Near-extremal gravitational collapse in 4+1 dimensions: Schwarzschild-de-Sitter space

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black hole thermodynamics

- 0th law:
  - black holes—The surface gravity, κ, is constant over the event horizon of a stationary black hole.
  - thermodynamics—The temperature, T, is constant throughout a body in thermal equilibrium.
- 1st law:
  - black holes—$\Delta M = \frac{\kappa}{8\pi}\Delta A + \Omega_H \Delta J$
  - thermodynamics—$\Delta E = T\Delta S - P\Delta V$
- 2nd law:
  - black holes—$\Delta A \geq 0$
  - thermodynamics—$\Delta S \geq 0$
- 3rd law:
  - black holes—It is impossible to reduce the surface gravity κ to zero by any finite sequence of processes.
  - thermodynamics—It is impossible to reduce the temperature T to zero by any finite sequence of processes.

3rd law of black hole thermodynamics

- proposed (as a conjecture): Bardeen, Carter, Hawking 1973
  [Commun. Math. Phys. **31**, 161 (1973)]
  - "It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations."
- "proved": Israel 1986
  [Phys. Rev. Lett. **57**, 397 (1986)]
  - continuous process, bounded $T_{\mu\nu}$ satisfying the weak energy condition near the apparent horizon
- disproved: Kehle, Unger 2022
  [arXiv:2211.15742]
  - gravitational collapse of regular data to an exactly extremal Reissner–Nordström black hole in finite time

3rd law of black hole thermodynamics

- collapse to an extremal Reissner–Nordström black hole is a critical phenomenon: Kehle, Unger 2024 [arXiv:2402.10190]
  - exactly extremal end states lie, conjecturally, on a codimension-one threshold in families of initial data
- the counterexamples require a highly charged scalar field; a third law is restored for supersymmetric black holes in theories with a local mass–charge inequality: Reall 2024 [Phys. Rev. D **110**, 124059 (2024)]
- in vacuum ($4+1$, $\Lambda = 0$): Crump, Gadioux, Reall, Santos 2026 [Phys. Rev. Lett. **136**, 171405 (2026)]

the Bizoń–Chmaj–Schmidt ansatz

- Bizoń, Chmaj, Schmidt 2005
  [Phys. Rev. Lett. **95**, 071102 (2005)]
  $$g = -Ae^{-2\delta}dt^2 + A^{-1}dr^2 + \frac{1}{4}r^2\left(e^{2B}({\sigma_1}^2 + {\sigma_2}^2) + e^{-4B}{\sigma_3}^2\right)$$
- A, B, δ are functions of (r, t); σ_i — left-invariant one-forms on $S^3 \cong SU(2)$
- B measures the squashing of S^3; isometry group $SU(2) \times U(1)$
- radial symmetry is preserved, but time dependence is allowed — the Birkhoff theorem is evaded

Einstein equations with Λ

- vacuum Einstein equations with Λ reduce to four PDEs for (A, B, δ) on the $1{+}1$-dimensional quotient:

$$
\begin{aligned}
A' &= -\frac{2A}{r} + \frac{1}{3r}(8e^{-2B} - 2e^{-8B}) - 2r(e^{2\delta}A^{-1}\dot{B}^2 + AB'^2) - \frac{2}{3}\Lambda r \\
\delta' &= -2r(e^{2\delta}A^{-2}\dot{B}^2 + B'^2) \\
0 &= \left(e^{\delta}A^{-1}r^3\dot{B}\right)^{\cdot} - (e^{-\delta}Ar^3B')' + \frac{4}{3}e^{-\delta}r(e^{-2B} - e^{-8B}) \\
\dot{A} &= -4rA\dot{B}B'
\end{aligned}
$$

- $A' = \partial_r A$, $\dot{A} = \partial_t A$

double null coordinates

- the BCS ansatz in the double null form, F, r, B functions of (u, v):
$$g = -e^{2F} du dv + \frac{1}{4} r^2 \Big(e^{2B} ({\sigma_1}^2 + {\sigma_2}^2) + e^{-4B} {\sigma_3}^2 \Big)$$

