Overview of standing gravitational wave solutions

Adam Zychowicz

Astronomical Observatory of the Jagiellonian University

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Plan

1. Definition of standing gravitational waves (SGWs)
2. Overview of derivation of two families of SGWs
3. Similarities in the high frequency limit
4. Differences in the weak field limit

What even are standing gravitational waves?

Standing waves in classical theories (mechanical waves, E-M wave):

- Oscillates in time, but does not appear to travel through space
- Does not carry energy (at all, or on average)
- Has nodes (0 amplitude) and anti-nodes (maximum amplitude)

How to apply it to gravitational waves?

What even are standing gravitational waves?

Stephani (2003) – attempt at a definition:

1. metric functions should depend on the timelike coordinate only through the periodic factor
2. A suitable analogue of the Poynting vector (if there is any) should be divergence-free and its time average should be 0.

Both conditions are rather vague and not invariantly formulated.

Setup

A cylindrically symmetric solution:

$$ds^2 = e^{2(\gamma-\psi)}(-dt^2 + d\rho^2) + \rho^2 e^{-2\psi} d\phi^2 + e^{2\psi}(dz + \omega d\phi)^2$$

$$\rho > 0;\ t, z \in (-\infty, \infty);\ \phi \in [0, 2\pi)$$

The vacuum field equations:

$$\begin{aligned}
\dot{\gamma} =& 2\rho\dot{\psi}\psi' + \frac{e^{4\psi}}{2\rho}\dot{\omega}\omega' \ , \\
\gamma' =& \rho\left(\dot{\psi}^2 + {\psi'}^2\right) + \frac{e^{4\psi}}{4\rho}\left(\dot{\omega}^2 + {\omega'}^2\right) \ , \\
-\ddot{\psi}+\psi'/\rho + \psi'' =& \frac{e^{4\psi}}{2\rho^2}\left({\omega'}^2 - \dot{\omega}^2\right) \ , \\
-\ddot{\omega}-\omega'/\rho + \omega'' =& 4\left(\dot{\psi}\dot{\omega} - \psi'\omega'\right) \ .
\end{aligned} \tag{1}$$

Wave-like solutions

A cylindrically symmetric solution:

$$ds^2 = e^{2(\gamma-\psi)}(-dt^2 + d\rho^2) + \rho^2 e^{-2\psi} d\phi^2 + e^{2\psi}(dz + \omega d\phi)^2$$

$$\rho > 0;\ t, z \in (-\infty, \infty);\ \phi \in [0, 2\pi)$$

Assume some functions satisfy the harmonic oscillator equation. Ex. ω:

$$\ddot{\omega} + \frac{1}{\lambda^2}\omega = \frac{Y}{\lambda}\ .$$

Different further assumptions can lead to various solutions:

1. $Y(\rho) = 0$ leads to the Halilsoy solution
2. $\dot{\gamma} = 0$ leads to the Chandrasekhar solution

Obtaining Halilsoy solution

Fixing ω to satisfy the harmonic oscillator equation and putting $Y(\rho) = 0$:

$$\frac{1}{\lambda^2}\omega - \omega'/\rho + \omega'' = 4\left(\dot{\psi}\dot{\omega} - \psi'\omega'\right) .$$

Introducing $x = \rho/\lambda$, an auxiliary function $\hat{\omega} = \rho\,\omega$ and the simplifying ansatz $\dot{\psi}\dot{\omega} - \psi'\omega' \; = 0$ brings part of the solution to the Bessel equation of order 1:

$$\frac{d^2\hat{\omega}}{dx^2} + \frac{1}{x}\frac{d\hat{\omega}}{dx} + \left(1 - \frac{1}{x^2}\right)\hat{\omega} = 0 .$$

