Positivity of holographic mass

joint work with Raphaela Wutte

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Spoiler

Theorem (PTC, Raphaela Wutte (2026))

Consider a 4d asymptotically locally AdS spacetime $(\mathscr{M}, g)$ solving the Einstein equations with $\Lambda = -3$, with sources decaying sufficiently fast and satisfying the dominant energy condition. Suppose:

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2. *$\mathscr{M}$ contains a complete spacelike hypersurface $\boldsymbol{\mathcal{S}}$*
3. *The conformal boundary $\mathscr{I}$ is conformally static*

Then the weighted holographic energy is non-negative:

$$\boxed{Q[\boldsymbol{\mathcal{S}} \cap \mathscr{I}, e^{-u}\partial_t](g) := -\int_S t^A{}_0 e^{-u} \, dS_A \geq 0}$$

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- We consider four-dimensional spacetimes $(\mathscr{M}, g)$ solving the Einstein equations with $\Lambda < 0$, and with matter fields, decaying sufficiently fast, that satisfy the dominant energy condition.

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- We assume $\mathscr{M}$ contains a complete spacelike hypersurface $\boldsymbol{\mathcal{S}}$, possibly with black hole boundaries
- We introduce a time coordinate such that $\boldsymbol{\mathcal{S}} = \{t = 0\}$

Holographic energy-momentum

Near $\mathscr{I} = \{x = 0\}$ there exist Fefferman-Graham coordinates in which

$$g = x^{-2}\Big(dx^2 + \big(\mathring{\gamma}_{AB} + x^2\overset{(2)}{\gamma}_{AB} + x^3\overset{(3)}{\gamma}_{AB} + O(x^4)\big)dy^A dy^B\Big)$$

where $\mathring{\gamma}_{AB}$ is the boundary metric, $\mathscr{I} = \{x = 0\}$, $(y^A) = (t, y^a)$

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- $\overset{(3)}{\gamma}_{AB}$: globally defined; trace-free and divergence-free

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Holographic energy-momentum tensor: $t_{AB} = \frac{3}{16\pi}\overset{(3)}{\gamma}_{AB}$

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Holographic charges: $Q[\mathcal{C}, X](g) = -\int_{\mathcal{C}} \mathring{\gamma}^{AC} t_{CB} X^B dS_A$

given vector field X on $\mathscr{I}$ and a section $\mathcal{C}$ of $\mathscr{I}$ [Papadimitriou, Skenderis, '05]

Schrödinger-Lichnerowicz-Sen-Witten-Gibbons-Hawking-Horowitz-Perry (SLSWGHHP) identity

We have the identity, for $\epsilon > 0$:

$$\int_{\mathcal{S}\setminus\{x\leq\epsilon\}} \left(|\hat{\nabla}\psi|^2 + \underbrace{\langle\psi,(\rho + J^i\gamma_i\gamma_0)\psi\rangle} - |\gamma^j\hat{\nabla}_j\psi|^2\right) d\mu = \Re \int_{\partial\left(\mathcal{S}\setminus\{x\leq\epsilon\}\right)} B^i(\psi)\, dS_i$$

where

$$\hat{\nabla}_j = \nabla_j + \frac{i}{2}\gamma_j\,, \qquad B^i(\psi) = \langle\hat{\nabla}^i\psi + \gamma^i\gamma^j\hat{\nabla}_j\psi,\psi\rangle\,.$$

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With dominant energy condition and when $\gamma^j\hat\nabla_j\psi = 0$:

$$0 \le \int_{\mathcal{S}} |\hat\nabla\psi|^2\, d\mu \le \Re \int_{\partial\mathcal{S}} B^i(\psi)\, dS_i$$

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Thus, when the Witten equation holds, a finite boundary integral, together with a finite contribution of the matter fields, implies square-integrability of $|\hat\nabla\psi|$.

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Thus, when the Witten equation holds, a finite boundary integral, together with a finite contribution of the matter fields, implies square-integrability of $|\hat\nabla\psi|$. Conversely, square-integrability of $|\hat\nabla\psi|$ together with a finite contribution of the matter fields implies a finite boundary integral.

Asymptotic expansion of solutions of the SLSWGHHP equation

We use the orthonormal frame[1]

$$e_{\hat{3}} = x\,\partial_x\,, \quad e_{\hat{A}} = x\big(\mathring{e}_{\hat{A}} + x^2 f_{\hat{A}}\big)$$

with $f_{\hat{A}}$ smooth on conformally completed manifold. In this frame

$$\psi(x, y^A) = x^{-1/2}\big(\psi_{-\frac{1}{2}}(y^A) + x\,\chi(x, y^A)\big)$$

with χ a bounded, polyhomogeneous spinor field.

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The exponent $-1/2$ arises from the indicial exponents of the Fuchsian SLSWGHHP equation

$$\gamma^i \hat{\nabla}_i \psi = 0\,.$$

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Finiteness of boundary integral, space-dimension three

$|\hat{\nabla}\psi| = O(x^{1+\epsilon})$, $\epsilon > 0$, is necessary and sufficient for a finite boundary integral when the matter fields provide a finite contribution, e.g. in vacuum.

