Hawking: In stationary ST, satisfying EEs BH horizon is a Killing horizon

Killing horizon: Let ℓ be spacetime KVF $\mathcal{L}_\ell g = 0$

$$\mathcal{H}_K = \mathcal{N}_\ell := \{g(\ell,\ell) = 0, \ell \neq 0\} \quad \text{(and } \ell \text{ must be tangent to } \mathcal{N}\text{)}$$

Usually (e.g. Schwarzschild, Kerr), spherical BH has topology: $\underbrace{S^2}_{\text{positive}} \times \underbrace{\mathbb{R}}_{\text{null}}$

Isolated horizons. Why?

The "proper" definition:

$$\underbrace{\mathcal{B}}_{\text{black hole region}} = \mathcal{M} \setminus J^-(\mathcal{I}^+), \qquad \underbrace{\mathcal{H}^+}_{\text{black hole event horizon}} = \partial\mathcal{B}$$

- Problem: teleological nature - we know it is there only at the end
- Hawking: In stationary ST, satisfying EEs BH horizon is a Killing horizon

Killing horizon: Let ℓ be spacetime KVF $\mathcal{L}_\ell g = 0$

$$\mathcal{H}_K = \mathcal{N}_\ell := \{g(\ell, \ell) = 0, \ell \neq 0\} \quad \text{(and } \ell \text{ must be tangent to } \mathcal{N}\text{)}$$

Usually (e.g. Schwarzschild, Kerr), spherical BH has topology: $\underbrace{S^2}_{\text{positive}} \times \underbrace{\mathbb{R}}_{\text{null}}$

- $\mathcal{N}_\ell$ is a null surface $\Rightarrow$ past and future lightcones are on one side
- Non-expanding: S^2 part of the horizon does not expand $\approx$ time independent
- Problem: non-local - required symmetries (e.g. stationarity) outside of the horizons

Intrinsic structure of Isolated Horizons

Definition: *Non-expanding horizon* $(H, {}^{(3)}g, {}^{(3)}\nabla)$

- $H =$ 3D null surface with ${}^{(3)}g = \Pi^* g$ of signature $(0++)$ and topology of bundle $\Pi : H \to S$. (S, g) compact, Riemannian, dim 2.
- Non-expanding: any null normal $\ell \in \Gamma(TH)$ preserves the metric $\mathcal{L}_\ell {}^{(3)}g = 0$
- H-null $\implies$ ${}^{(3)}\nabla$ is not unique. Has to be prescribed s.t. : $\nabla\, g = 0$ or inherited from ST.

Intrinsic structure of Isolated Horizons

Definition: *Non-expanding horizon* $(H, {}^{(3)}g, {}^{(3)}\nabla)$

- $H =$ 3D null surface with ${}^{(3)}g = \Pi^* g$ of signature $(0++)$ and topology of bundle $\Pi : H \to S$. (S, g) compact, Riemannian, dim 2.
- Non-expanding: any null normal $\ell \in \Gamma(TH)$ preserves the metric $\mathcal{L}_\ell {}^{(3)}g = 0$
- H-null $\implies$ ${}^{(3)}\nabla$ is not unique. Has to be prescribed s.t. : $\nabla\, g = 0$ or inherited from ST.

More data:

- Surface gravity: ${}^{(3)}\nabla_\ell \ell = \kappa^{(\ell)} \ell$. Constant on $\mathcal{H}$ assuming EEs.
- Rotation 1-form $\omega^{(\ell)}$: ${}^{(3)}\nabla \ell =: \omega^{(\ell)} \otimes \ell$, $\mathcal{L}_\ell \omega^{(\ell)} = 0$.
- Pseudo-scalar invariant: $\mathrm{d}\omega^{(\ell)} = \Omega\, \eta$. η -area form. Intrinsic measure of angular momentum, $\Omega \neq 0 \iff$ rotation

Intrinsic structure of Isolated Horizons

Definition: *Non-expanding horizon* $(H, {}^{(3)}g, {}^{(3)}\nabla)$

- $H =$ 3D null surface with ${}^{(3)}g = \Pi^* g$ of signature $(0++)$ and topology of bundle $\Pi : H \to S$. (S, g) compact, Riemannian, dim 2.
- Non-expanding: any null normal $\ell \in \Gamma(TH)$ preserves the metric $\mathcal{L}_\ell {}^{(3)}g = 0$
- H-null $\implies$ ${}^{(3)}\nabla$ is not unique. Has to be prescribed s.t. : $\nabla\, g = 0$ or inherited from ST.