With the final solution for ω:

$$\omega = D\,\rho\,J_1 \sin\left(t/\lambda\right) ,$$

Obtaining Halilsoy solution

The $\dot{\psi}\dot{\omega} - \psi'\omega' = 0$ part now reads:

$$\dot{\psi} J_1 \cos\left(t/\lambda\right) = \psi' J_0 \sin\left(t/\lambda\right) ,$$

which can be partially solved with the method of characteristics to learn that ψ is of the form $\psi(y(t,\rho))$ with $y(t,\rho) = J_0 \cos\left(t/\lambda\right)$.
We use another Einstein equation:

$$-\ddot{\psi} + \psi'/\rho + \psi'' = \frac{e^{4\psi}}{2\rho^2}\left(\omega'^2 - \dot{\omega}^2\right) \implies \frac{d^2\psi}{dy^2} + \frac{D^2}{2} e^{4\psi} = 0 .$$

This equation can be solved with regular ODE solving methods. Similarly, γ can be simply obtained via quadratures.

Halfway there

A cylindrically symmetric solution:

$$ds^2 = e^{2(\gamma-\psi)}(-dt^2 + d\rho^2) + \rho^2 e^{-2\psi} d\phi^2 + e^{2\psi}(dz + \omega d\phi)^2$$

$$\rho > 0;\ t, z \in (-\infty, \infty);\ \phi \in [0, 2\pi)$$

Halilsoy solution - closed form

$$e^{-2\psi} = e^{AJ_0 \cos(t/\lambda)} \sinh^2 \frac{\alpha}{2} + e^{-AJ_0 \cos(t/\lambda)} \cosh^2 \frac{\alpha}{2}$$

$$\omega = -(A \sinh \alpha)\, \rho\, J_1 \sin(t/\lambda)$$

$$\gamma = \frac{1}{8} A^2 \left[\left(\frac{\rho}{\lambda}\right)^2 (J_0^2 + J_1^2) - 2\frac{\rho}{\lambda} J_0 J_1 \cos^2(t/\lambda) \right]$$

Chandrasekhar solution

???

Obtaining Chandrasekhar solution

Main assumption: $\dot{\gamma} = 0$ (constant C-Energy).
Introducing an auxiliary function $\psi = -\frac{1}{2} \ln \hat{\Psi}$:

$$\dot{\gamma} = 2\rho\dot{\psi}\psi' + \frac{e^{4\psi}}{2\rho}\dot{\omega}\omega' \implies 0 = \rho^2 \dot{\hat{\Psi}}\hat{\Psi}' + \dot{\omega}\omega' \; . \tag{2}$$

The metric function ω already satisfies the harmonic oscillator equation.
We require $\hat{\Psi}$ to solve it too:

$$\omega = \lambda \left(X(\rho) \cos(t/\lambda) + Y(\rho) \right) \; ,$$
$$\hat{\Psi} = Z(\rho) \cos(t/\lambda) + W(\rho) \; .$$

Obtaining Chandrasekhar solution

The Einstein equations written in terms of X, Y, W, Z depend on t only via powers of $\cos(t/\lambda)$. Requiring the coefficients to vanish leads to a system of 8 equations for X, Y, W, Z and their derivatives. One of them is:

$$0 = X^2(Z^2 - W^2) + \rho^2(ZW' - WZ')^2 \, .$$

We redefine $(W, Z) \to (R, F)$ using:

$$W = R\cosh P \, , \quad Z = R\sinh P \, , \quad P = 2\,\mathrm{arctanh}(-F) \, .$$

Obtaining Chandrasekhar solution

This allows us to solve the system to obtain X, Y, W, Z in terms of $F(\rho)$:

$$\begin{aligned} X =& 2\rho\frac{F'}{1-F^2}\,, \\ Y' =& 4\frac{\rho}{\lambda^2}\frac{F^2}{(1-F^2)^2}\,, \\ W =& \frac{1+F^2}{1-F^2}\,, \\ Z =& -\frac{2F}{1-F^2}\,. \end{aligned}$$