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$$\psi(x, y^A) = x^{-1/2}\psi_{-\frac{1}{2}}(y^A) + x^{1/2}\psi_{\frac{1}{2}}(y^A) + \ldots \tag{1}$$

$$(1 - i\gamma_{\hat{3}})\psi_{-\frac{1}{2}} = 0\,, \quad \mathring{\mathcal{D}}_{\mathring{e}_{\hat{a}}}\psi_{-\frac{1}{2}} + i\gamma_{\hat{a}}\psi_{\frac{1}{2}} = 0 \tag{2}$$

where $\mathring{\mathcal{D}}$ is the spinor covariant derivative of $\mathring{\gamma}_{AB}$.

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By multiplying (2) with $\gamma^{\hat{a}}$, we have $\psi_{\frac{1}{2}} = -\frac{i}{2}\gamma^{\hat{a}}\mathring{\mathcal{D}}_{\mathring{e}_{\hat{a}}}\psi_{-\frac{1}{2}}$. Inserting back into (2) yields

$$\boxed{\gamma^{\hat{b}}\gamma^{\hat{a}}\mathring{\mathcal{D}}_{\mathring{e}_{\hat{b}}}\psi_{-\frac{1}{2}} = 0}$$

x‘

Solutions?

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1. Siklos waves at $\mathscr{I}$ (later)
2. Conformally static $\mathscr{I}$ (coming up)
3. Else?

Solution to the twistor equation

Suppose the conformal class at $\mathscr{I}$ contains a static representative, which can be chosen to be *ultrastatic*:

$$\mathring{\gamma} = -dt^2 + \mathring{\gamma}_{ab}\, dx^a dx^b\,, \quad \partial_t \mathring{\gamma}_{ab} = 0$$

Then $\gamma^{\hat{b}}\gamma^{\hat{a}}\mathring{\mathcal{D}}_{\mathring{e}_{\hat{b}}}\psi_{-\frac{1}{2}} = 0$ is the twistor equation on $(\Sigma := \{t = 0\} \subset \mathscr{I}, \mathring{\gamma}_{ab})$.

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This equation is conformally covariant. If we write, on Σ,

$$\mathring{\gamma}_{ab} = e^{2u}\widetilde{\mathring{\gamma}}_{ab}$$

where $\widetilde{\mathring{\gamma}}$ has constant scalar curvature, we have

$$\widetilde{\psi}_{-\frac{1}{2}} = e^{u/2}\psi_{-\frac{1}{2}}\,, \quad \gamma^{\hat{b}}\gamma^{\hat{a}}\widetilde{\mathcal{D}}_{\tilde{e}_{\hat{b}}}\widetilde{\psi}_{-\frac{1}{2}} = e^{-u/2}\gamma^{\hat{b}}\gamma^{\hat{a}}\mathring{\mathcal{D}}_{\mathring{e}_{\hat{b}}}\psi_{-\frac{1}{2}}\,.$$

Solutions exist on $\mathbb{S}^2$, and $\mathbb{T}^2$ with trivial spin structure

Solution to the Witten equation

Given the asymptotic values $\psi_{-\frac{1}{2}}$ and $\psi_{\frac{1}{2}}$ determined so far we would now like to find a solution to

$$\gamma^j \hat{\nabla}_j \psi = 0 \,.$$

Write

$$\psi(x, y^A) = x^{-1/2}\psi_{-\frac{1}{2}}(y^A) + x^{1/2}\psi_{\frac{1}{2}}(y^A) + \phi(x, y^A)\,,$$

$$\gamma^j \hat{\nabla}_j \phi = -\gamma^j \hat{\nabla}_j \big(x^{-1/2}\psi_{-\frac{1}{2}} + x^{1/2}\psi_{\frac{1}{2}}\big)\,.$$

As the right-hand side is in $L^2(\boldsymbol{\mathcal{S}})$, one can show that a solution $\phi \in L^2(\boldsymbol{\mathcal{S}})$ exists, s.t. the contribution of ϕ to the boundary term vanishes (cf. [Bartnik, Chruściel '05], [Chruściel, Herzlich '03]).