- four independent Einstein equations:

$$\begin{aligned}
r_{vv} &= 2 r_v F_v - 2 r {B_v}^2 \\
r_{uu} &= 2 r_u F_u - 2 r {B_u}^2 \\
r_{uv} &= \frac{\Lambda r e^{2F}}{6} - \frac{2}{r} r_u r_v + \frac{e^{2F}}{6r} (e^{-8B} - 4 e^{-2B}) \\
B_{uv} &= \frac{e^{2F}}{3r^2} (e^{-8B} - e^{-2B}) - \frac{3}{2r} (r_v B_u + r_u B_v)
\end{aligned}$$

- the fifth equation, for F_{uv}, follows from these
- the constraints along null surfaces are ODEs

de Sitter and Schwarzschild–de Sitter solutions

- static solutions: $B = 0$, $\delta = 0$ and
$$A = A_0 = 1 - \frac{m}{r^2} - \frac{1}{6}\Lambda r^2 = \frac{\Lambda}{6r^2}({r_c}^2 - r^2)(r^2 - {r_s}^2)$$

- de Sitter ($m = 0$): only the cosmological horizon $r_c = \sqrt{6/\Lambda}$
- Schwarzschild–de Sitter ($0 < \Lambda m < \frac{3}{2}$): black hole horizon r_s and cosmological horizon r_c
$$r_s = \sqrt{\frac{3}{\Lambda}\Big(1 - \sqrt{1 - \frac{2}{3}\Lambda m}\Big)}, \qquad r_c = \sqrt{\frac{3}{\Lambda}\Big(1 + \sqrt{1 - \frac{2}{3}\Lambda m}\Big)}$$

- extreme Schwarzschild–de Sitter ($\Lambda m = \frac{3}{2}$): horizons coincide, $r_s = r_c \equiv r_N = \sqrt{3/\Lambda}$, maximal mass $m_{\max} = \frac{3}{2\Lambda}$

$r_s = r_c$ is a singular limit: extreme SdS

- substituting $\Lambda m = \frac{3}{2}$ into the family — extreme SdS:

$$A_0 = -\frac{\Lambda}{6r^2}\Big(r^2 - \frac{3}{\Lambda}\Big)^2 \le 0$$

  - no static region, r is timelike everywhere off the degenerate horizon $r = r_N$: a globally dynamical spacetime which retains the $r = 0$ singularity
    [$3{+}1$ Penrose diagram: Podolský, Gen. Rel. Grav. **31**, 1703 (1999)]

$r_s = r_c$ is a singular limit: Nariai

- taking the limit $m \to m_{\max}$ — Nariai: the coordinate width of the static region shrinks to zero, but its proper width does not,

$$\int_{r_s}^{r_c} \frac{dr}{\sqrt{A_0}} \longrightarrow \pi\sqrt{\frac{3}{2\Lambda}}$$

  - zooming into the throat ($r = r_N + \epsilon\cos\chi$, $t = \tau/(\epsilon c)$ with $c = \frac{2\Lambda}{3}$, $\epsilon \to 0$) gives $dS_2 \times S^3$ with radii $\sqrt{3/(2\Lambda)}$ and $\sqrt{3/\Lambda}$: homogeneous, nonsingular
    [$3{+}1$: Ginsparg, Perry, Nucl. Phys. B **222**, 245 (1983); D dim.: Cardoso, Dias, Lemos, Phys. Rev. D **70**, 024002 (2004)]
- evaluation and limit do not commute — two distinct spacetimes (as with extremal Reissner–Nordström vs. Bertotti–Robinson $AdS_2 \times S^2$)

renormalised Hawking (Misner–Sharp) mass

- in $4+1$ dimensions with radial symmetry the Hawking mass reduces to the Misner–Sharp mass $M = r^2(1 - |\nabla r|_g{}^2)$
- in the presence of Λ we use the renormalised mass

$$M_\Lambda = r^2\left(1 - |\nabla r|_g{}^2 - \frac{1}{6}\Lambda r^2\right)$$

- $M_\Lambda = 0$ for de Sitter, $M_\Lambda = m$ for Schwarzschild–de Sitter
- monotonicity along null directions controlled by the expansions $\theta_+ = 3e^{-2F}r^{-1}r_v$, $\theta_- = 3e^{-2F}r^{-1}r_u$
- a sphere of radius r is trapped if $\theta_\pm < 0$, and then