The remaining equations lead to an ODE constraining F:

$$F(1+F^2)/\lambda^2 + \frac{1-F^2}{\rho}(\rho F')' + 2F(F')^2 = 0$$

Two standing wave solutions

A cylindrically symmetric solution:

$$ds^2 = e^{2(\gamma-\psi)}(-dt^2 + d\rho^2) + \rho^2 e^{-2\psi} d\phi^2 + e^{2\psi}(dz + \omega d\phi)^2$$

$$\rho > 0;\ t, z \in (-\infty, \infty);\ \phi \in [0, 2\pi)$$

Halilsoy solution - closed form

$$e^{-2\psi} = e^{AJ_0 \cos(t/\lambda)} \sinh^2\frac{\alpha}{2} + e^{-AJ_0 \cos(t/\lambda)} \cosh^2\frac{\alpha}{2}$$
$$\omega = -(A \sinh\alpha)\,\rho\, J_1 \sin(t/\lambda)$$
$$\gamma = \frac{1}{8}A^2\left[\left(\frac{\rho}{\lambda}\right)^2 (J_0^2 + J_1^2) - 2\frac{\rho}{\lambda} J_0 J_1 \cos^2(t/\lambda)\right]$$

Chandrasekhar solution - numerical

$$e^{2\psi} = \frac{1 - F^2}{1 + F^2 - 2F\cos(t/\lambda)}$$
$$\omega = \frac{2\lambda\rho F'}{1 - F^2}\cos(t/\lambda) + \frac{4}{\lambda}\int_0^\rho \frac{\hat{\rho}F^2}{(1 - F^2)^2}\, d\hat{\rho}$$
$$\dot{\gamma} = 0$$
$$\gamma' = \frac{\rho}{(1 - F^2)^2}(F^2/\lambda^2 + (F')^2)$$
$$F(1 + F^2)/\lambda^2 + \frac{1 - F^2}{\rho}(\rho F')' + 2F(F')^2 = 0$$

Green-Wald framework (Green & Wald 2011)

For a family of solutions $g_{\alpha\beta}(\lambda, x)$, we define the background metric:

$$g^{(0)}_{\alpha\beta}(x) = g_{\alpha\beta}(0, x) = \underset{\lambda\to 0}{\text{w-lim}}\, g_{\alpha\beta}(\lambda, x) \ ,$$

$$h_{\alpha\beta}(\lambda, x) = g_{\alpha\beta}(\lambda, x) - g_{\alpha\beta}(0, x) \ .$$

Where w-lim can be defined as: $B_{\alpha_1 \ldots \alpha_n}$ is a weak limit of $A_{\alpha_1 \ldots \alpha_n}$ if:

$$\lim_{\lambda\to 0} \int f^{\alpha_1 \ldots \alpha_n} A_{\alpha_1 \ldots \alpha_n}(\lambda) \sqrt{|g^{(0)}|} d^n x = \int f^{\alpha_1 \ldots \alpha_n} B_{\alpha_1 \ldots \alpha_n} \sqrt{|g^{(0)}|} d^n x$$

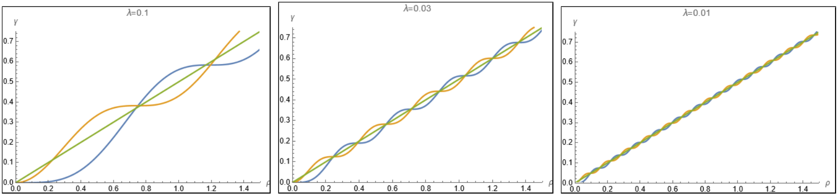
for all smooth $f^{\alpha_1 \cdots \alpha_n}$ of compact support.

The simplest example of the operational procedure:
$\text{w-lim}_{\lambda\to 0} \cos^2\left(\frac{x}{\lambda}\right) = \frac{1}{2}$.

