Exotic singularities in cosmology – classical and quantum

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References

- M. P. Dąbrowski, Spacetime averaging of exotic singularity universes, Physics Letters B 702, 320 (2011).
- M.P. Dąbrowski, Are Singularities the Limits of Cosmology?, in: Mathematical Structures of the Universe, M. Eckstein, M. Heller, S.J. Szybka (eds.), CC Press, Kraków, Poland, p. 99 (2014); arXiv:1407.4851.
- A. Balcerzak, S. Barroso-Bellido, MPD, S. Robles-Perez, Phys. Rev. D 103, 043507 (2021).
- S. Barroso-Bellido & MPD, Eur. Phys. J. C82, 990 (2022).
- MPD & T.B. Vasilev, Finite time Pseudo-Rip singularity in cosmology, Phys. Lett. B 878, 140516 (2026).
- S. Robles-Perez, MPD - in progress.

Content:

- 1. Singularities in relativity. Standard Big-Bang/Crunch.
- 2. Properties of non-standard (exotic) singularities.
- 3. Classifications of singularities.
- 4. A new finite-time Pseudo-Rip singularity.
- 5. Quantum entanglement and standard/exotic singularities.
- 6. Remarks.

1. Singularities in relativity.

- Singularities - best definition is by geodesic incompleteness (e.g. Hawking and Penrose 1973, Wald 1983) which allows to practically detect them without "adopting" them into the theory.
- Spacetime is singular if there exists **at least one geodesic which is incomplete** i.e. which cannot be extended in at least one direction and has only a finite range of affine parameter (proper time or length for non-null geodesics).
- This is a kind of "minimalistic" approach which does not tell us the full nature of these singularities: e.g. how they influence the physical and geometrical quantities.
- It is worth to investigate more subtleties of the matter and apply some other definitions to isolate them.

General requirements: the obedience of energy conditions.

The strong energy condition (SEC)

$$R_{\mu\nu}V^{\mu}V^{\nu} \geq 0, \quad V^{\mu} - \mathrm{a\ \ timelike\ \ vector}\ , \ \ \mathrm{i.e.,} \quad \varrho + p \geq 0, \quad \varrho + 3p \geq 0\ , \quad (1)$$

($R_{\mu\nu}$ - Ricci tensor) guarantees that **gravity is attractive**, the null (NEC)

$$T_{\mu\nu}k^{\mu}k^{\nu} \quad \geq \quad 0, \quad k^{\mu} - \mathrm{a\ \ null\ \ vector}\ , \ \ \mathrm{i.e.,} \quad \varrho + p \geq 0\ , \qquad (2)$$

guarantees that the **flux of matter is non-spacelike**, the weak (WEC)

$$T_{\mu\nu}V^{\mu}V^{\nu} \quad \geq \quad 0, \quad V^{\mu} - \mathrm{a\ \ timelike\ \ vector}\ , \ \ \mathrm{i.e.,}\ \varrho + p \geq 0, \ \ \rho \geq 0\ , \quad (3)$$

guarantees **positivity of energy**, and the dominant energy (DEC)

$$T_{\mu\nu}V^{\mu}V^{\nu} \quad \geq \quad 0, \quad T_{\mu\nu}V^{\mu} - \mathrm{not\ \ spacelike}\ , \ \ \mathrm{i.e.,}\ \mid p \mid \leq \varrho, \ \ \varrho \geq 0\ . \quad (4)$$

requires that the pressure p **is smaller** than the mass density ρ.

Standard Big-Bang/Crunch singularities.

Standard Einstein-Friedmann equations are two equations for three unknown functions of time $a(t), p(t), \varrho(t)$

$$\varrho = 3\left(\frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right) , \quad (5)$$

$$p = -\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right) . \quad (6)$$

plus an equation of state, e.g., of a barotropic type ($w =$ const. ≥ -1):

$$p(t) = w\varrho(t) . \quad (7)$$

Since very recently most of cosmologists studied only simplest - say "standard" solutions - each of them starts with Big-Bang singularity in which $a \to 0$, $\varrho, p \to \infty$
– one of them (of $K = +1$) terminates at the second singularity (Big-Crunch) where $a \to 0$, $\varrho, p \to \infty$ – the other two ($K = 0, -1$) continue to an asymptotic emptiness $\varrho, p \to 0$ for $a \to \infty$.

BB and BC exhibit geodesic incompleteness and curvature blow-up.

Dark energy and phantom - a Big-Rip exotic singularity and more.

Violation of SEC leads to the acceleration of the universe since from (5) and (6) one has

$$-\frac{4\pi G}{3}(\varrho + 3p) \quad = \quad \frac{\ddot{a}}{a} = -qH^2 \, , \quad (q = -\frac{\ddot{a}a}{\dot{a}^2}; \; H = \frac{\dot{a}}{a}) \, , \tag{8}$$

which together with (1)) means that

$$\ddot{a} \geq 0, \quad \text{or} \quad q \leq 0 \, , \tag{9}$$

and the universe reaches standard Big-Crunch singularity.. However, phantom as dark energy of a very large negative pressure (Caldwell 2002)

$$p < -\varrho, \qquad \text{or} \quad w < -1 \, , \tag{10}$$

violates all the energy conditions and the universe **reaches exotic** Big-Rip singularity (type I).

