Universality and Criticality in a Massless Scalar Field Collapse

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Outline

Motivation

Methodology

Main results

Takeaway

Motivation

Cosmic Censorship Conjecture

- Does a singularity necessarily entail an event horizon?
- Which symmetries are necessary?
- Which kind of matter fields?

Scalar fields and cosmic censorship

Massless scalar fields (MSF): naked singularities[a].

Massive scalar fields: naked singularities[b].

Criticality in MSF collapse: Applications

- Primordial black hole seeds.
- Dark matter particles.

[a] D. Christodoulou, Communications in Mathematical Physics **93**, 171–195 ([1984]).

[b] R. Goswami and P. S. Joshi, gr-qc/0410144 **arXiv** ([2004]), K. Mosani et al., Phys. Rev. D **108**, 044049 ([2023]).

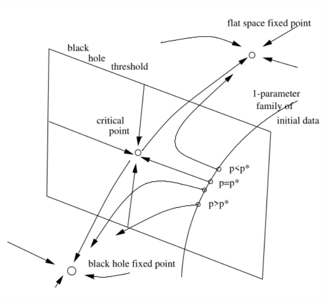
Criticality in MSF collapse

Choptuik[a]

- Black-hole formed mass:

  $$M \sim (p - p_*)^\gamma,$$

- $\gamma \approx 0.37$: universal across families.
- p_*: attractor in parameter space.
- Self-similarity: *echoing*.



Phase-space of critical parameter[a]

[a] M. W. Choptuik, Phys. Rev. Lett. **70**, 9–12 ([1993]).

[a] C. Gundlach, Phys. Rept. **376**, 339–405 ([2003]).

Criticality in MSF collapse

Past works

- Criticality and self-similarity[a].
- Criticality for specific scalar fields[b].
- Criticality in massive scalar field collapse[c].

Limitations

- MSF families: ansatz dependent.
- Formulation: coordinate dependent.
- Approach: Numerical.

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[a] P. R. Brady, Class. Quant. Grav. **11**, 1255–1260 ([1994]).

[b] S. Bhattacharya and P. S. Joshi, 1009.3897 **arXiv** ([2010]).

[c] C. Gundlach, Phys. Rept. **376**, 339–405 ([2003]).

Questions

- MSFs: Type-I or Type-II matter fields?
- LRS-II decomposition: Suffice to capture the problem?
- Ansatz independence: formulation be based on gradient of the MSF?

Motivation

Methodology

Main results

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Symmetries and local decomposition

Symmetries

- Time-orientability.
- Spherical symmetry.

Resulting reductions

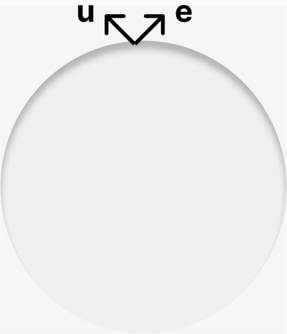
- $g_{ab} = -u_a u_b + e_a e_b + N_{ab}$
- $T_{ab} = \mu u_a u_b + (p + \Pi) e_a e_b + 2Q u_{(a} e_{b)} + (p - \frac{1}{2}\Pi) N_{ab}$.

Detailed expressions

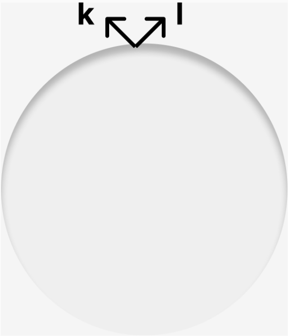
Divergence of the scalar field:

- In the $[u, e]$ basis, $\nabla_a \Psi = f u_a + g e_a$.
- In the $[k, l]$ basis, $\nabla_a \Psi = \Psi_+ k_a + \Psi_- l_a$.

Non-exclusive bases



$\Longleftrightarrow$



Equations for motion

Along radial and temporal unit vectors

$$
\begin{aligned}
\dot{f} + \hat{g} + f\Theta + g(\mathcal{A} + \phi) &= 0, \\
\dot{g} + \hat{f} + \left(\frac{\Theta}{3} + \Sigma\right) g + \mathcal{A} f &= 0.
\end{aligned}
$$

Along double null unit vectors

$$
\begin{aligned}
\dot{\Psi}_+ + \hat{\Psi}_+ + \Psi_+(\Theta + \mathcal{A}) + \frac{\Psi_+ - \Psi_-}{2}(\phi + \Sigma - \frac{2}{3}\Theta) &= 0\,, \\
\dot{\Psi}_- - \hat{\Psi}_- + \Psi_-(\Theta + \mathcal{A}) + \frac{\Psi_+ - \Psi_-}{2}(\phi - \Sigma + \frac{2}{3}\Theta) &= 0\,.
\end{aligned}
$$

$$
\mathcal{D} \equiv (\Theta, \mathcal{A}, \Sigma, \mathcal{E}, \mu, \phi, p, \Pi, \mathcal{Q})\ .
$$