High frequency limit of Halilsoy solution (Szybka et al. 2026)

To fulfill G-W framework requirements, we chose a family of solutions by setting the amplitude $A = 2\beta\sqrt{\lambda}$. The resulting weak limit $\lambda \to 0$ leads to:

$$
\begin{aligned}
\psi^{(0)} &= -\ln\sqrt{\cosh\alpha}\ , \\
\omega^{(0)} &= 0\ , \\
\gamma^{(0)} &= \frac{\beta^2\rho}{\pi}\ , \\
g^{(0)} &= \cosh\alpha\left(e^{\frac{2\beta^2\rho}{\pi}}(-dt^2 + d\rho^2) + \rho^2 d\phi^2 + \operatorname{sech}^2\alpha\, dz^2\right)\ .
\end{aligned}
$$



High frequency limit of Chandrasekhar solution (Szybka et al. 2026)

The main component of the Chandrasekhar standing gravitational wave is given by a numerical solution to the equation:

$$F(1+F^2)/\lambda^2 + \frac{1-F^2}{\rho}(\rho F')' + 2F(F')^2 = 0\ .$$

For small λ the solution can be approximated as:

$$F(\rho) \approx \beta\sqrt{\lambda/2}J_0(\rho/\lambda) \approx \frac{\beta\lambda}{\sqrt{\pi\rho}}\cos(\pi/4-\rho/\lambda)\ ,$$

$$F'(\rho) \approx \frac{\beta}{\sqrt{\pi\rho}}\sin(\pi/4-\rho/\lambda)\ .$$

This leads, in the weak limit to a similar metric:

$$g^{(0)} = \left(e^{\frac{2\beta^2\rho}{\pi}}(-dt^2+d\rho^2)+\rho^2 d\phi^2 + dz^2\right)\ .$$

High frequency limit (Szybka et al. 2026)

For both solutions, the weak limit leads to a simple and traceless stress-energy tensor:

$$t^{(0)} = \frac{\beta^2}{8\pi^2\rho}\left(dt^2 + d\rho^2\right) \ .$$

Linear approximation (Nikiel & Szybka 2025)

Halilsoy wave:

$$h_{\alpha\beta}^{TT}\Big|_{y=0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\epsilon\lambda Q_1/x & 0 & 0 \\ 0 & 0 & -\epsilon\left(Q_0 - \lambda Q_1/x\right) & -\epsilon\sinh(\alpha)\,W \\ 0 & 0 & -\epsilon\sinh(\alpha)\,W & \epsilon Q_0 \end{pmatrix}.$$

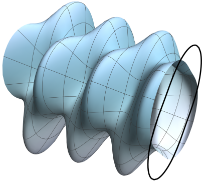
Chandrasekhar wave:

$$h_{\alpha\beta}^{TT}\Big|_{y=0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\epsilon\lambda Q_1/x & 0 & 0 \\ 0 & 0 & -\epsilon\left(Q_0 - \lambda Q_1/x\right) & -\epsilon Q_1 \\ 0 & 0 & -\epsilon Q_1 & \epsilon Q_0 \end{pmatrix}.$$

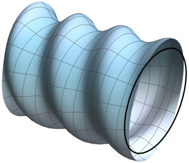
$$Q_i = J_i\left(\frac{\rho}{\lambda}\right)\cos\left(\frac{t}{\lambda}\right),$$

$$W = J_1\left(\frac{\rho}{\lambda}\right)\sin\left(\frac{t}{\lambda}\right).$$

Linear approximation (Nikiel & Szybka 2025)

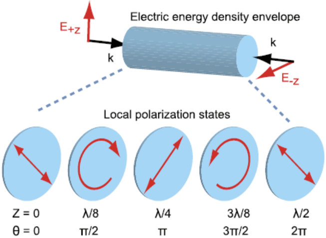


Halilsoy



Chandrasekhar

Fig. Nikiel & Szybka (2025)



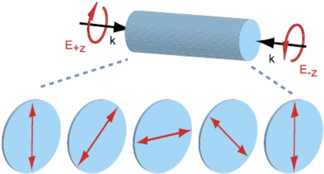


Fig. Fang et al. (2016)

Summary - similarities and differences between Halilsoy and Chandrasekhar SGWs

Similarities:

- Cylindrically symmetric
- Metric functions periodic in the timelike coordinate t
- The same high frequency limit (null dust).

Differences:

- The Halilsoy family includes the Einstein-Rosen gravitational wave. Chandrasekhar waves always need non-trivial polarization.
- In the linear approximation, solutions are analogs of two different electromagnetic polarization standing waves.

The End
...questions?

Mentioned literature:

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