Big-Rip exotic singularity approach:

In a Big-Rip scenario everything is pulled apart on the approach to a Big-Rip in a reverse order (Caldwell et al. PRL '03).
Specifically, for $w = -3/2$ Big-Rip will happen in 20 Gyr from now and the scenario will be as follows:

- in 1 Gyr before BR - clusters are erased
- in 60 Myr before BR - Milky Way is destroyed
- 3 months before BR - Solar System becomes unbound
- 30 min before BR - Earth explodes
- 10^{-19} s before BR - atoms are dissociated
- nuclei etc. .....

All this comes from the formula $t \approx P\sqrt{2 \mid w+1 \mid}/[6\pi \mid 1+w \mid]$, where P is the period of a circular orbit around the system at radius R, mass M.

2. Non-standard (exotic) singularities and their properties.

Sudden Future Singularity (SFS): – manifests as a singularity of pressure (or $\ddot{a}$) only

– leads to the **DEC violation only**.

The scale factor can be as follows (Barrow 2004)

$$a(t) = a_s \left[\delta + (1-\delta)\, y^m - \delta\, (1-y)^n\right] , \qquad y \equiv \frac{t}{t_s} \tag{11}$$

where $a_s \equiv a(t_s) = \text{const.}$ and $\delta, m, n = \text{const.}$

Obviously, for $t = 0$ one has a Big-Bang singularity. If we assume that

$$1 < n < 2, \tag{12}$$

then using Einstein equations (38)-(40) we get

$$a = \text{const.}, \qquad \dot{a} = \text{const.} \qquad \varrho = \text{const.}$$
$$\ddot{a} \to \mp\infty \qquad p \to \pm\infty \qquad \text{for} \qquad t \to t_s \tag{13}$$

More examples are:

- generalized sudden future singularities (GSFS - Barrow 2005),
- finite scale factor singularities (FSF or type III),
- big-separation singularities (BS or type IV) (Nojiri et al. PRD 71, 063004 (2005)
- w-singularities (type V) (MPD. Denkiewicz 2009).
- some versions of a Big-Rip known as: a Little-Rip - LR (Frampton et al. 2011) and a Pseudo-Rip - PR (Frampton et al. 2012).

Type IV - Big Separation (BS)

Type IV singularity according to Nojiri et al. ’05 is when:

$$a = a_s = \text{const.},\ \varrho \to 0,\ p \to 0,\ \dddot{a},\ \ddot{H} \to \infty \text{ etc.}$$

and so it is similar to Generalized Sudden Future singularity with only one exception: it also gives the divergence of the barotropic index in the barotropic equation of state

$$p(t) = w(t)\varrho(t)$$

i.e.,

$$w(t) \to \infty$$

Barotropic index $w-$singularity

Another exotic is a $w-$singularity only (without the divergence of the higher-derivatives of the scale factor). Choose

$$a(t) = A + B\left(\frac{t}{t_s}\right)^{\frac{2}{3\gamma}} + C\left(D - \frac{t}{t_s}\right)^{n} , \tag{14}$$

where A, B, C, D, γ, n, and t_s are constants and impose the conditions:

$$a(0) = 0,\ a(t_s) = const. \equiv a_s,\ \dot{a}(t_s) = 0,\ \ddot{a}(t_s) = 0 , \tag{15}$$

The strength of singularities - definitions.

They are different from BB/BC and can be characterized w.r.t. a blow-up of various physical quantities and property of geodesic completeness and compared using some mathematical tools.

- Tipler's (Phys. Lett. A64, 8 (1977)) definition (of a strong singularity):
  $I^i_j(\tau) = \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' |R^i_{ajb} u^a u^b|$
  diverges on the approach to a singularity at $\tau = \tau_s$
- i.e. an extended object is crushed to zero volume (represented by three linearly independent, vorticity-free geodesic deviation vectors at p parallely transported along causal geodesic l) at the singularity by infinite tidal forces
- Królak's (CQG 3, 267 (1988)) definition (of a strong singularity):
  $I^i_j(\tau) = \int_0^\tau d\tau' |R^i_{ajb} u^a u^b|$
  diverges on the approach to a singularity at $\tau = \tau_s$
- i.e. the expansion of every future-directed congruence of null (timelike) geodesics emanating from point p and containing l becomes negative somewhere on l

SFS as a "weak" singularity

Geodesics (point particles) do not feel SFS at all, since **geodesic equations are not singular** for $a_s = a(t_s) = \text{const}$. (Fernandez-Jambrina, Lazkoz PRD 74, 064030 (2006) ((gr-qc/0607073))

$$\left(\frac{dt}{d\tau}\right)^2 = A + \frac{P^2 + KL^2}{a^2(t)} , \tag{16}$$

$$\frac{dr}{d\tau} = \frac{P_1 cos\phi + P_2 \sin\phi}{a^2(t)}\sqrt{1 - Kr^2} , \tag{17}$$

$$\frac{d\phi}{d\tau} = \frac{L}{a^2(t) r^2} . \tag{18}$$

Geodesic deviation equation

$$\frac{D^2 n^\alpha}{d\lambda^2} + R^\alpha_{\ \beta\gamma\delta} u^\beta n^\gamma u^\delta = 0 , \tag{19}$$

feels SFS since at $t = t_s$ we have the Riemann tensor $R^\alpha_{\ \beta\gamma\delta} \to \infty$ (extended objects may suffer **instantaneous infinite tidal forces** but still may not be crushed)

Weak singularities do not exhibit geodesic incompleteness.