Matter-field type

Thermodynamic observables

$$
\begin{aligned}
\mu &= \frac{1}{2}f^2 + \frac{1}{2}g^2\,, \\
p &= \frac{1}{2}f^2 - \frac{1}{6}g^2\,, \\
\mathcal{Q} &= fg\,, \\
\Pi &= \frac{2}{3}g^2\,.
\end{aligned}
$$

Scalar field $\Rightarrow$ type-I matter field. ▸ Show proof

Choice of basis

Type-I matter field

- $g = 0 \;\Rightarrow\; \nabla_a \Psi$: purely time-like.
- $f = 0 \;\Rightarrow\; \nabla_a \Psi$: purely space-like.

- Type-II matter fields $\Rightarrow$ either $\Psi_+ = 0$ or $\Psi_- = 0$.
- Heat flux $\mathcal{Q} = 0$.

Coordinate free

- $\nabla_a \Psi$: purely time-like $\Rightarrow \nabla_a \Psi \parallel u_a$.
- $\nabla_a \Psi$: purely space-like $\Rightarrow \nabla_a \Psi \perp u_a$.

Description of the system

Collapsing scenario

- Regularity condition $\Rightarrow$ Frobenius's criterion.

  ▸ Elaborated expression

- Focusing $\Rightarrow \Theta \to -\infty$.
- Genuine space-time singularity $\Rightarrow \mu \to \infty$ and $p \to \pm\infty$.

Description of the system

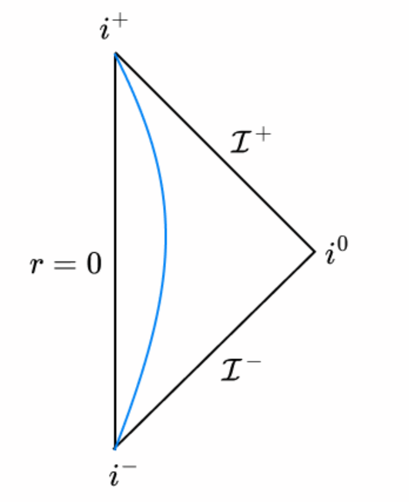
Visibility

- Locate apparent horizon $\Rightarrow$ Misner-Sharp mass.
  ▸ Show Detailed Proof
- Slope of the apparent horizon: $\mathcal{X} = \frac{1-\eta^2}{1+\eta^2}$. ▸ Show Detailed Proof
- Critical parameter: $\eta = 2\mathcal{M}\Psi_-$.

End-state singularity

$$\eta_0 \ = \ 2\mathcal{M}_0\Psi_{-0} \qquad \mathcal{M} \ \to \ \mathcal{M}_0 \qquad \Psi_- \ \to \ \Psi_{-0}$$

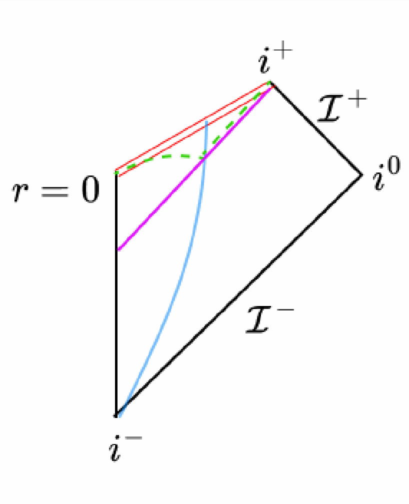
End-state: Dispersion



$\Psi_{-0} \to$ finite

- Energy densities $\Rightarrow$ finite.
- Space-time $\Rightarrow$ flat.

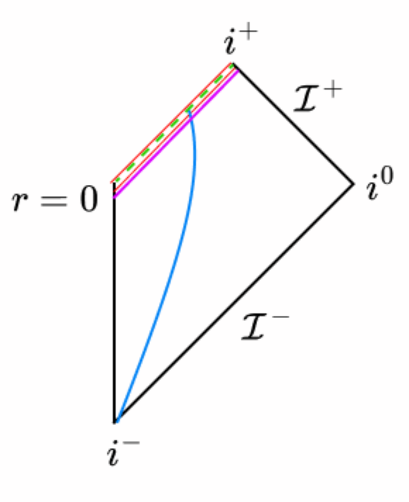
End-state: Zero mass black-hole



$0 < \eta_0 \leq 1$

- $\mathcal{M}_0 \to$ *zero*.
- $\Psi_{-0} \to$ infinite $\Rightarrow$ Energy densities diverge.
- $1 > \mathcal{X}_0 \geq 0$.
- AH $\Rightarrow$ outgoing space-like.
- Singularity $\Rightarrow$ space-like.

