Statistical reconstruction algorithm based on the method of moments in positronium imaging

Roman Y. Shopa

17-02-2025

Outline

Positronium decay

Ps lifetime spectrum

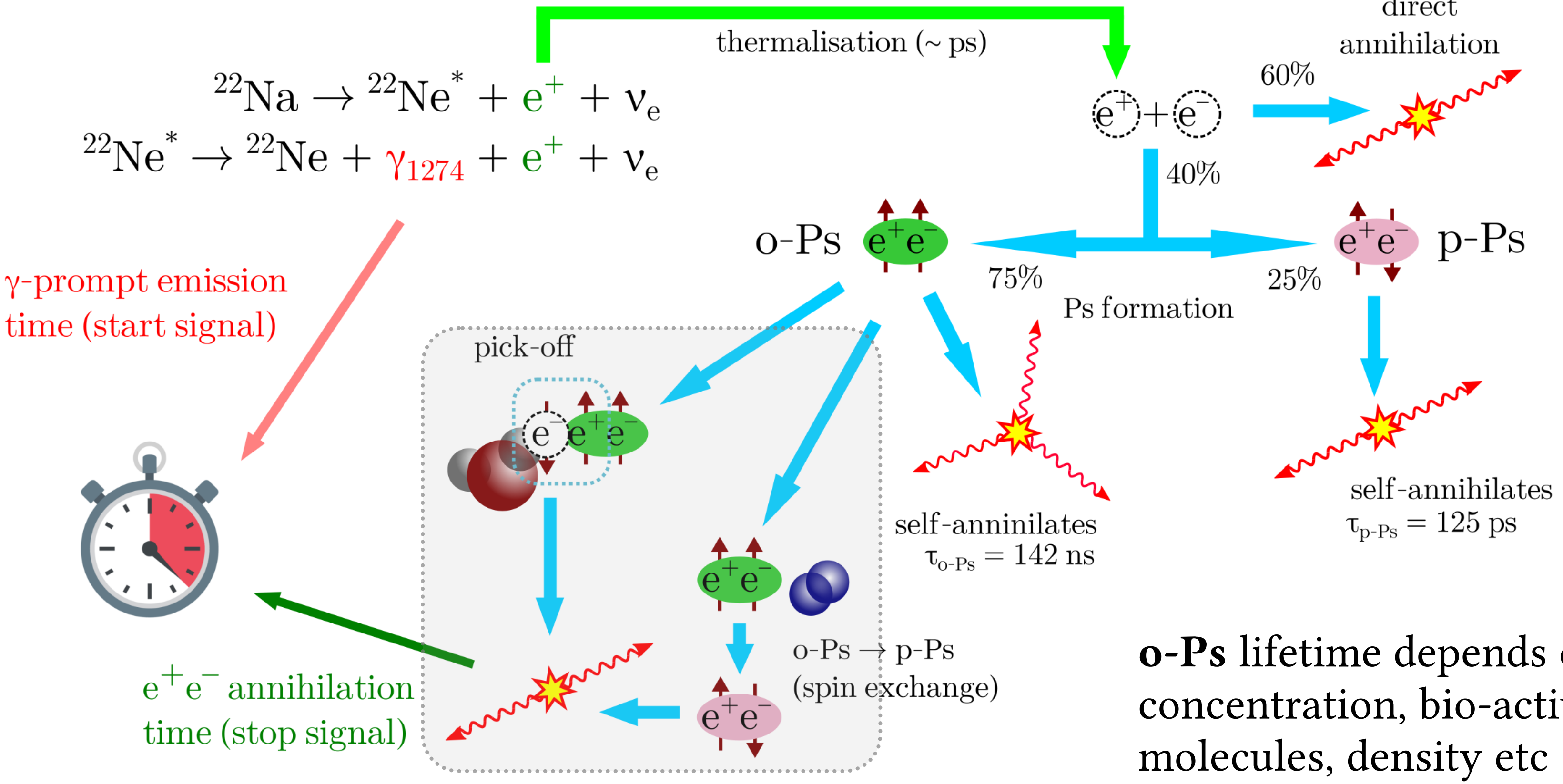
The SIMPLE algorithm

Method of moments

SIMPLE-Moment

Preliminary results for the simulated study

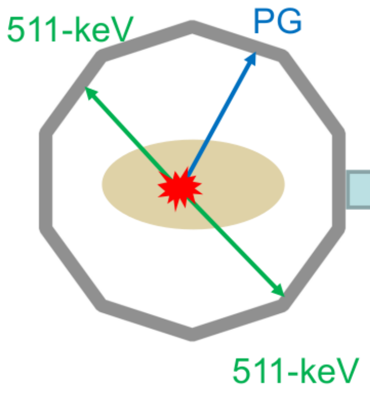
Positronium decay



o-Ps lifetime depends on O_2 concentration, bio-active molecules, density etc

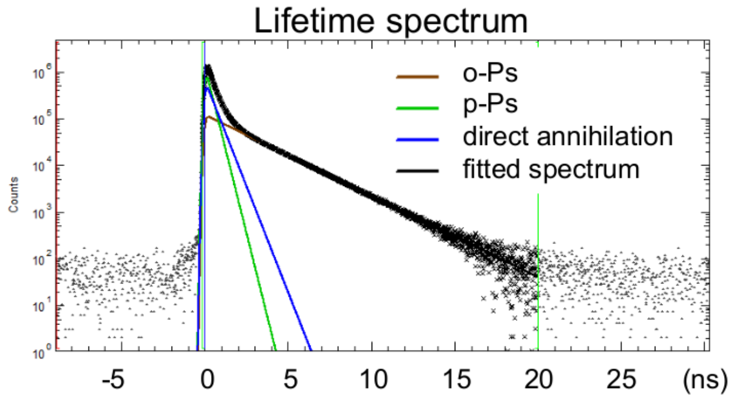
Ps lifetime spectrum

From J.Qi presentation (Jagiellonian symposiun 2024)



Lifetime measurement
(after travel-time correction)

$$\tau = \frac{(t_{1st\ 511} + t_{2nd\ 511})}{2} - t_{PG}$$



List-mode event parameters

- LOR i_k, including TOF information
- Time delay τ_k

Likelihood model of time measurement

$$p(\tau|\boldsymbol{\lambda}, \boldsymbol{A}) = \sum_l g(\tau) * [A_l \lambda_l \exp(-\lambda_l \tau) u(\tau)] + B$$

- λ_l: 1/lifetime of the l^{th} pathway
- A_l: Fraction of the l^{th} pathway
- $g(\tau)$: Detector timing response
- $u(\tau)$: Heaviside function
- B: random background events

The SIMPLE algorithm

SIMPLE – Statistical IMage reconstruction of Positron annihilation LifetimE

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Article

Fast High-resolution Lifetime Image Reconstruction for Positron Lifetime Tomography