- No geodesic incompleteness ($a =$ const. and r.h.s. of geodesic eqs. do not diverge) $\Rightarrow$ SFS are not the final state of the universe. They are weak curvature singularities i.e. **in-falling observers or detectors are not destroyed by tidal forces**.
- Physically, it means that the tidal forces which manifest here as the (infinite) impulse which reverses (or stops) the increase of separation of geodesics and the geodesics themselves can evolve further – the universe can continue its evolution through a singularity. This behaviour is **like a turning point** of a harmonic oscillator.
- A bigger surprise follows if one considers an extended object such as a classical string (Balcerzak, MPD 2006) The point is that an invariant string size $S(\tau) = 2\pi a(\eta(\tau))R(\tau)$ (assuming circular ansatz with radius R) shows that they are: infinitely stretched $S \to \infty$ at a Big-Rip (a string is destroyed) while at SFS the scale factor is finite at η-time so that the invariant string size is also finite.

3. Classification of singularities.

- Type 0 - Big-Bang (Big-Crunch) $a \to 0$, $p \to \infty$, $\varrho \to \infty$
- Type I - Big-Rip $a(t_s) \to \infty$ $(t_s < \infty)$, $p \to \infty$, $\varrho \to \infty$ (Caldwell 2002)
- Type II - Sudden Future (includes Big Boost and Big-Brake) $a(t_s) = \text{const.}$, $\varrho = \text{const.}$, $p \to \infty$ (Barrow 2004)
- Type IIg - Generalized Sudden Future $a(t_s)$= const., $\varrho =$ const., $p =$const., $\dddot{a} \to \infty$ etc., $w < \infty$ (Barrow 2004)
- Type III - Finite Scale Factor (also Big-Freeze) $a(t_s)$ = const., $\varrho \to \infty$, $p \to \infty$ (Nojiri et al. 2005, Denkiewicz 2012)
- Type IV - Big Separation: $a(t_s)$= const., $p = \varrho = 0$, $w \to \infty$, $\dddot{a} \to \infty$ etc. (Nojiri et al. 2005) (and generalizations $p = \varrho =$const. Yurov 2010)
- Type V - w-singularity $a(t_s)$= const., $p = \varrho = 0$, $w \to \infty$ (MPD, Denkiewicz 2009) (and generalizations $p =$const. Yurov 2010)
- More subtleties: Little-Rip $a(t_s) \to \infty$, $\varrho(t_s) \to \infty$ $(t_s \to \infty)$ and Pseudo-Rip $\varrho(t_s) < \infty$ $(t_s \to \infty)$ (Frampton et al. 2011, 2012)

Spacetime averaging - new strength of singularities evaluator.

The idea is that one may **average physical and kinematical scalars χ** over the whole open spacetime (provided the scalars vanish rapidly at spatial and temporal infinity) as follows (Raychaudhuri, PRL 1998)

$$< \chi >= \lim_{x^a \to \infty} \frac{\int \int \int \int_{-x^a}^{x^a} \chi \sqrt{-g} d^4 x}{\int \int \int \int_{-x^a}^{x^a} \sqrt{-g} d^4 x} \quad , \tag{20}$$

where g is the determinant of the spacetime metric tensor. By an open model it is meant that the **ratio** of the 3-volume hypersurfaces to a 4-volume of spacetime vanishes, i.e.,

$$\frac{\int \int \int \sqrt{|^3 g \,|} d^3 x}{\int \int \int \int \sqrt{-g} d^4 x} = 0 \quad , \tag{21}$$

where $^3 g$ is the determinant of the spatial part of the metric tensor. The vanishing of the average $< \chi >$ **was related to singularity avoidance** in cosmology, but also may serve a **new evaluator** of the "strength" of singularities (MPD, 2011).

Spacetime averaging - example strengths.

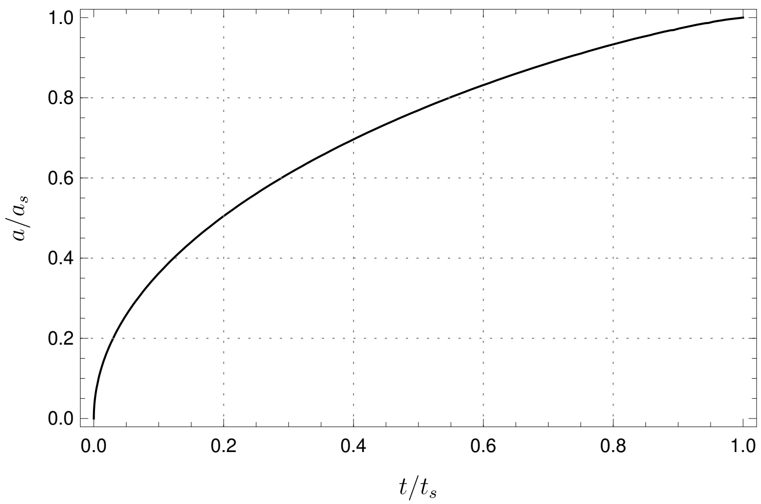
Type	S-T Average	Rank
BB/BC	0	mild
BR	∞	strong
SFS	0	mild
FSF	∞	strong
w	finite	weak

4. A new finite time Pseudo-Rip (FTRP) singularity.

Applied a new method of generating solutions for exotic singularities starting from the Hubble parameter. **Model 1 is decelerating SFS** solution (MPD, Vasilev 2026):

$$H[a(t)] \equiv \frac{\dot{a}}{a} = H_\star \frac{a_s^n}{a^{\frac{m}{2}}(t)} \left(1 - \frac{a(t)}{a_s}\right)^n , \tag{22}$$

where $H_\star$ and a_s are positive constants. Here m corresponds to a standard fluid ($m \geq 0$) to avoid phantom and n is responsible for the exotic behaviour. SFS is obtained for $0 < n < \frac{1}{2}$.