More data:

- Surface gravity: ${}^{(3)}\nabla_\ell \ell = \kappa^{(\ell)} \ell$. Constant on $\mathcal{H}$ assuming EEs.
- Rotation 1-form $\omega^{(\ell)}$: ${}^{(3)}\nabla \ell =: \omega^{(\ell)} \otimes \ell$, $\mathcal{L}_\ell \omega^{(\ell)} = 0$.
- Pseudo-scalar invariant: $\mathrm{d}\omega^{(\ell)} = \Omega\, \eta$. η -area form. Intrinsic measure of angular momentum, $\Omega \neq 0 \iff$ rotation

Definition: *Isolated Horizon* $(H, {}^{(3)}g, [\ell], {}^{(3)}\nabla)$

- $[\mathcal{L}_\ell, {}^{(3)}\nabla] = 0$. Restrict equivalence class of ℓ: $\ell \mapsto c_0 \ell$, $c_0 \in \mathbb{R}$
- ω - unique, ℓ and κ up to a real const $\implies$ either $\underbrace{\text{extremal}}_{\kappa^{(\ell)}=0}$ or $\underbrace{\text{not}}_{\kappa^{(\ell)}\neq 0}$

Principle bundle structure

- So far: fibre bundle $\Pi : H \to S$
- More: principle bundle $G \hookrightarrow H \xrightarrow{\Pi} S$ (trivial or not)
  - G acts via the flow of some horizon symmetry (KVF)
  - Connected group: $G = \mathbb{R}$ or $U(1)$.
  - Hopf bundle ($\propto S^3$) for $G = U(1)$ and $S = S^2$
- Can we define the $U(1)$ connection (choice of horizontal v.f. via kernel) using natural IH objects?
  - There is a choice if $U(1)$ is generated by ℓ:
  - Connection: $A = \frac{\omega}{\kappa} \otimes \ell^*$. Bundle curvature $F = \frac{1}{\kappa}\,\mathrm{d}\omega \otimes \ell^*$
  - Breaks for $\kappa = 0$, we need other approach. Also ω is the same on any 2-surface transversal to ℓ

Principle bundle structure

- So far: fibre bundle $\Pi : H \to S$
- More: principle bundle $G \hookrightarrow H \xrightarrow{\Pi} S$ (trivial or not)
  - G acts via the flow of some horizon symmetry (KVF)
  - Connected group: $G = \mathbb{R}$ or $U(1)$.
  - Hopf bundle ($\propto S^3$) for $G = U(1)$ and $S = S^2$
- Can we define the $U(1)$ connection (choice of horizontal v.f. via kernel) using natural IH objects?
  - There is a choice if $U(1)$ is generated by ℓ:
  - Connection: $A = \frac{\omega}{\kappa} \otimes \ell^*$. Bundle curvature $F = \frac{1}{\kappa} \mathrm{d}\omega \otimes \ell^*$
  - Breaks for $\kappa = 0$, we need other approach. Also ω is the same on any 2-surface transversal to ℓ

The Main Goal

Classify IH geometries according to S and G

Part II: Isolated Horizon geometry constraints

Horizon reduction of Λ-vacuum Einstein Equations

[1] Also known as Near-Horizon Geometry Equation, quasi-Einstein Equation

Horizon reduction of Λ-vacuum Einstein Equations

1. Petrov Type D horizons: (If embeddable then) Weyl $\big|_{\mathcal{H}}$ is of type D
   - Subject to Petrov type D equation in suitable "restricted null tetrad" $(\ell, m, \bar{m})$

$$\bar{m}^A \bar{m}^B \nabla_A \nabla_B \left(K + \mathrm{i}\Omega - \frac{\Lambda}{3} \right)^{-1/3} = 0$$

where K is Gaussian curvature of g and Ω is the rotation invariant.

