New Einstein–Maxwell gravitational instantons

Maciej Dunajski

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- Joint work on Einstein–Maxwell instantons with Bernardo Araneda (PRL 2025, in progress 2026).
- . . . with Sean Hartnoll (CQG 2007), with Jan Gutowski,Wafic Sabra & Paul Tod (CQG 2011, JHEP 2011) on SUSY EM instantons.
- Links to projective geometry with Thomas Mettler (JGA 2018).
- . . . with Paul Tod (CQG 2024) on twistor theory for Chen–Teo.
- . . . in progress with Lionel Mason and Paul Tod (2026) on complete twistor theory for ALE and ALF toric instantons.

Gravitational Instantons

Gravitational Instantons (Hawking 1977): solutions to the four-dimensional Einstein or Einstein–Maxwell equations in Riemannian signature which give complete metrics and asymptotically 'look-like' flat space.

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- Einstein–Maxwell theory $(M, g, A \in \Lambda^1(M))$

$$F = dA, \quad d \star_g F = 0, \quad \text{Ricci} = 2F_+ \circ F_-$$

$$\text{where} \quad F_\pm \equiv \frac{1}{2}(F \pm \star_g F), \quad (G \circ H)_{ab} \equiv G_{ac} H_b{}^c.$$

Euclidean Reissner–Nordström Instanton

- Reissner–Nordström (RN) metric (Reissner 1916)

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\begin{aligned}
g &= f^{-1}dr^2 - f dt^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad A = \frac{Q}{r}dt \\
f &= 1 - \frac{2m}{r} + \frac{Q^2}{r^2} \\
&= \frac{1}{r^2}(r - r_+)(r - r_-), \quad r_\pm = m \pm \sqrt{m^2 - Q^2}, \quad |Q| \le m.
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- RN instanton: $t = i\tau, F \to \star F, r_+ < r < \infty$. Near $r = r_+$

$$\begin{aligned} g &\approx d\rho^2 + {\kappa_+}^2\rho^2 d\tau^2 + {r_+}^2(d\theta^2 + \sin^2\theta d\phi^2), \quad \text{where} \\ \kappa_+ &= \frac{1}{2}f'(r_+), \quad \rho^2 = 2\frac{r-r_+}{\kappa_+}, \quad \tau \sim \tau + \frac{2\pi}{\kappa_+}. \end{aligned}$$

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- Kerr–Newman black holes $\to$ Euclidean Kerr–Newman instantons.

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- $Q = 0$. Schwarzschild black hole. $\tau \sim \tau + 8\pi m$.

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  - τ not periodic. Set $\rho = m^2/(r-m)$. $\mathbb{H}^2 \times S^2$ near $r = m$.

$$g \sim m^2 \Big[\frac{(r-m)^2}{m^4} d\tau^2 + \frac{1}{(r-m)^2} dr^2 + (d\theta^2 + \sin^2\theta d\phi^2) \Big].$$

- A complete regular four-dimensional Riemannian manifold (M, g) which solves the $\Lambda = 0$ Einstein equations is called ALF (asymptotically locally flat) if it approaches S^1 bundle over S^2 at infinity.

$$\lim_{r\to\infty} g = (d\tau + 2n\cos\theta d\phi)^2 + dr^2 + r^2(d\theta^2 + \sin\theta^2 d\phi^2).$$

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- Lorentzian black hole uniqueness (Hawking, Carter, D. Robinson, . . .).
- Riemannian 'black hole uniqueness' conjecture: Euclidean Schwarzschild and Kerr are the only AF gravitational instantons.

The Chen–Teo instanton

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- Exploits lack of positivity (needed for the Mazur identity) of the harmonic map formulation: Riemannian target $SL(2,\mathbb{R})/SO(2)$ in $(3+1)$, Lorentzian target $SL(2,\mathbb{R})/SO(1,1)$ in $(4,0)$.