Accelerating FTPR singularity

Model 2 is accelerating, but not SFS. It is **a FINITE TIME Pseudo-Rip** solution:

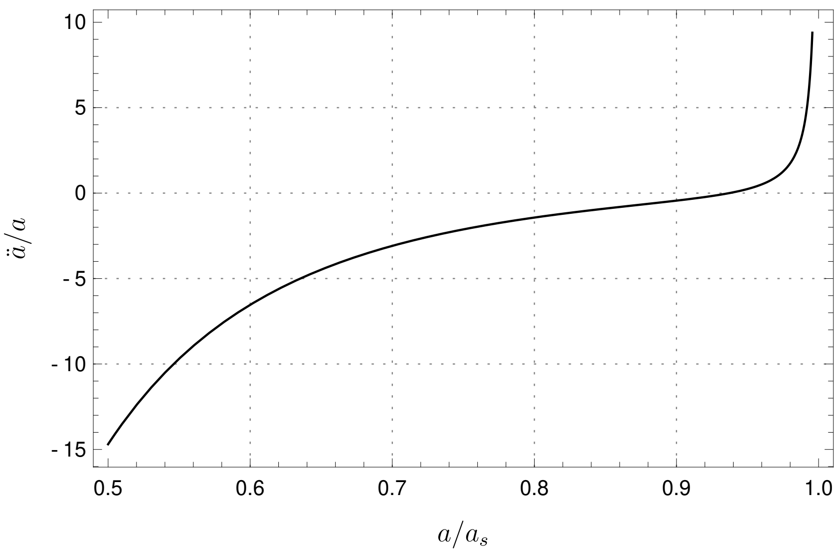
$$H[a(t)] \equiv \frac{\dot{a}}{a} = \frac{H_1}{a^{\frac{m}{2}}} - H_2 a_s^n \left(1 - \frac{a}{a_s}\right)^n, \tag{23}$$

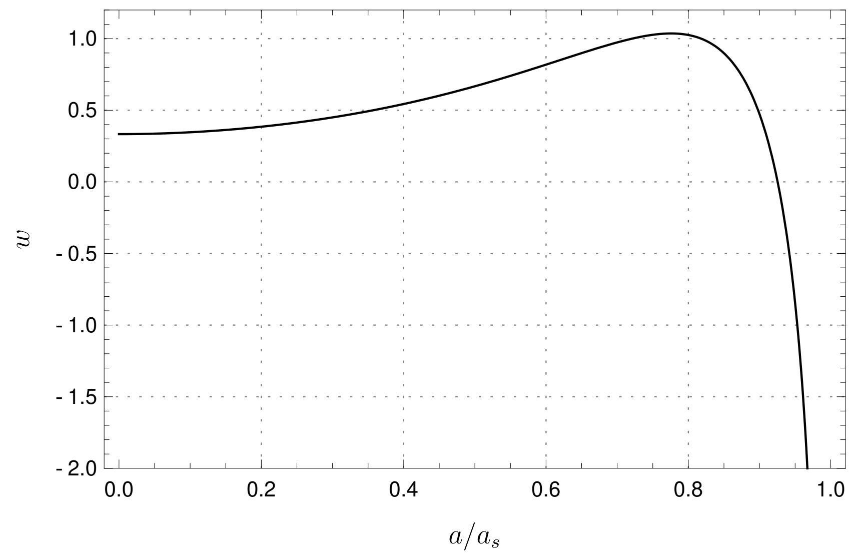
$$\begin{aligned}
\frac{\ddot{a}}{a} &= \left(1 - \frac{m}{2}\right) H_1^2 a^{-m} + n H_1 H_2 a_s^{n-1} a^{1-\frac{m}{2}} \left(1 - \frac{a}{a_s}\right)^{n-1} \\
&+ H_2^2 a_s^{2n} \left(1 - \frac{a}{a_s}\right)^{2n} - n H_2^2 a a_s^{2n-1} \left(1 - \frac{a}{a_s}\right)^{2n-1} \\
&- H_1 H_2 a_s^n a^{-\frac{m}{2}} \left(2 - \frac{m}{2}\right) \left(1 - \frac{a}{a_s}\right)^n .
\end{aligned} \tag{24}$$

Dominating in (24) second term always accelerates and, eventually, blows up at $a = a_s$ for $0 < n < 1$.

An accelerating FTPR.

Contrary to the usual decelerating SFS behaviour discussed before, this signals the presence of an extreme acceleration event at finite time t_s for $n \in (0, 1)$ with $\ddot{a} \to \infty$ and $w \to -\infty$.





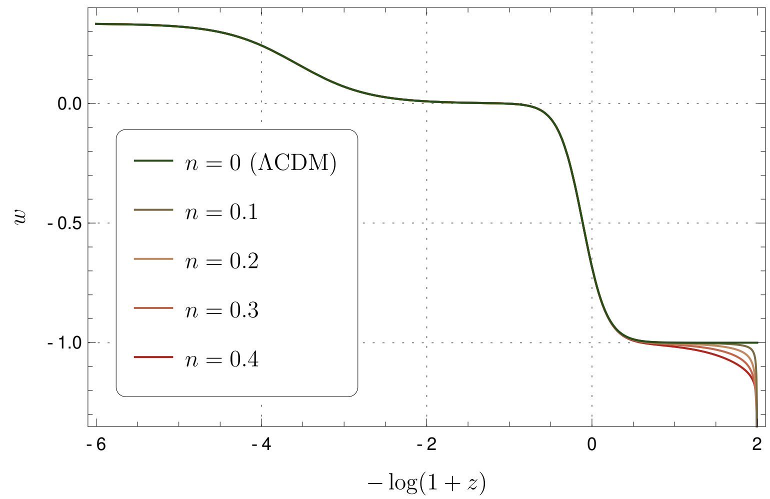
An FTPR model mimicking ΛCDM.