Jinyi Qi, Bangyan Huang, Zipai Wang, Xinjie Zeng, Amir Goldan

This is a preprint; it has not been peer reviewed by a journal.

https://doi.org/10.21203/rs.3.rs-5045821/v1

PI event model

Positronium imaging (PI), also – positronium lifetime imaging (PLI)

J.Qi (doi.org/10.21203/rs.3.rs-5045821/v1)

The distribution of the lifetime measurement of true PI events (no randoms, no background):

$$p(\tau|\boldsymbol{\lambda}, \boldsymbol{A}) = g(\tau - \mu) * \sum_{l \in \{o,p,d\}} A_l \lambda_l \, exp(-\lambda_l \tau) \, u(\tau) \qquad (1)$$

where $u(t)$ is the unit step function. The subscript $l \in \{o, p, d\}$ denotes an annihilation pathway corresponding to o-Ps, p-Ps and direct annihilation, respectively. A_l denotes the intensity of the l^{th} pathway and λ_l is the annihilation rate (i.e., the inverse of the lifetime). $g(\tau - \mu)$ is a Gaussian function accounting for the detector timing response, where μ is the timing offset. The

The Ps lifetime *m* at voxel *j*:

$$m_j = \int_{-\infty}^{+\infty} \tau \times p(\tau|\boldsymbol{\lambda}_j, \boldsymbol{A}_j) d\tau - \mu.$$

NOT o-Ps, with the expected *m* covering *all decay mechanisms* (*j* is omitted here)

$$m = E[\tau] = \int_{-\infty}^{\infty} \tau \, p(\tau|\boldsymbol{\lambda}, \boldsymbol{A}) d\tau = \frac{A_o}{\lambda_o} + \frac{A_p}{\lambda_p} + \frac{A_d}{\lambda_d}$$