$$\frac{M_\Lambda}{r^2} + \frac{1}{6}\Lambda r^2 - 1 > 0$$

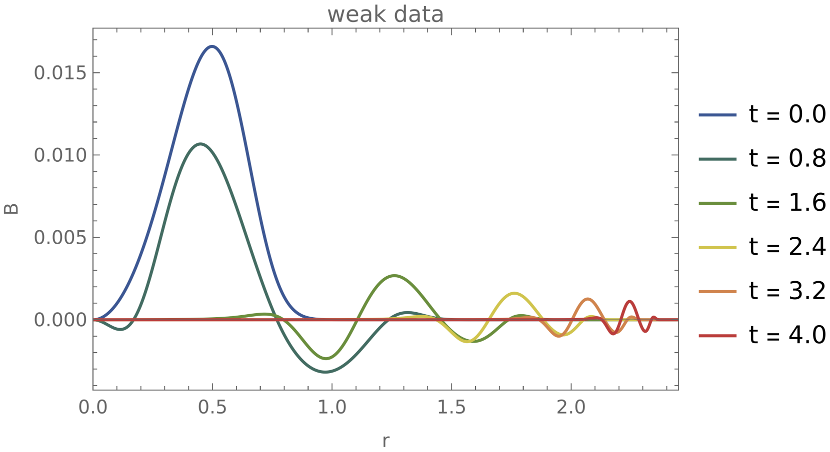
- the formation of black holes is controlled by the LHS of this inequality

animations

	A	B	M
weak	video	video	video
intermediate	video	video	video
strong	video	video	video

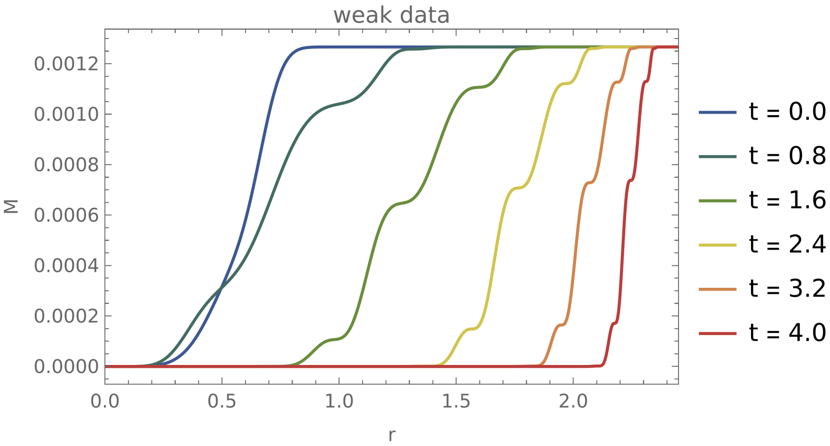
the function A, the squashing function B, and the Hawking mass M for the three sets of initial data

weak data: the squashing function B



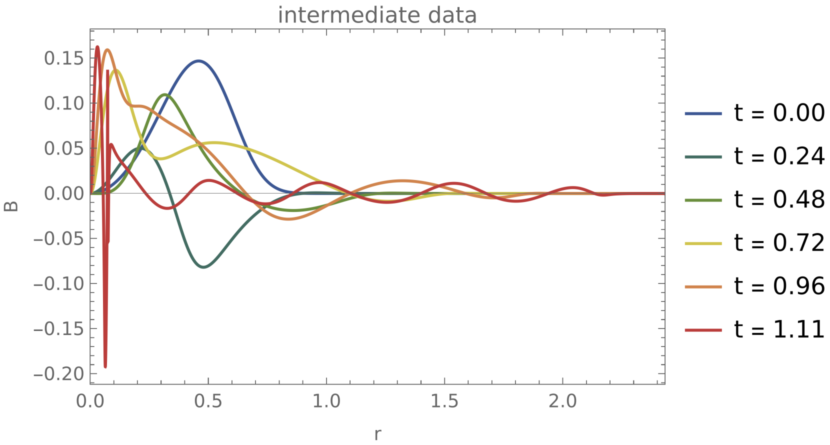
a small ingoing perturbation of de Sitter space reflects from the centre and propagates outwards — the solution settles back to de Sitter