[1] Also known as Near-Horizon Geometry Equation, quasi-Einstein Equation

Horizon reduction of Λ-vacuum Einstein Equations

1. Petrov Type D horizons: (If embeddable then) Weyl$\big|_{\mathcal{H}}$ is of type D
   - Subject to Petrov type D equation in suitable "restricted null tetrad" $(\ell, m, \bar{m})$

$$\bar{m}^A \bar{m}^B \nabla_A \nabla_B \left(K + \mathrm{i}\Omega - \frac{\Lambda}{3} \right)^{-1/3} = 0$$

   where K is Gaussian curvature of g and Ω is the rotation invariant.

2. Extremal horizons: vanishing surface gravity $\kappa = 0$
   - Extremal horizon Einstein Equations: [1]

$$\nabla_{(A}\omega_{B)} + \omega_A \omega_B - K g_{AB} + \frac{\Lambda}{2} g_{AB} = 0$$

   where R_{AB} is Ricci tensor on $(\mathcal{S}, g)$
   - Equivalent to Einstein Equations for the Near-Horizon Geometry

$$\mathrm{d}s^2 = 2\mathrm{d}v \left(\mathrm{d}r + r\omega_A \mathrm{d}x^A + \frac{1}{2} r^2 F \mathrm{d}v \right) + g_{AB} \mathrm{d}x^A \mathrm{d}x^B$$

[1] Also known as Near-Horizon Geometry Equation, quasi-Einstein Equation

Horizon reduction of Λ-vacuum Einstein Equations

JAGIELLONIAN UNIVERSITY IN KRAKÓW

1. Petrov Type D horizons: (If embeddable then) Weyl$\big|_{\mathcal{H}}$ is of type D
   - Subject to Petrov type D equation in suitable "restricted null tetrad" $(\ell, m, \bar{m})$

$$\bar{m}^A \bar{m}^B \nabla_A \nabla_B \left(K + \mathrm{i}\Omega - \frac{\Lambda}{3}\right)^{-1/3} = 0$$

   where K is Gaussian curvature of g and Ω is the rotation invariant.
2. Extremal horizons: vanishing surface gravity $\kappa = 0$
   - Extremal horizon Einstein Equations: [1]

$$\nabla_{(A}\omega_{B)} + \omega_A \omega_B - K g_{AB} + \frac{\Lambda}{2} g_{AB} = 0$$

   where R_{AB} is Ricci tensor on $(\mathcal{S}, g)$
   - Equivalent to Einstein Equations for the Near-Horizon Geometry

$$\mathrm{d}s^2 = 2\mathrm{d}v\left(\mathrm{d}r + r\omega_A \mathrm{d}x^A + \frac{1}{2} r^2 F \mathrm{d}v\right) + g_{AB}\mathrm{d}x^A \mathrm{d}x^B$$

Type D Equation is an integrability condition of EEH Equation; Any solution of EEH is a solution of Type D Eq, but may not satisfy Type D condition: $\Psi_2 = K + \mathrm{i}\Omega - \frac{\Lambda}{3} \neq 0$ everywhere at $\mathcal{H}$.

[1] Also known as Near-Horizon Geometry Equation, quasi-Einstein Equation

Petrov Type D Equation - overview of the solutions

Assumption: Local section S is an orientable and compact surface $\implies$ topologies may be characterized by the genus.
Known Λ-vacuum solutions (with structure group generated by ℓ):

[2] Physics Letters B 783 (2018) 415–420, Denis Dobkowski-Ryłko, Wojciech Kaminski, Jerzy Lewandowski, Adam Szereszewski
[3] Phys. Rev. D 110 (2024), 024071, Jerzy Lewandowski, and **MO**
[4] Phys. Rev. D 98 (2018), Denis Dobkowski-Ryłko, Jerzy Lewandowski, and Tomasz Pawłowski
[5] Phys. Rev. D 100 (2019), Denis Dobkowski-Ryłko, Jerzy Lewandowski, and István Rácz

Petrov Type D Equation - overview of the solutions

Assumption: Local section S is an orientable and compact surface $\implies$ topologies may be characterized by the genus.
Known Λ-vacuum solutions (with structure group generated by ℓ):

- $H = S \times \mathbb{R}$, S - smooth Riemann surface with genus > 0: $K = \frac{4\pi(1-\text{genus})}{\text{Area}} \neq \frac{\Lambda}{3}$, $\Omega = 0$ (ω closed), [2]
  - Embeddable in Schwarzschild-(a)dS with planar/hyperbolic slices mod identification