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- One–sided type D (Aksteiner-Andersson 2024), Asymptotic charges (Kunduri-Lucietti 2021), Twistor Theory (MD-Tod 2024), Rigidity (Biquard–Gauduchon 2023), Tip of the iceberg (Li-Sun 2025).

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- Question: Does there exist an Einstein–Maxwell AF analogue?

- ALF multicentered EM instantons

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g &= \frac{1}{U_+U_-}(d\tau+\omega)^2 + U_+U_- d\mathbf{x}^2 \\
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U_\pm &= \frac{4\pi}{\beta_\pm} + \sum_{m=1}^{N_\pm}\frac{(a^\pm)_m}{|\mathbf{x}-(\mathbf{x}^\pm)_m|}, \quad d\omega = U_-\star_3 dU_+ - U_+\star_3 dU_-.
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- Regularity: need $\beta_+ = \beta_-$ and
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- $N_\pm = 1, U_+ = U_-$: Extreme Reissner–Nordström instanton.
- Theorem (MD-Hartnoll CQG 2007): Riemannian IWP are the most general EM instantons with super-covariantly constant Killing spinor.

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- MD-Bernardo Araneda (PRL 2025): This conjecture is wrong:
  - Six–parameter family of toric, ALF, Einstein–Maxwell metrics.
  - Three–parameter sub–family (Q, ν, k) of toric, AF, one–sided type D Einstein–Maxwell instantons. $Q = 0$: Chen-Teo's Ricci-flat instantons.

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- Integrability techniques
  - Riemannian Harrison transformation of Chen–Teo Ricci–flat metrics.
  - Conformal to Kähler: special $SU(\infty)$ Toda monopole.
  - Reductions of anti–self–dual Yang–Mills equations with $SL(3, \mathbb{C})$ gauge group : twistor theory and harmonic maps.

- A quartic f with four real roots. Set

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\begin{aligned}
f &= f(\xi) = a_4\xi^4 + a_3\xi^3 + a_2\xi^2 + a_1\xi + a_0 \\
\Phi &= f(x)y^2 - f(y)x^2 \\
H &= (\nu x + y)[(\nu x - y + Q(x+y))(a_1 - a_3 xy) \\
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G &= [\nu^2 a_0 + 2\nu a_3 y^3 + (2\nu - 1)a_4 y^4 \\
&\quad -Q(\nu a_1 y + (2\nu - 1)a_3 y^3 + 2(\nu - 1)a_4 y^4)]f(x) \\
&\quad +[(1-2\nu)a_0 - 2\nu a_1 x - \nu^2 a_4 x^4 \\
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Details (more than you wish for)

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- Einstein–Maxwell metric

$$g = \frac{kH}{(x-y)^3}\Big(\frac{dx^2}{f(x)} - \frac{dy^2}{f(y)} - \frac{f(x)f(y)}{k\Phi}d\phi^2\Big) + \frac{1}{\Phi H(x-y)}(\Phi d\tau + G d\phi)^2.$$

- Torus action $K_i = \partial/\partial\phi^i$ where $\phi^i = (\phi, \tau)$,

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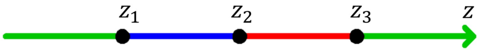

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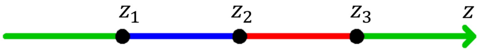

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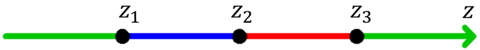

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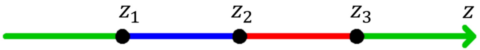
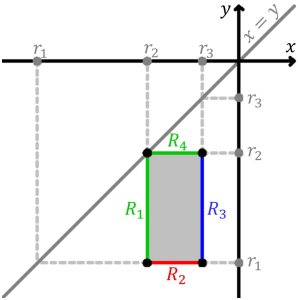


- Eliminate conical and orbifold singularities at the rods.
- Make sure that $\int_M |F|^2 \text{vol}_g < \infty$.
- 3–dimensional (Q, k, ν) moduli space of AF Einstein–Maxwell instantons on $M = \mathbb{CP}^2 \setminus S^1$.