Taking into account the structure of the FTPR model 2 as in (23), we can build the model following ΛCDM-like expansion profile

$$H^2(a) = \mu_1 a^{-4} + \mu_2 a^{-3} + \mu_3 \left[1 - n\left(1 - \frac{a}{a_s}\right)^{2n}\right], \qquad (25)$$

where μ_i are positive constants.

As seen from the plot below, this model **only differs** from ΛCDM while on the approach to an FTPR singularity in future:



Properties of cosmological singularities (including FTPR).

t_s is the time when a singularity appears, $w = p/\rho$ is the barotropic index, T - Tipler's definition, K - Królak definition.

Type	Name	t_s	a(t$_s$)	$\varrho(t_s)$	p(t$_s$)	$\dot{p}(t_s)$ etc.	w(t$_s$)	T	K
0	Big-Bang (BB)	0	0	∞	∞	∞	finite	strong	strong
I	Big-Rip (BR)	t_s	∞	∞	∞	∞	finite	strong	strong
I$_l$	Little-Rip (LR)	∞	∞	∞	∞	∞	finite	strong	strong
I$_p$	Pseudo-Rip (PR)	∞	∞	finite	finite	finite	finite	weak	weak
I$_f$	**Future Time Pseudo-Rip (FTPR)**	t_s	a_s	ϱ_s	∞	∞	∞	**weak**	**weak**
II	Sudden Future (SFS)	t_s	a_s	ϱ_s	∞	∞	finite	weak	weak
II$_g$	Gen. Sudden Future (GSFS)	t_s	a_s	ϱ_s	p_s	∞	finite	weak	weak
III	Finite Scale Factor (FSF)	t_s	a_s	∞	∞	∞	finite	weak	strong
IV	Big-Separation (BS)	t_s	a_s	0	0	∞	∞	weak	weak
V	w-singularity (w)	t_s	a_s	0	0	0	∞	weak	weak

5. Multiverse quantum entanglement and exotic singularities.

- Assume that **the universes 1 and 2 are quantum mechanically entangled** and there are **critical points** of their evolution when the entanglement matters (e.g. maximum expansion/inflection points, singularities) and influences the behaviour of individual universes, though **most of the evolution of the individual universes is classical** (Balcerzak, Barroso-Bellido, MPD, Robles-Perez, 2021).
- These universes are classically disconnected, but quantum mechanically entangled (Robles-Perez et al. 2010, 2014, 2015) and the effect of entanglement can be **imprinted in individual universes** (e.g. Mersing-Houghton et al. 2009, 2017 (CMB temperature); Kinney 2016 (power spectrum), S. Barroso-Bellido & MPD, 2022 (CMB)).
- We will focus briefly on **the entanglement** issues taking the entanglement entropy (and/or temperature) **as a measure of the quantumness** of the critical points of the evolution - including exotic singularities.

Third quantization - 3Q

- QFT formalism in which WdW wave function ϕ becomes an operator.
- This is due to fact that WdW equation can be obtained from the Hamilton eqs of the (third-quantized) Hamiltonian

$$\mathrm{H} = \frac{1}{2} P_\phi^2 + \frac{\omega^2(a)}{2} \phi^2, \tag{26}$$

where, $P_\phi \equiv \dot{\phi}$.

- 3Q makes ϕ and P_ϕ operators.
- We have

$$\hat{\phi}(a) = \frac{1}{\sqrt{2\omega}} e^{iS(a)} \hat{b}_+ + \frac{1}{\sqrt{2\omega}} e^{-iS(a)} \hat{b}_-^\dagger, \tag{27}$$

where, $\hat{b}_+ \equiv \hat{b}_+(a_{\text{min}})$ and $\hat{b}_-^\dagger \equiv \hat{b}_-^\dagger(a_{\text{min}})$, are constant operators given at some initial value, $a = a_{\text{min}}$.

Invariant representation of vacuum

- Vacuum state for $(b_\pm, b_\pm^\dagger)$ representation – given by $|0_+, 0_-\rangle$.
- Not stable because of scale factor dependence of $\omega = \omega(a)$ (along minisuperspace geodesic).
- An invariant representation for the harmonic oscillator like WdW equation is (Lewis and Riesenfeld JMP, 1458 (1969), Robles-Perez and Gonzalez-Diaz 2010, 2014)

$$c_+ = \sqrt{\frac{1}{2}}\left(\frac{1}{R}\phi + i(RP_\phi - \dot{R}\phi)\right), \tag{28}$$

$$c_-^\dagger = \sqrt{\frac{1}{2}}\left(\frac{1}{R}\phi - i(RP_\phi - \dot{R}\phi)\right), \tag{29}$$

where $R = \sqrt{\phi_1^2 + \phi_2^2}$, with ϕ_1 and ϕ_2 being two real solutions of WdW equation satisfying