weak data: the Hawking mass M



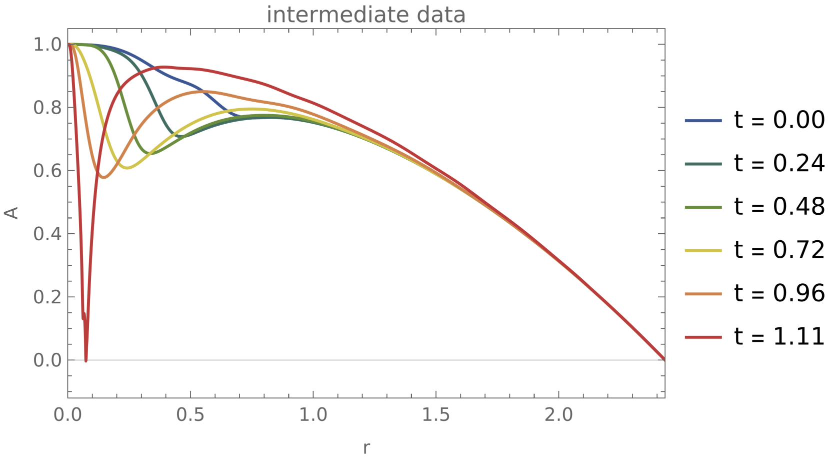
the energy is radiated away from the centre; the coordinates are singular at the cosmological horizon $r_c = 2.449$

intermediate data: the squashing function B



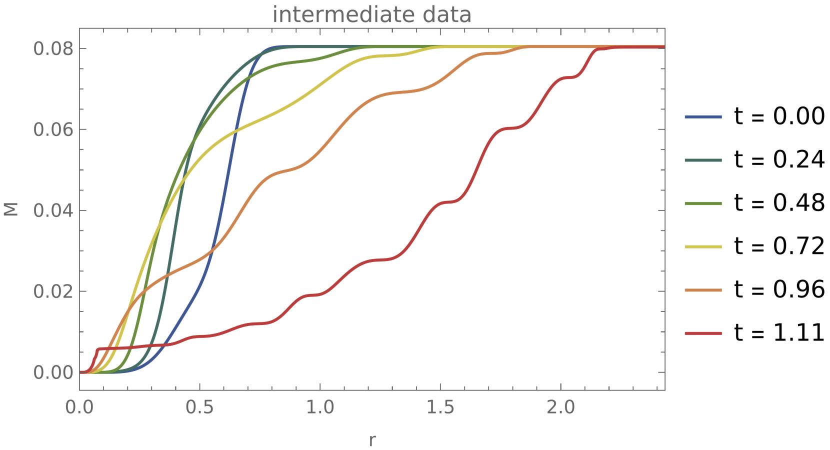
collapse to a small black hole in the spirit of critical collapse

intermediate data: the function A



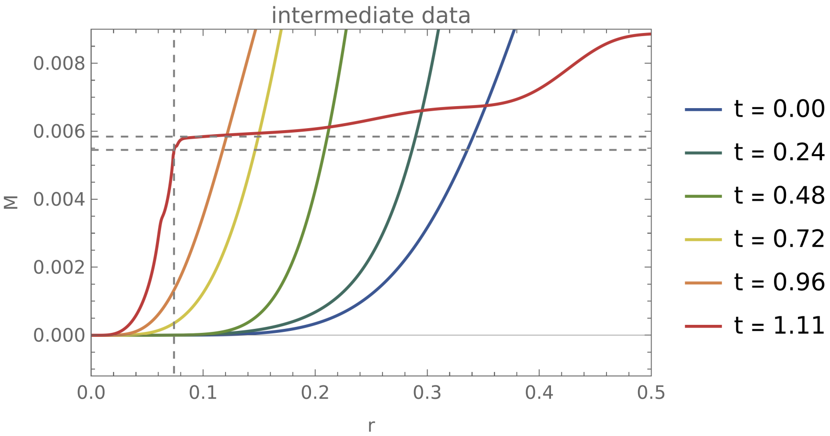
the two zeros of A correspond to the apparent horizon $r_s = 0.074$ and the cosmological horizon $r_c = 2.433$