Boundary integral

All calculations have already been done by Cheng and Skenderis `arXiv:hep-th/0506123`

The Witten boundary integral, before passing to $x \to 0$, gives:

$$x^{-1}\Re \int_{\mathcal{S}\cap\{x=\epsilon\}} (\psi_{-\frac{1}{2}})^\dagger \left[(\gamma^{\hat{a}}\mathring{\mathcal{D}}_{\mathring{e}_{\hat{a}}})^2 + \left(\mathring{R}_{\hat{b}\hat{A}} - \tfrac{\mathring{R}}{4}\mathring{\gamma}_{\hat{b}\hat{A}}\right)\gamma^{\hat{b}}\gamma^{\hat{A}}\right]\psi_{-\frac{1}{2}}\, d\mu_{\mathring{\gamma}}$$
$$+ Q[\mathcal{S}\cap\mathscr{I}, X](g) + o(1)$$

with $X^A = (\psi_{-\frac{1}{2}})^\dagger\gamma^{\hat{0}}\gamma^A\psi_{-\frac{1}{2}}$

When the asymptotic conditions are satisfied by ψ:

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When the asymptotic conditions are satisfied by ψ:

- The first, seemingly divergent, term vanishes
- The finite term $Q[\mathcal{S}\cap\mathscr{I},X](g)$ is the holographic charge associated with the boundary vector-field X^A

Boundary integral = Holographic Charge

$$\Re \int_{\partial \boldsymbol{\mathcal{S}}} B^i(\psi)\, dS_i = Q[\boldsymbol{\mathcal{S}} \cap \mathscr{I}, X](g)$$

with $X^A = (\psi_{-\frac{1}{2}})^\dagger \gamma^{\hat{0}} \gamma^A \psi_{-\frac{1}{2}}$

- Under $\mathring{\gamma}_{ab} = e^{2u} \widetilde{\mathring{\gamma}}_{ab}$, the vector field transforms as $X^A = e^{-u} \widetilde{X}^A$

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- When $\widetilde{X}^A = \partial_t$, this is the holographic energy weighted by e^{-u}

The main result

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Here $\mathring{\gamma}_{ab} = e^{2u} \widetilde{\mathring{\gamma}}_{ab}$ with $\widetilde{\mathring{\gamma}}$ of constant curvature, when using an ultrastatic representative $\mathring{\gamma} = -dt^2 + \mathring{\gamma}_{ab}\, dx^a dx^b\,, \quad \partial_t \mathring{\gamma}_{ab} = 0$
$\widetilde{X}$ causal, arises from spinor fields (e.g. ∂_t)

In fact

In case of a spherical $\mathscr{I}$, letting $\psi_{-\frac{1}{2}}$ run over all possible solutions of the twistor equations above one concludes, that in the zero-space-momentum frame the energy m, center of mass $\vec{c}$, and angular momentum $\vec{j}$ satisfy

$$m \geq \sqrt{|\vec{c}|^2 + |\vec{j}|^2 + 2|\vec{c} \times \vec{j}|}\,, \tag{3}$$

where $\vec{c} \times \vec{j}$ is the vector product, while $|\vec{j}| = \sqrt{(j^1)^2 + (j^2)^2 + (j^3)^2}$, etc.

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For a toroidal $\mathscr{I}$ with trivial induced spin structure one has instead

$$m \geq |\vec{j}|\,, \qquad |\vec{j}| := \sqrt{(j^1)^2 + (j^2)^2}\,;$$

Comments

$$Q[\boldsymbol{\mathcal{S}} \cap \mathscr{I}, e^{-u}\partial_t](g) \geq 0$$

- For any conformal Killing vector X of the conformal geometry at infinity the integral

$$Q[\boldsymbol{\mathcal{S}} \cap \mathscr{I}, X](g)$$

is time-independent

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- $e^{-u}\partial_t$ is not a conformal Killing vector unless u is constant
- When u is not constant, $\partial_t Q$ does not have to vanish: the weighted energy can be radiated away, but there is a lower bound by zero in any case

A surprise

$$Q[\mathcal{S} \cap \mathscr{I}, e^{-u}\partial_t](g) \geq 0$$

- For static vacuum $(3+1)$-dimensional metrics $\bar{g}$ with conformally compact spacelike sections $\mathcal{S}$ (no conditions on the topology of $\mathcal{S} \cap \mathscr{I}$)

$$Q[\mathcal{S} \cap \mathscr{I}, \partial_t](\bar{g}) \leq 0\,,$$

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A perturbative inequality

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- By continuity $D^2 Q|_{\bar{g}} > 0$ at nearby metrics.
- It follows that when g and $\bar{g}$ share the same conformal infinity, are both close to anti-de Sitter $\check{g}$, with $g \neq \check{g}$, vacuum, $\bar{g}$ static, it holds that

$$\underbrace{Q[\mathcal{S} \cap \mathscr{I}, \partial_t](g)}_{sign?} > \underbrace{Q[\mathcal{S} \cap \mathscr{I}, \partial_t](\bar{g})}_{<0} . \tag{4}$$

Relation between holographic and Hamiltonian energy

Our positivity bound does not require existence of a static background metric

When $\mathscr{I}$ is conformally static, $g, \bar{g}$ are vacuum, and $\overline{g}$ satisfies static vacuum equations to sufficiently high order

- The relative holographic energy equals the Hamiltonian energy [Chruściel, RW '26]:

$$Q[\boldsymbol{\mathcal{S}} \cap \mathscr{I}, \partial_t](g) - Q[\boldsymbol{\mathcal{S}} \cap \mathscr{I}, \partial_t](\bar{g}) = H[\boldsymbol{\mathcal{S}}, \partial_t, \bar{g}](g)$$