[2] Physics Letters B 783 (2018) 415–420, Denis Dobkowski-Ryłko, Wojciech Kaminski, Jerzy Lewandowski, Adam Szereszewski
[3] Phys. Rev. D 110 (2024), 024071, Jerzy Lewandowski, and **MO**
[4] Phys. Rev. D 98 (2018), Denis Dobkowski-Ryłko, Jerzy Lewandowski, and Tomasz Pawłowski
[5] Phys. Rev. D 100 (2019), Denis Dobkowski-Ryłko, Jerzy Lewandowski, and István Rácz

Petrov Type D Equation - overview of the solutions

Assumption: Local section S is an orientable and compact surface $\implies$ topologies may be characterized by the genus.
Known Λ-vacuum solutions (with structure group generated by ℓ):

- $H = S \times \mathbb{R}$, S - smooth Riemann surface with genus > 0: $K = \frac{4\pi(1-\text{genus})}{\text{Area}} \neq \frac{\Lambda}{3}$, $\Omega = 0$ (ω closed), [2]
    - Embeddable in Schwarzschild-(a)dS with planar/hyperbolic slices mod identification
- $H \to S$, with structure of $U(1)-$non -trivial bundle over S, smooth Riemann surface with genus > 0, K as above, $\Omega =$ const $\propto$ Chern class [3]
    - Embeddable in Taub-NUT-(a)dS with planar/hyperbolic slices mod identification

[2] Physics Letters B 783 (2018) 415–420, Denis Dobkowski-Ryłko, Wojciech Kaminski, Jerzy Lewandowski, Adam Szereszewski
[3] Phys. Rev. D 110 (2024), 024071, Jerzy Lewandowski, and **MO**
[4] Phys. Rev. D 98 (2018), Denis Dobkowski-Ryłko, Jerzy Lewandowski, and Tomasz Pawłowski
[5] Phys. Rev. D 100 (2019), Denis Dobkowski-Ryłko, Jerzy Lewandowski, and István Rácz

Petrov Type D Equation - overview of the solutions

Assumption: Local section S is an orientable and compact surface $\implies$ topologies may be characterized by the genus.
Known Λ-vacuum solutions (with structure group generated by ℓ):

- $H = S \times \mathbb{R}$, S - smooth Riemann surface with genus > 0: $K = \frac{4\pi(1-\text{genus})}{\text{Area}} \neq \frac{\Lambda}{3}$, $\Omega = 0$ (ω closed), [2]
  - ▶ Embeddable in Schwarzschild-(a)dS with planar/hyperbolic slices mod identification
- $H \to S$, with structure of $U(1)-$non -trivial bundle over S, smooth Riemann surface with genus > 0, K as above, $\Omega =$ const $\propto$ Chern class [3]
  - ▶ Embeddable in Taub-NUT-(a)dS with planar/hyperbolic slices mod identification
- $H = S^2 \times \mathbb{R}$ with axial symmetry[4]
  - ▶ Non-rotating ($\Omega = 0$): $K =$ const embeddable in Schwarzschild-(a)dS
  - ▶ Rotating ($\Omega \neq 0$): $K \neq$ const, 2 parameter family, embeddable in Kerr-(a)dS

[2] Physics Letters B 783 (2018) 415–420, Denis Dobkowski-Ryłko, Wojciech Kaminski, Jerzy Lewandowski, Adam Szereszewski
[3] Phys. Rev. D 110 (2024), 024071, Jerzy Lewandowski, and **MO**
[4] Phys. Rev. D 98 (2018), Denis Dobkowski-Ryłko, Jerzy Lewandowski, and Tomasz Pawłowski
[5] Phys. Rev. D 100 (2019), Denis Dobkowski-Ryłko, Jerzy Lewandowski, and István Rácz

Petrov Type D Equation - overview of the solutions

Assumption: Local section S is an orientable and compact surface $\implies$ topologies may be characterized by the genus.
Known Λ-vacuum solutions (with structure group generated by ℓ):