intermediate data: the Hawking mass M



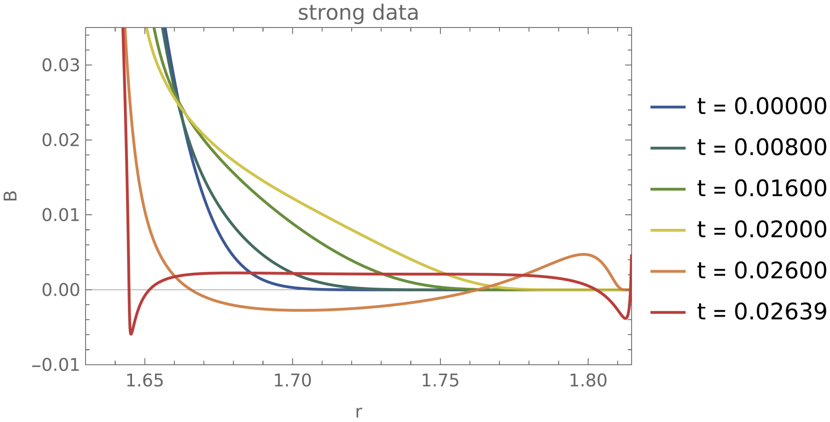
the apparent horizon forms with $M = 0.0055$

intermediate data: the Hawking mass M



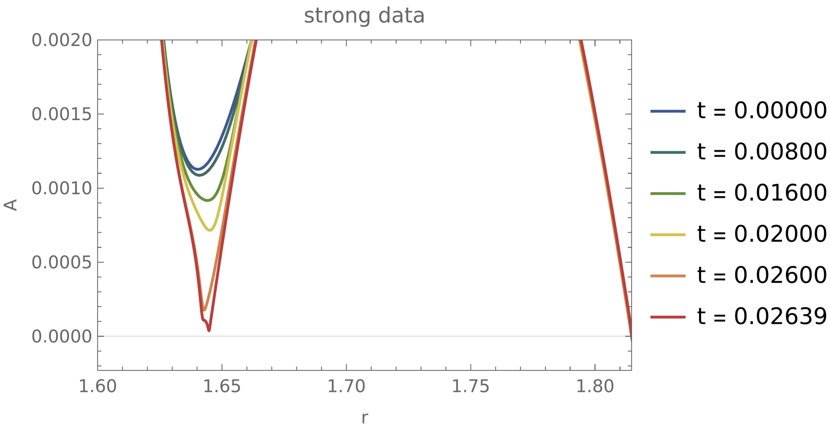
dashed lines: the apparent horizon mass $M = 0.0055$, the late-time plateau at $M = 0.0058$, and the position of the apparent horizon — the extra mass should be absorbed by the growing horizon

strong data: the squashing function B



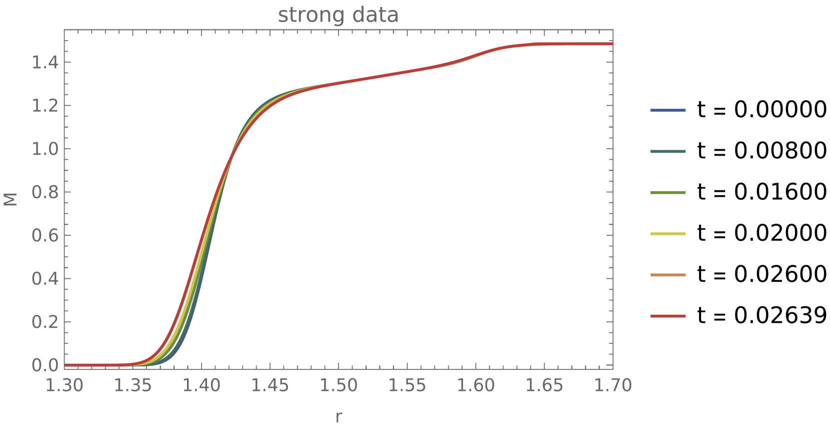
initial data already on the verge of forming a trapped surface

strong data: the function A



the apparent horizon forms at $r_s = 1.645$ where A vanishes; the cosmological horizon is located at $r_c = 1.815$

strong data: the Hawking mass M



M is close to the extremal value already at $t = 0$ and approaches it at late times

strong data: parameters

- initial data:

$$B(0,r) = b_0\, r^2\, e^{-\left(\frac{r-r_0}{s}\right)^4}, \qquad \partial_t B(t,r)|_{t=0} = -v_0\, \partial_r B(0,r)$$