Real roots of f: $r_1 < r_2 < r_3 < r_4$



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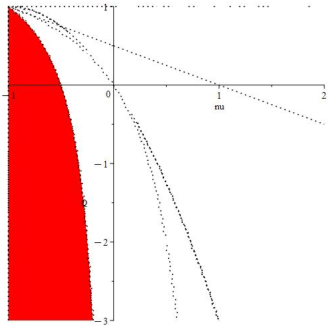

$$r_2 = -1, r_3 = \frac{(1-\nu)(1+r_1)Q - \nu^2 + 2\nu r_1 - 1}{(1+r_1)((\nu-1)Q - 2\nu)}, r_4 = 0$$

$$r_1{}^2 + 2\frac{\nu(Q+\nu)}{Q\nu - 2\nu - 1} r_1 + \frac{\nu\left(Q\nu + \nu^2 - Q + 1\right)(Q+\nu)}{(Q\nu - 2\nu - 1)(Q\nu - Q - 2\nu)} = 0.$$

Projective structures and Einstein metrics

- $P = [P^1, P^2, P^3] \in \mathbb{RP}^2, L = [L_1, L_2, L_3] \in \mathbb{RP}_2{}^*$. Incidence relation
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- Infinitesimal: $(2, 2)$ Einstein $(R \neq 0)$ metric with ASD Weyl tensor.

$$G = (P \cdot L)^{-1} dP \cdot dL - (P \cdot L)^{-2} (L \cdot dP)(P \cdot dL) =$$
$$f^{-2}(\Phi^+ d\Phi^- d\mathcal{E}^+ + \Phi^- d\Phi^+ d\mathcal{E}^- - (\mathcal{E}^- + \mathcal{E}^+) d\Phi^+ d\Phi^- - d\mathcal{E}^+ d\mathcal{E}^-)$$

where $\quad P = [\Phi^-, \mathcal{E}^-, 1], \quad L = [-\Phi^+, 1, \mathcal{E}^+], \quad f = P \cdot L.$

- Projective structure: An equivalence class of affine connections sharing the same unparametrised geodesics. Theorem (MD-Mettler JGA 2018). There is a $(2, 2)$ ASD Einstein metric associated to any projective structure.

Integrable techniques: $U(1)$ symmetry.

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- Special cases: extremal RN$\rightarrow$ geodesics in (X, G).

- Harmonic maps: $\exists(r, z, \phi^1, \phi^2), \Omega(r,z), G = G_{ij}(r,z)$ such that

$$\begin{aligned} g &= \Omega^2(dr^2 + dz^2) + G_{ij} d\phi^i d\phi^j, \quad F_\pm = dA_\pm, \quad A_\pm = (A_\pm)_i d\phi^i \\ 0 &= r^{-1}\partial_r(rJ^{-1}\partial_r J) + \partial_z(J^{-1}\partial_z J). \quad (J) \end{aligned}$$

where $\det J = r^2$ and

$$J = \begin{pmatrix} -G_{11} - 2(A_-)_1(A_+)_1 & -G_{12} - 2(A_-)_1(A_+)_2 & \sqrt{2}i(A_-)_1 \\ -G_{12} - 2(A_-)_2(A_+)_1 & -G_{22} - 2(A_-)_2(A_+)_2 & \sqrt{2}i(A_-)_2 \\ \sqrt{2}i(A_+)_1 & \sqrt{2}i(A_+)_2 & 1 \end{pmatrix}.$$

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- Twistor Theory: (J) is symmetry reductions of self–dual Yang–Mills with $G = SL(3,\mathbb{C})$. Ward transform: Holomorphic vector bundles over $\mathbb{CP}^3 \setminus \mathbb{CP}^1$ trivial on twistor lines . Programme: classify 3×3 patching matrices for such vector bundles.

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Thank You