- $H = S \times \mathbb{R}$, S - smooth Riemann surface with genus > 0: $K = \frac{4\pi(1-\text{genus})}{\text{Area}} \neq \frac{\Lambda}{3}$, $\Omega = 0$ (ω closed), [2]
  - Embeddable in Schwarzschild-(a)dS with planar/hyperbolic slices mod identification
- $H \to S$, with structure of $U(1)-$non -trivial bundle over S, smooth Riemann surface with genus > 0, K as above, $\Omega = $ const $\propto$ Chern class [3]
  - Embeddable in Taub-NUT-(a)dS with planar/hyperbolic slices mod identification
- $H = S^2 \times \mathbb{R}$ with axial symmetry[4]
  - Non-rotating ($\Omega = 0$): $K =$ const embeddable in Schwarzschild-(a)dS
  - Rotating ($\Omega \neq 0$): $K \neq$ const, 2 parameter family, embeddable in Kerr-(a)dS
- $H = S^3 \to S^2$, Hopf bundle with axial symmetry [5]
  - 2 - parameter family with $K =$ const, $\Omega =$ const $\neq 0$ embeddable in Taub-NUT-(a)dS
  - 3 - parameter family with $K \neq$ const, $\Omega \neq$ const embeddable in accelerated Kerr-NUT-(a)dS
  - (Mysterious) 2 - parameter family with $K =$ const, $\Omega \neq$ const embeddable in acc. Kerr-NUT-(a)dS at $r = 0$

Spherical and non-axially symmetric solution - open problem!

[2] Physics Letters B 783 (2018) 415–420, Denis Dobkowski-Ryłko, Wojciech Kaminski, Jerzy Lewandowski, Adam Szereszewski
[3] Phys. Rev. D 110 (2024), 024071, Jerzy Lewandowski, and **MO**
[4] Phys. Rev. D 98 (2018), Denis Dobkowski-Ryłko, Jerzy Lewandowski, and Tomasz Pawłowski
[5] Phys. Rev. D 100 (2019), Denis Dobkowski-Ryłko, Jerzy Lewandowski, and István Rácz

Extremal Horizon equation - overview of the solutions

Known Λ-vacuum solutions:

[6] Physics Letters B 783 (2018) 415–420, Denis Dobkowski-Ryłko, Wojciech Kaminski, Jerzy Lewandowski, Adam Szereszewski
[7] Phys. Rev. D 103, 104004, E. Buk and J. Lewandowski
[8] M. Dunajski and J. Lucietti, Intrinsic rigidity of extremal horizons (2023)
W. Kamiński and J. Lewandowski, Extreme horizon equation (2024)

Known Λ-vacuum solutions:

- $H = S \times \mathbb{R}$, S - smooth Riemann surface with genus > 0: $K = \Lambda$, $\Omega = 0$ $(\omega = 0)$, [6]
  - Embeddable in extremal Schwarzschild-(a)dS with planar/hyperbolic slices mod identification
  - Natural extension to non-trivial $U(1)-$bundle - no-go

[6] Physics Letters B 783 (2018) 415–420, Denis Dobkowski-Ryłko, Wojciech Kaminski, Jerzy Lewandowski, Adam Szereszewski
[7] Phys. Rev. D 103, 104004, E. Buk and J. Lewandowski
[8] M. Dunajski and J. Lucietti, Intrinsic rigidity of extremal horizons (2023)
W. Kamiński and J. Lewandowski, Extreme horizon equation (2024)

Known Λ-vacuum solutions:

- $H = S \times \mathbb{R}$, S - smooth Riemann surface with genus > 0: $K = \Lambda$, $\Omega = 0$ $(\omega = 0)$, [6]
  - Embeddable in extremal Schwarzschild-(a)dS with planar/hyperbolic slices mod identification
  - Natural extension to non-trivial $U(1)-$bundle - no-go
- $H = S^2 \times \mathbb{R}$ with axial symmetry[7]
  - Non-rotating $(\Omega = 0)$: $K =$ const embeddable in Schwarzschild-(a)dS
  - Rotating $(\Omega \neq 0)$: $K \neq$ const, 2 parameter family, embeddable in Kerr-(a)dS

[6] Physics Letters B 783 (2018) 415–420, Denis Dobkowski-Ryłko, Wojciech Kaminski, Jerzy Lewandowski, Adam Szereszewski
[7] Phys. Rev. D 103, 104004, E. Buk and J. Lewandowski
[8] M. Dunajski and J. Lucietti, Intrinsic rigidity of extremal horizons (2023)
W. Kamiński and J. Lewandowski, Extreme horizon equation (2024)