$$\phi_1\dot{\phi}_2 - \dot{\phi}_1\phi_2 = 1. \tag{30}$$

Invariant representation of vacuum

In terms of the invariant representation (28)–(29), the Hamiltonian (26) reads

$$H = H_0^- + H_0^+ + H_I, \tag{31}$$

where

$$H_0^{\pm} = \Omega(a)\left(c_{\pm}^{\dagger} c_{\pm} + \frac{1}{2}\right), \tag{32}$$

and,

$$H_I = \gamma(a) c_+^{\dagger} c_-^{\dagger} + \gamma^* c_+ c_-, \tag{33}$$

is the **Hamiltonian of interaction** (describing a non-local interaction by an entangled pair of the universes) of the universes while

$$\Omega(a) = \frac{1}{4}\left(\frac{1}{R^2} + R^2\omega^2 + \dot{R}^2\right), \tag{34}$$

$$\gamma(a) = -\frac{1}{4}\left\{\left(\dot{R} + \frac{i}{R}\right)^2 + \omega^2 R^2\right\}. \tag{35}$$

Entanglement thermodynamics - general framework

- The multiverse is in the vacuum state described by the ground state of the invariant representation in the minisuperspace, $|0_+0_-\rangle_c$ - a pure state with zero entropy - evolves unitary so that the entropy is constantly zero.
- However, there is non-zero entropy of entanglement in each single universe (for an internal observer) which eventually can give rise to an arrow of time.
- The state of the universe for an internal observer in terms of diagonal representation reads (Robles-Perez and Gonzalaez-Diaz 2014)

$$|0_+0_-\rangle_c = \frac{1}{|\alpha|}\sum_{n=0}^{\infty}\left(\frac{|\beta|}{|\alpha|}\right)^n |n_-, n_+\rangle_b, \tag{36}$$

where $|n_-, n_+\rangle_b$ are the entangled mode states of the diagonal representation, and α and β are the Bogoliubov coefficients that relate both representations ($|\alpha|^2 - |\beta|^2 = 1$)

$$\hat{c}_- = \alpha\hat{b}_- - \beta\hat{b}_+^\dagger, \quad \hat{c}_-^\dagger = \alpha^*\hat{b}_-^\dagger - \beta^*\hat{b}_+. \tag{37}$$

Entanglement thermodynamics

The quantum state of a single universe of the entangled pair can be obtained by tracing out the degrees of freedom of the partner universe

$$\rho_- = \mathrm{Tr}_+ \rho \equiv \sum_{n=0}^{\infty} {}_b\langle n_+|\rho|n_+\rangle_b, \tag{38}$$

where the density matrix is

$$\rho = |0_+ 0_-\rangle_c \langle 0_+ 0_-| \quad = \frac{1}{|\alpha|^2} \sum_{n,m} \left(\frac{|\beta|}{|\alpha|}\right)^{n+m} |n_-, n_+\rangle_b \langle m_-, m_+|. \tag{39}$$

As a result we get a thermal state (Robles-Perez and Gonzalez-Diaz 2014)

$$\rho_- = \frac{1}{|\alpha|^2} \sum_{n,m,l} \left(\frac{|\beta|}{|\alpha|}\right)^{n+m} \langle l_+|m_+\rangle |n_-\rangle_b \langle n_-| \langle m_+|l_+\rangle = \frac{1}{Z} \sum_n e^{-\frac{\omega}{T}(n+\frac{1}{2})} |n_-\rangle_b \langle n_-|, \tag{40}$$

where, $Z^{-1} = 2\sinh\frac{\omega}{2T}$.

Entanglement thermodynamics

Now we have the temperature of entanglement

$$T \equiv T(a) = \frac{\omega(a)}{2 \ln \coth r}, \quad \tanh r \equiv \frac{|\beta|}{|\alpha|}, \tag{41}$$

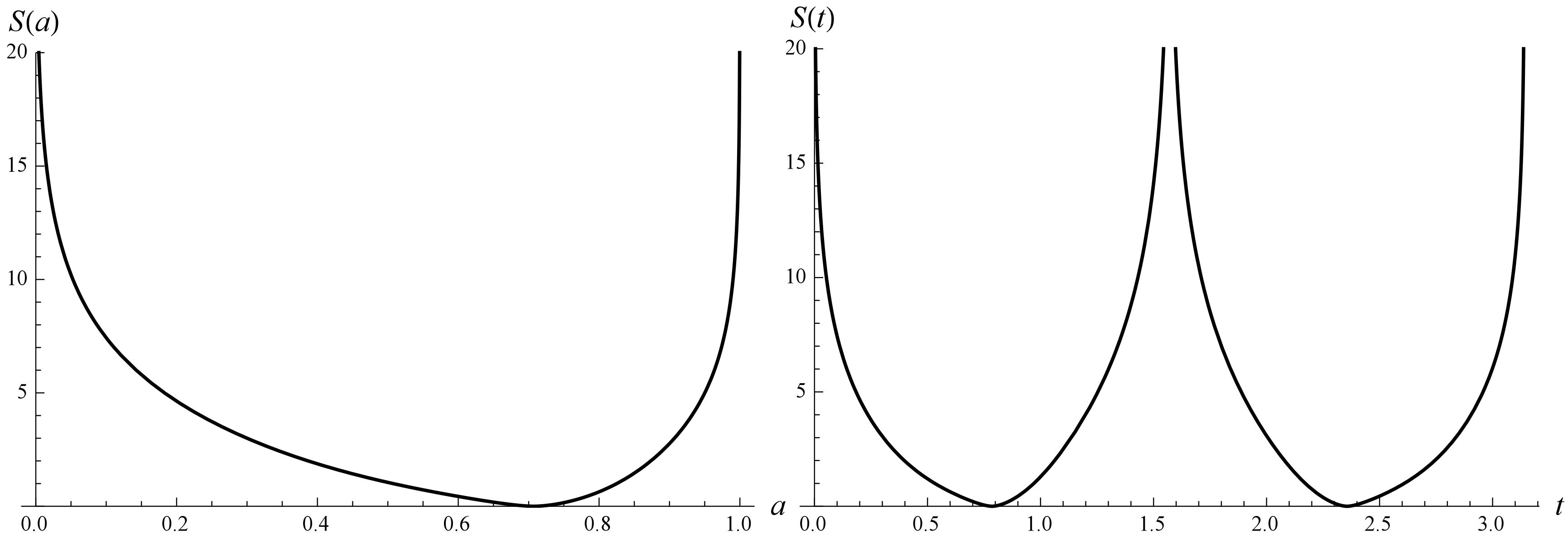
where r – the entanglement parameter (Nakagawa 2016, Baskal 2016).
The **entropy of entanglement** is given by the von Neumann formula (Horodecki et al. 2009, Nakagawa 2016, Baskal 2016)