- parameters: $b_0 = 0.14$, $r_0 = 1.52$, $s = 0.11$, $v_0 = 0$ (time-symmetric), $\Lambda = 1$
- numerics: $\Delta t = 2 \cdot 10^{-7}$, $r_{\max} = 2.0$, $\Delta r \approx 4.5 \cdot 10^{-5}$ ($n = 44800$ nodes), the grid reaches beyond the cosmological horizon
- results: the apparent horizon at $r_s = 1.645$, the cosmological horizon at $r_c = 1.815$
  - near-extremal black hole: $r_s/r_c = 0.906$, $M = 1.485 = 0.99\, M_{\max}$, where $M_{\max} = \frac{3}{2\Lambda} = 1.5$
- the coordinates are singular at both horizons and ill-defined in the limit $r_s \to r_c$: the extremal black hole cannot be reached numerically, but can be approached arbitrarily closely

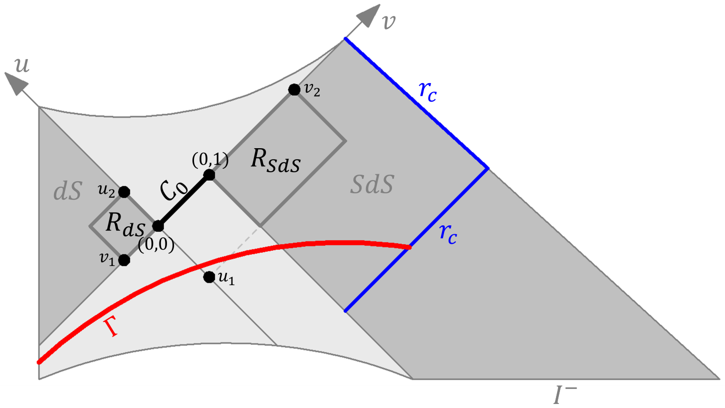
characteristic gluing

- gluing of initial data on *spacelike* slices:
  Corvino 2000; Chruściel, Delay 2003; Corvino, Schoen 2006
- gluing along a *null* hypersurface — characteristic gluing:
  Aretakis, Czimek, Rodnianski 2021 [Ann. Henri Poincaré **25**, 3081 (2024)]
- the Kehle–Unger strategy adapted to $\Lambda > 0$: glue an outgoing null cone of de Sitter ($v \leq 0$) to the horizon of Schwarzschild–de Sitter ($v \geq 1$) into one cone $C_0 = \{u = 0\}$; here the extreme case $r_s = r_c = r_N$, i.e. $\Lambda m = \frac{3}{2}$
- ansatz for the squashing factor in the gluing region $v \in [0, 1]$:

$$B(0, v) = v^2(1 - v)^2(1 + \alpha_1 v + \alpha_2 v^2)$$

- gauge $F(0, v) = 0$; the constraint $r_{vv} = -2r{B_v}^2$ solved backwards from the horizon sphere: $r(0, 1) = r_s$, $r_v(0, 1) = 0$
- the remaining Einstein equations: first order ODEs for r_u, B_u, F_u integrated forward from the de Sitter values at $v = 0$

characteristic gluing



a de Sitter null cone in R_{dS} and the black hole horizon in R_{SdS} are glued along the cone C_0 $(u = 0)$; extending backwards allows to read off the Cauchy data on the spacelike surface Γ

characteristic gluing to extreme SdS

- matching to the extreme SdS sphere data requires $B_u(0,1) = 0$: shooting on α_2 at $\alpha_1 = 0$ gives C^1 ($\Lambda = 0.1$, $m = m_{\max} = 15$: $\alpha_2 = -3.2017$); the two parameter matching on (α_1, α_2) with $B_u = B_{uu} = 0$ gives C^2 ($\alpha_1 = -2.89671$, $\alpha_2 = 1.63196$)
- Luk's extension theorem: the data on C_0 and on the transverse null surfaces determine a spacetime region, hence the Cauchy data at $t = 0$ evolved numerically

summary

- Λ affects gravitational collapse: slows it down if $\Lambda > 0$, enhances it if $\Lambda < 0$
- $\Lambda > 0$: dispersion or collapse depending on the size of the initial data; the size of the black hole is limited by the cosmological horizon
- large initial data result in a near-extremal collapse, $M = 0.99\, M_{\max}$ — numerical evidence, not a proof, that the extremal collapse takes place
- open: small data and critical cosmological collapse with $\Lambda > 0$