Known Λ-vacuum solutions:

- $H = S \times \mathbb{R}$, S - smooth Riemann surface with genus > 0: $K = \Lambda$, $\Omega = 0$ $(\omega = 0)$, [6]
  - Embeddable in extremal Schwarzschild-(a)dS with planar/hyperbolic slices mod identification
  - Natural extension to non-trivial $U(1)-$bundle - no-go
- $H = S^2 \times \mathbb{R}$ with axial symmetry[7]
  - Non-rotating $(\Omega = 0)$: $K =$ const embeddable in Schwarzschild-(a)dS
  - Rotating $(\Omega \neq 0)$: $K \neq$ const, 2 parameter family, embeddable in Kerr-(a)dS
- $\mathcal{S} = S^2$ without axial symmetry and smooth? No-go! Rigidity theorem: axial symmetry always exists [8]

[6] Physics Letters B 783 (2018) 415–420, Denis Dobkowski-Ryłko, Wojciech Kaminski, Jerzy Lewandowski, Adam Szereszewski
[7] Phys. Rev. D 103, 104004, E. Buk and J. Lewandowski
[8] M. Dunajski and J. Lucietti, Intrinsic rigidity of extremal horizons (2023)
W. Kamiński and J. Lewandowski, Extreme horizon equation (2024)

Part III: Towards new solutions: transversal IH

Taub-NUT spacetime and Misner interpretation

$$\mathrm{d}s^2 = -f(r)(\mathrm{d}t - \underbrace{2l(\cos\theta - 1)}_{A(\theta)}\,\mathrm{d}\phi)^2 + \frac{\mathrm{d}r^2}{f(r)} + (r^2 + l^2)g_{S^2}$$

$f(r) = \frac{r^2-2mr-l^2}{r^2+l^2}$, $\omega := \mathrm{d}t - A\mathrm{d}\phi$ not continuous at $\theta = \pi \implies$ singular half-axis.

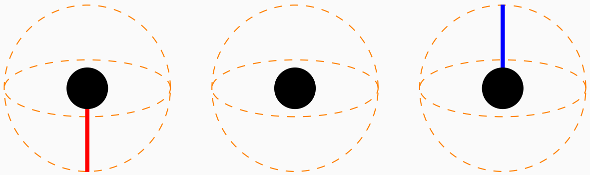
Taub-NUT spacetime and Misner interpretation

$$\mathrm{d}s^2 = -f(r)(\mathrm{d}t - \underbrace{2l(\cos\theta - 1)}_{A(\theta)}\,\mathrm{d}\phi)^2 + \frac{\mathrm{d}r^2}{f(r)} + (r^2 + l^2)g_{S^2}$$

$f(r) = \frac{r^2 - 2mr - l^2}{r^2 + l^2}$, $\omega := \mathrm{d}t - A\mathrm{d}\phi$ not continuous at $\theta = \pi \implies$ singular half-axis.

Misner interpretation

$t' := t - 4l\phi \implies \omega := \mathrm{d}t - A'\mathrm{d}\phi, \quad A' = 2l\,(\cos\theta + 1)$
t is cyclic with period $8\pi l$! $\partial_t, \partial_{\phi_\pi}$ - cyclic
Spacetime topology: $S^3 \times \mathbb{R}$, $U(1)$-bundle over $S^2 \times \mathbb{R}$



Plebański-Demiański black hole solutions

- Most general Petrov Type D Lambda-electro-vacuum solution to EEs in 4D, with Weyl and Maxwell field aligned
- Generalisation of Schwarzschild, Kerr etc:

$$\underbrace{M}_{\text{"mass"}},\ \underbrace{a}_{\text{Kerr}},\ \underbrace{\alpha}_{\text{acceleration}},\ \underbrace{e,\ g}_{\text{e.m. charges}},$$

$$\underbrace{\Lambda}_{\text{cosmological constant}},\ \underbrace{l}_{\text{N(ewmn)-U(unti)-T(amburino)}}$$

[9] Jerzy Lewandowski, **MO**, Nonsingular extension of the Kerr-NUT–(anti–)de Sitter spacetimes, Phys. Rev. D 104, 024022 (2021)
Jerzy Lewandowski, **MO**, Projectively nonsingular horizons in Kerr-NUT–de Sitter spacetimes, Phys. Rev. D 102, 124055 (2020)
Jerzy Lewandowski, **MO**, Non-singular Kerr-NUT-de Sitter spacetimes, Classical and Quantum Gravity 37, 205007 (2020)