$$S(\rho) = -\mathrm{Tr}\left(\rho \ln \rho\right), \tag{42}$$

applied to the thermal state ρ_-, and yields (Robles-Perez and Gonzlez-Diaz 2014)

$$S_{\text{ent}}(a) = \cosh^2 r \; \ln \cosh^2 r - \sinh^2 r \; \ln \sinh^2 r. \tag{43}$$

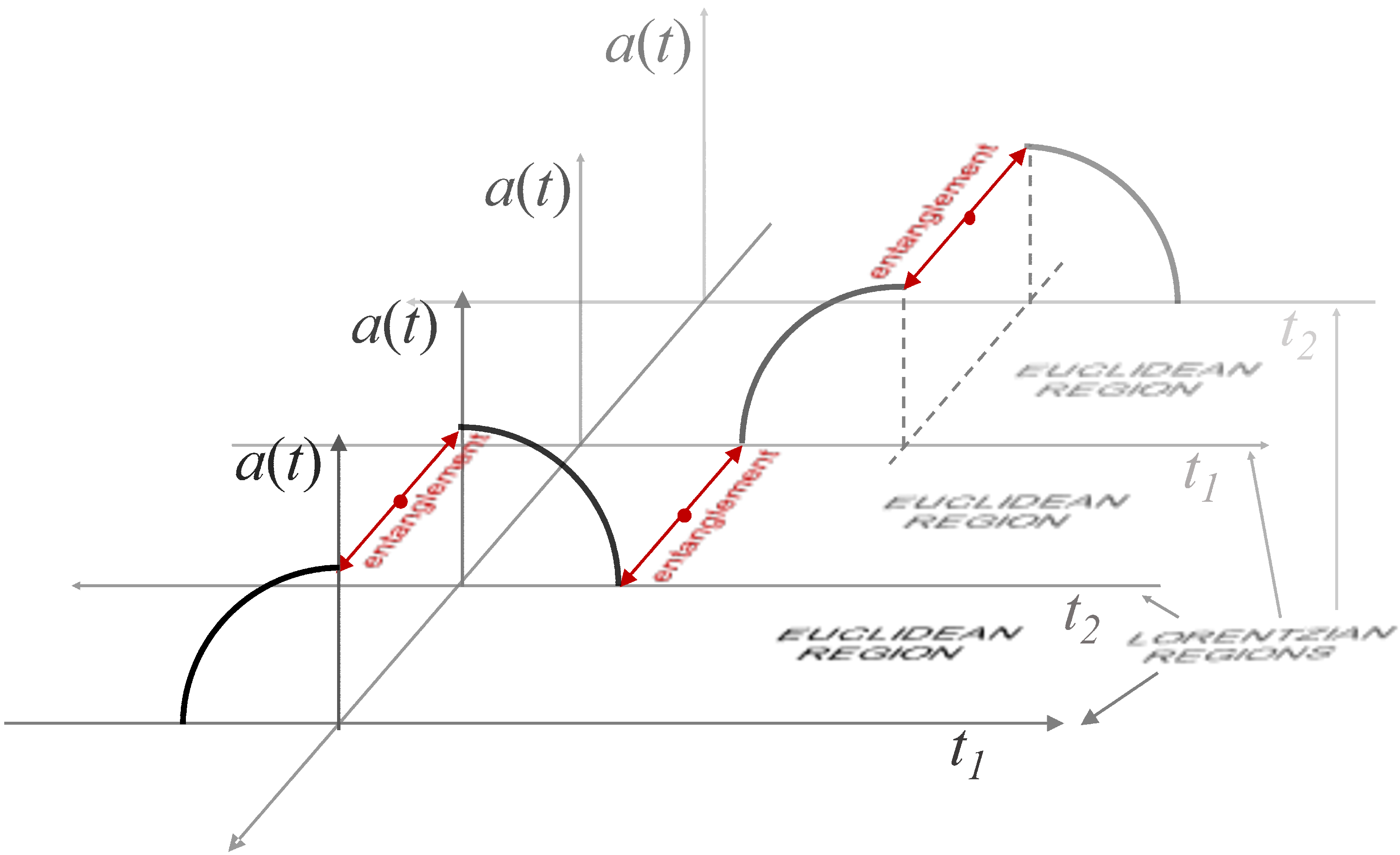
Entropy of entanglement - BB/BC/Maximum expansion