End-state: Zero mass locally naked singularity

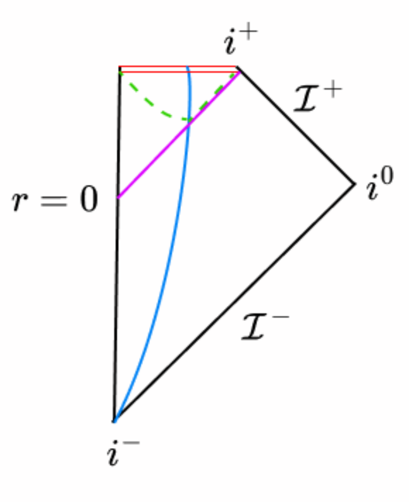


$\eta_0 \to 0$

- $\mathcal{M}_0 \to zero.$
- $\Psi_{-0} \to$ infinite $\Rightarrow$ Energy densities diverge.
- $\mathcal{X}_0 \to 1.$
- AH $\Rightarrow$ outgoing null.
- Singularity $\Rightarrow$ outgoing null.
- Singularity $\Rightarrow$ at least locally naked[a].

[a]Koushiki and P. S. Joshi, 2508.14663 **arXiv** ([2025]).

End-state: Non-zero mass black-hole



$\eta_0 > 1.$

- $\mathcal{M}_0 \rightarrow non - zero.$
- $\Psi_{-0} \rightarrow$ infinite $\Rightarrow$ Energy densities diverge.
- $\mathcal{X}_0 < 1.$
- AH $\Rightarrow$ incoming space-like.
- Singularity $\Rightarrow$ future space-like.

To summarise...

Novelties

- Completely analytic formulation.
- Co-ordinate independent.
- Scalar field families are not fine-tuned.

Key findings[1]

- $\eta_0 \Rightarrow$ end-state singularity and visibility:
  - Dispersion,
  - Zero-mass black-hole,
  - Non-zero mass black-hole and
  - Zero-mass null locally naked singularity.
- $\eta_0 \Rightarrow$ dimension-less: self-similarity?[a]
- Scalar field families $\Rightarrow$ universal.

[1] Koushiki et al., 2512.20998 **arXiv** ([2025]).

[a] P. R. Brady, Class. Quant. Grav. **11**, 1255–1260 ([1994]).

Moving forward...

- Symmetric structure near the singularity.
- Stability of the end-state singularity.
- Phase transition and criticality.

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Thank you!

Decomposition

The volume element on the $[u,e]$ plane:

$$\varepsilon_{ab} \equiv \epsilon_{abc}e^c = u^d\eta_{dabc}e^c.$$

A vector is decomposed as:

$$v_a = Ve_a + V_a, \quad \text{where} \quad V \equiv V^a e_a \ , \quad \text{and} \quad V_a \equiv N_{ab}V^b \equiv v_{\bar{a}}.$$

A tensor can also be split into its irreducible parts similarly:

$$\rho_{ab} = \rho_{\langle ab\rangle} = Pe_ae_b - 12N_{ab} + 2P_{(a}e_{b)} + P_{ab} \ ,$$

where

$$\begin{aligned} P &\equiv e^ae^b\rho_{ab} = -N^{ab}\rho_{ab} \ , \\ P_a &\equiv N_a{}^be^c\rho_{bc} = P_{\bar{a}} \ , \\ P_{ab} &\equiv \left(N_{(a}{}^cN_{b)}{}^d - \frac{1}{2}N_{ab}N^{cd}\right)\rho_{cd} \equiv P_{\{ab\}} \ . \end{aligned}$$

Decomposition

The covariant derivatives along u^a and e^a are:

$$\dot{\rho}^{c..d}{}_{a..b} = u^k \nabla_k \rho^{c..d}{}_{a..b} \text{ and } \hat{\rho}^{c..d}{}_{a..b} = e^k D_k \rho^{c..d}{}_{a..b} \ .$$

The derivative along the orthogonal surface is:

$$\delta_l \rho^{c..d}{}_{a..b} = N^f{}_a N^c{}_p ... N^g{}_b N^d{}_q N^r{}_l \nabla_r \rho^{p..q}{}_{f..g} \ ,$$

the full covariant derivative of e_a and u_a is now written as:

$$\begin{aligned}
\nabla_a e_b &= -\mathcal{A} u_a u_b + \left(\Sigma + \frac{1}{3}\Theta\right) e_a u_b + \frac{1}{2}\phi N_{ab}, \\
\nabla_a u_b &= -u_a e_b \mathcal{A} + e_a e_b \left(\frac{1}{3}\Theta + \Sigma\right) + \frac{1}{2} N_{ab} \left(\frac{2}{3}\Theta - \Sigma\right), \\
\text{with } \mathcal{A} &= e_a \dot{u}^a \ , \ \Sigma = e_a e_b \sigma^{ab} \ , \Theta = D_a u^a \ , \ \phi = \delta_a e^a \ .
\end{aligned}$$