Plebański-Demiański black hole solutions

- Most general Petrov Type D Lambda-electro-vacuum solution to EEs in 4D, with Weyl and Maxwell field aligned
- Generalisation of Schwarzschild, Kerr etc:

$$\underbrace{M}_{\text{"mass"}},\ \underbrace{a}_{\text{Kerr}},\ \underbrace{\alpha}_{\text{acceleration}},\ \underbrace{e,\ g}_{\text{e.m. charges}},$$

$$\underbrace{\Lambda}_{\text{cosmological constant}},\ \underbrace{l}_{\text{N(ewmn)-U(unti)-T(amburino)}}$$

Misner interpretation maybe extended to generic Plebański-Demiański spacetime with right choice of cyclic KVF - not necessarily ℓ. Including singularity free sols, and $S^3 \times \mathbb{R}$ cosmologies [9].

Generically up to 4 Killing horizons, possibly extremal

[9] Jerzy Lewandowski, **MO**, Nonsingular extension of the Kerr-NUT–(anti–)de Sitter spacetimes, Phys. Rev. D 104, 024022 (2021)
Jerzy Lewandowski, **MO**, Projectively nonsingular horizons in Kerr-NUT–de Sitter spacetimes, Phys. Rev. D 102, 124055 (2020)
Jerzy Lewandowski, **MO**, Non-singular Kerr-NUT-de Sitter spacetimes, Classical and Quantum Gravity 37, 205007 (2020)

Construction of IH with transversal symmetry

Idea: Construct the horizon explicitly in coordinates by glueing two local trivializations, with $U(1)$ symmetry not null and find admissible (g, ω)

Assume that projection to the space of null generators (orbits of ℓ) is differentiable

$$\pi : H \to \mathcal{S}$$

and reduce the problem to 2D

$$^{(3)}g = \pi^* g \quad \text{and} \quad ^{(3)}\omega = \pi^* \omega.$$

- g a metric tensor on S^2 with conical singularities
- Extremal: ω a regular 1-form on S^2 [10]. Type D: two local sections of regular 1-form on S^3.

[10] One may show that for $\kappa = 0$ a rescaling s.t. ω is regular is always possible.

- $H = S^3 \to S^2$, Hopf bundle with axial symmetry and conical singularities [11]

Assume axial symmetry, but ϕ not necessarily 2π-periodic, possible conical singularities

$$g = R^2\left(\frac{dx^2}{P(x)^2} + P(x)^2 d\phi^2\right), \quad \omega = \frac{dB}{B} + \star dU = \frac{B_{,x}(x)}{B(x)}dx + P^2(x)U_{,x}(x)d\phi.$$

[11] Phys. Rev. D 108 (2023), Denis Dobkowski-Ryłko, Jerzy Lewandowski, and **MO**
Phys. Rev. D 113, 044004 (2026) Eryk Buk, Jerzy Lewandowski, Denis Dobkowski-Ryłko, **MO**

- $H = S^3 \to S^2$, Hopf bundle with axial symmetry and conical singularities [11]

Assume axial symmetry, but ϕ not necessarily 2π-periodic, possible conical singularities

$$g = R^2\left(\frac{dx^2}{P(x)^2} + P(x)^2 d\phi^2\right), \quad \omega = \frac{dB}{B} + \star dU = \frac{B_{,x}(x)}{B(x)}dx + P^2(x)U_{,x}(x)d\phi.$$

Trace part of EEH solves for ω, trace-free part for g:

- Non-rotating: $d\omega = 0$, 2 parameter family necessarily $\Lambda > 0$
- Rotating: $d\omega \neq 0$, 3 parameter family, any sign of Λ

Type D Equation:

- Rotating: 4 parameters family (not 3 as before + 2 conical singularities), any Λ

[11] Phys. Rev. D 108 (2023), Denis Dobkowski-Ryłko, Jerzy Lewandowski, and **MO**
Phys. Rev. D 113, 044004 (2026) Eryk Buk, Jerzy Lewandowski, Denis Dobkowski-Ryłko, **MO**