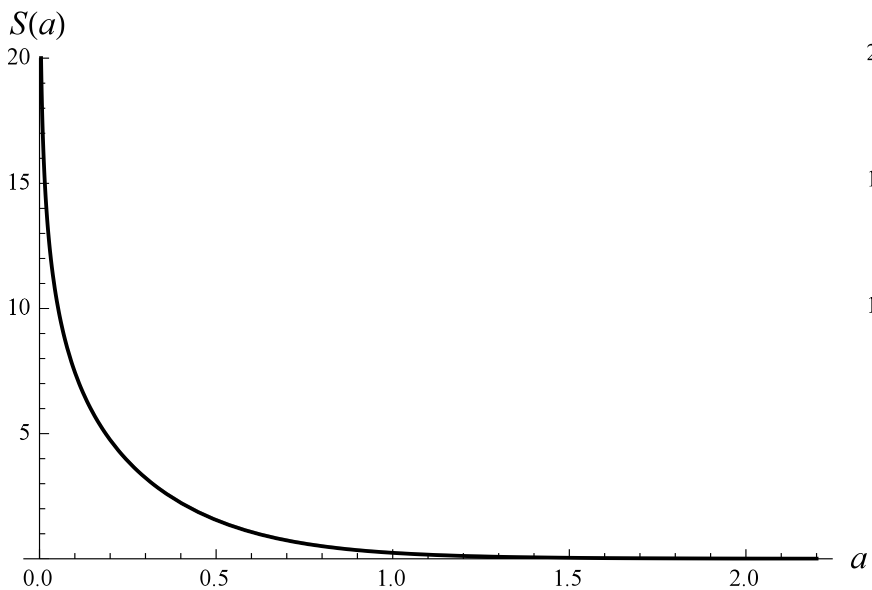
Result: Entropy is **large** (infinite) for big-bang ($a = 0$), big-crunch ($a = 2$ - not plotted) and also **maximum expansion** ($a = 1$) regions!

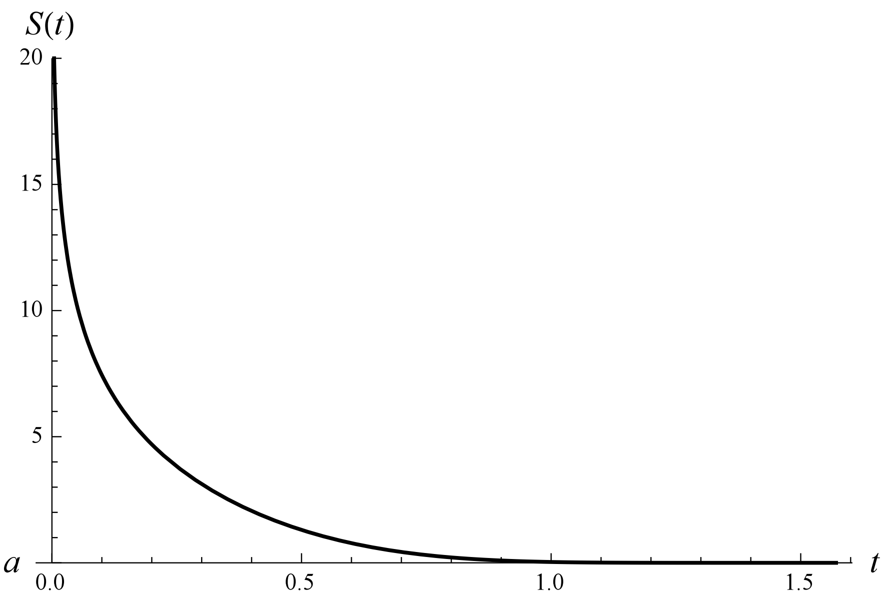
Interestingly, entropy of entanglement is **small and finite at the inflection point** of non-singular simple harmonic oscillator evolution due to negative Λ and domain walls (Balcerzak et al. 2021)

Creation of entangled pairs at BB/BC/maximum



Entropy of entanglement - Big-Rip.





- Result: Entropy is **large** (infinite) for big-bang ($a = 0$) region, but **vanishes** for big-rip ($a = \infty$) region!
- Temperature of entanglement is, however, infinite, and can be taken as the measure of **quantumness**.

Quantumness of critical points - summary

Type	Ent. entropy	Ent. temp.
maximum expansion	∞	∞
inflection point	finite	finite
BB/BC	∞	∞
BR	0	∞
SFS	∞	finite
FSF	∞	∞
Big-Separation	∞	finite

6. Remarks

- Exotic singularities **differ from ”standard” Big-Bang/Big-Crunch singularities** in their properties related to geodesic incompleteness, energy density and pressure (curvature) behaviour and the timescale of their appearance. They can be ”stronger’ or ”weaker” according to some measures (Tipler criterion, Królak criterion, Raychaudhuri criterion etc.)
- Exotic singularities **are related** to new physical sources of gravity which can **serve as the dark energy**.
- Despite the set of exotic singularities has been **explored strongly** (both theoretical and observational), some **new cases appea**r: a Finite Time Pseudo-Rip (FTPR) is an example.
- They do not only allow a number of subtleties on a classical level, but also on quantum - **strong quantum entanglement properties at their critical points** (maximum expansion, BB/BC, BR, SFS etc.).