The electric and magnetic parts of the Weyl tensor:

$$\begin{aligned}
E_{ab} &= \mathcal{E}\left(e_a e_b - \frac{1}{2} N_{ab}\right) + 2\mathcal{E}_{(a} e_{b)} + \mathcal{E}_{ab}, \\
H_{ab} &= \mathcal{H}\left(e_a e_b - \frac{1}{2} N_{ab}\right) + 2\mathcal{H}_{(a} e_{b)} + \mathcal{H}_{ab}.
\end{aligned}$$

Back to Previous Slide

Eigen equations

The eigen-equation for the stress-energy tensor:

$$T_{ab}v^b = \lambda v_a.$$

If $v \in [u, e]$ plane, then:

$$\begin{aligned} T_{ab}u^a &= -\mu u_b - \mathcal{Q} e_b \ , \\ T_{ab}e^a &= (p + \Pi) e_b + \mathcal{Q} u_b \ . \end{aligned}$$

$\mathcal{Q} = 0 \Rightarrow f = 0$ or $g = 0 \Rightarrow$ type-I matter field $\Rightarrow \Psi_+ = \pm\Psi_-$.
Type-II scalar field $\Rightarrow$ either $\Psi_+ = 0$ or $\Psi_+ = 0$.

Back to Previous Slide

Frobenius' theorem

Involute submanifold is identified by:

$$\dot{\hat{\mathrm{Y}}} - \hat{\dot{\mathrm{Y}}} = \left(\frac{\Theta}{3} + \Sigma\right)\hat{\mathrm{Y}} - \mathcal{A}\dot{\mathrm{Y}}\,,$$

- f and g have to abide by this.
- $[u, e]$ plane is involute.
- $[k, l]$ plane is involute.
- The system is regular other than the singular epoch.

Back to Previous Slide

Misner-Sharp mass and AH

Misner-Sharp mass:

$$\mathcal{M} = \frac{\mathcal{R}}{2}(1 - \nabla_a \mathcal{R} \nabla^a \mathcal{R}),$$

Gaussian curvature:

$$K = \frac{1}{\mathcal{R}^2} = \frac{1}{6}(f^2 - g^2) - \mathcal{E} + \frac{1}{4}\phi^2 - \frac{1}{9}\left(\Theta - \frac{3}{2}\Sigma\right)^2.$$

Misner-Sharp mass in terms of Gaussian curvature and the thermodynamic scalars:

$$\mathcal{M} = \frac{1}{2K^{\frac{3}{2}}}\left(\frac{1}{6}(f^2 - g^2) - \mathcal{E}\right).$$

With $\hat{\mathcal{R}} > 0$, the definition of AH becomes:

$$\tilde{h}^{ab}\nabla_a k_b = 0 \implies \varphi \equiv \frac{2}{3}\Theta + \phi - \Sigma = 0.$$

And on AH, the Misner-Sharp mass is given by:

$$\mathcal{M} = \frac{1}{2\sqrt{K}}.$$

Back to Previous Slide

Slope of the AH

Tangent to the AH:

$$\nabla_a \varphi = -\dot{\varphi} u_a + \hat{\varphi} e_a$$

The slope:

$$\mathcal{X} = \frac{\hat{\varphi}}{\dot{\varphi}} = \frac{\frac{1}{3}(f^2 + 2g^2) + \mathcal{E} - fg}{-\frac{1}{3}(2f^2 + g^2) + \mathcal{E} + fg} = \frac{1 - \dfrac{\Psi_-^2}{2\mathcal{M}K^{\frac{3}{2}}}}{1 + \dfrac{\Psi_-^2}{2\mathcal{M}K^{\frac{3}{2}}}}.$$

Two cases arise:

$$\frac{\hat{\varphi}}{\dot{\varphi}} > 0 \quad \Longrightarrow \quad \text{AH is future outgoing}$$

$$\frac{\hat{\varphi}}{\dot{\varphi}} < 0 \quad \Longrightarrow \quad \text{AH is future incoming.}$$

Equation of slope becomes:

$$\mathcal{X} = \frac{1 - \eta^2}{1 + \eta^2}, \text{ where, } \eta = 2\mathcal{M}\Psi_-.$$

Back to Previous Slide