Embedding into Plebański-Demiański spacetimes

(Partial) embedding into the Plebański-Demiański family

- Any extremal/Type D horizon of accelerated Kerr-NUT-(anti-) de Sitter is of the rotating $d\omega \neq 0$ type
- Non-rotating $d\omega = 0$ horizons embed as
  - Horizons at $r = 0$ of accelerated Kerr-NUT-dS
  - Taub-NUT-dS if conical singularities vanish
  - Probably also accelerated Taub-NUT-dS

Embedding into Plebański-Demiański spacetimes

(Partial) embedding into the Plebański-Demiański family

- Any extremal/Type D horizon of accelerated Kerr-NUT-(anti-) de Sitter is of the rotating $d\omega \neq 0$ type
- Non-rotating $d\omega = 0$ horizons embed as
  - Horizons at $r = 0$ of accelerated Kerr-NUT-dS
  - Taub-NUT-dS if conical singularities vanish
  - Probably also accelerated Taub-NUT-dS

Dual purpose horizons:

1. Construct smooth (!) horizon with conically singular space of null orbits of ℓ, but the horizon is compact $\implies$ Misner interpretation
2. Forget about the transversal construction and construct product $S^2 \times \mathbb{R}$ horizons, but non smooth $\implies$ no periodic time

Related solutions?

- Podolsky-Matejov solutions: full classification of g for extremal horizons with conical singularities, parametrization by angle deficits $\delta_{\pm}$, agreement for vanishing charges. No explicit ω [12]
- Lucietti-Kunduri solutions: no assumption on topology, possibly conical singularities, not clear where (and if!) there are axis. Agreement if their solution has two poles (i.e. is on a sphere)
- Lewandowski-Buk, similar methodology. Complete agreement if conical singularities vanish [13]
- Type D equation on a sphere with conical singularities. The same construction of horizon, but subject to Petrov Type D equation. Solutions to EEH must solve type D Equation, but are not necessarily type D. [14]
- Constraints on the non-embedded horizon parameters

[12] D. Matejov and J. Podolsk´y, Uniqueness of extremal isolated horizons and their identification with horizons of all type d black holes*, Classical and Quantum Gravity 38, 135032 (2021)
D. Matejov and J. Podolsk´y, Extremal isolated horizons with a cosmological constant and the related unique type d black holes, Phys. Rev. D 105, 064016 (2022)

[13] E. Buk and J. Lewandowski, Axisymmetric, extremal horizons in the presence of a cosmological constant, Phys. Rev. D 103, 104004

[14] D. Dobkowski-Rylko, J. Lewandowski, and M. Ossowski, Isolated horizons of the hopf bundle structure transversal to the null direction, the horizon equations, and embeddability in nut-like space-times, Phys. Rev. D 108, 104057 (2023)

Related solutions?

- Podolsky-Matejov solutions: full classification of g for extremal horizons with conical singularities, parametrization by angle deficits $\delta_{\pm}$, agreement for vanishing charges. No explicit ω [12]
- Lucietti-Kunduri solutions: no assumption on topology, possibly conical singularities, not clear where (and if!) there are axis. Agreement if their solution has two poles (i.e. is on a sphere)
- Lewandowski-Buk, similar methodology. Complete agreement if conical singularities vanish [13]
- Type D equation on a sphere with conical singularities. The same construction of horizon, but subject to Petrov Type D equation. Solutions to EEH must solve type D Equation, but are not necessarily type D. [14]
- Constraints on the non-embedded horizon parameters

Thank you for your attention!

[12] D. Matejov and J. Podolsk´y, Uniqueness of extremal isolated horizons and their identification with horizons of all type d black holes*, Classical and Quantum Gravity 38, 135032 (2021)
D. Matejov and J. Podolsk´y, Extremal isolated horizons with a cosmological constant and the related unique type d black holes, Phys. Rev. D 105, 064016 (2022)

[13] E. Buk and J. Lewandowski, Axisymmetric, extremal horizons in the presence of a cosmological constant, Phys. Rev. D 103, 104004

[14] D. Dobkowski-Rylko, J. Lewandowski, and M. Ossowski, Isolated horizons of the hopf bundle structure transversal to the null direction, the horizon equations, and embeddability in nut-like space-times, Phys. Rev. D 108, 104057 (2023)