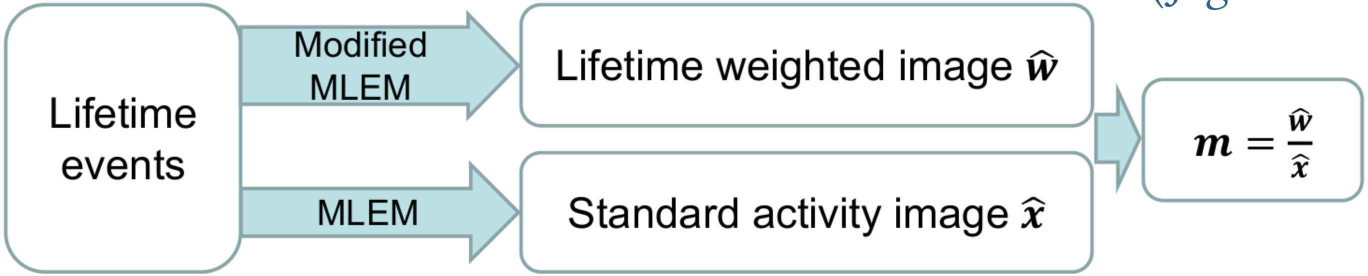
The SIMPLE algorithm

J.Qi (doi.org/10.21203/rs.3.rs-5045821/v1)

Ps lifetime image m -- from activity $\boldsymbol{x}$
as a weighted $\boldsymbol{w}$:

$$w_j = x_j m_j,$$

From J.Qi presentation
(Jagiellonian symposiun 2024)



- Computation cost equivalent to reconstructing two PET images

[1]Huang, et al., Statistical Image Reconstruction of Positron Lifetime via Time-Weighting (SIMPLE), 2022 IEEE MIC, Milano, Italy, 2022.
[2]Huang and Qi, Fast Reconstruction of Positronium Lifetime Image by the Method of Moments, 17th International Meeting on Fully 3D Image Reconstruction in Radiology and Nuclear Medicine, Stony Brook, NY, USA, 2023.

The SIMPLE algorithm

J.Qi (doi.org/10.21203/rs.3.rs-5045821/v1)

Projection data vector $\boldsymbol{z}$ whose <u>*expectation*</u> is the forward projection of $\boldsymbol{w}$.

$$\bar{z}_i = \sum_j^N H_{ij} w_j = \sum_j^N H_{ij} x_j m_j = \sum_j^N E[y_{ij}] m_j$$

$$E[y_{ij}] m_j = E\left[\sum_{k \in K_{ij}} \tau_k\right]$$

y_{ij} is the number of PI events originated in voxel j and detected in TOF sinogram bin i; N is the number of voxels; $\boldsymbol{H}$ and is a standard TOF system matrix whose element H_{ij} denotes the probability of detecting a coincident event originated in voxel j in TOF sinogram bin i.

$$\bar{z}_i = \sum_j^N E[y_{ij}] m_j = \sum_j^N E\left[\sum_{k \in K_{ij}} \tau_k\right] = E\left[\sum_{k \in K_i} \tau_k\right],$$

$$K_i = \bigcup_j K_{ij}$$

Projection data is a sum of the measured lifetimes:

$$z_i = \sum_{k \in K_i} \tau_k$$

K_{ij} denotes the set of list-mode indices of the events originated in voxel j and detected in LOR i

The SIMPLE algorithm

J.Qi (doi.org/10.21203/rs.3.rs-5045821/v1)

minimizing the cross-entropy (Kullback-Leibler) distance between the data vector z and the forward projection $\boldsymbol{Hw}$

$$\hat{\boldsymbol{w}} = \underset{\boldsymbol{w}}{\operatorname{argmin}}\, KL(\boldsymbol{z}, \boldsymbol{Hw})$$

Surrogate function (via optimisation transfer):

$$\phi(\boldsymbol{w}; \boldsymbol{w}^n)$$

$$= \sum_{i=1}^{M} \left[-z_i \sum_{j=1}^{N} \frac{H_{ij} w_j^n}{\sum_{l=1}^{N} H_{il} w_l^n} \log(w_j) + \sum_{j=1}^{N} H_{ij} w_j \right]$$

As a result – MLEM-like iterative update

$$w_j^{(n+1)} = \frac{w_j^{(n)}}{\sum_i H_{ij}} \sum_i \frac{H_{ij} z_i}{\sum_m H_{im} w_m^{(n)}}$$

In list-mode:

$$w_j^{(n+1)} = \frac{w_j^{(n)}}{\sum_i H_{ij}} \sum_k \frac{H_{i_k j} \tau_k}{\sum_m H_{i_k m} w_m^{(n)}}$$

Each event is weighted by its time delay

The SIMPLE (experiment)

J.Qi (doi.org/10.21203/rs.3.rs-5045821/v1)

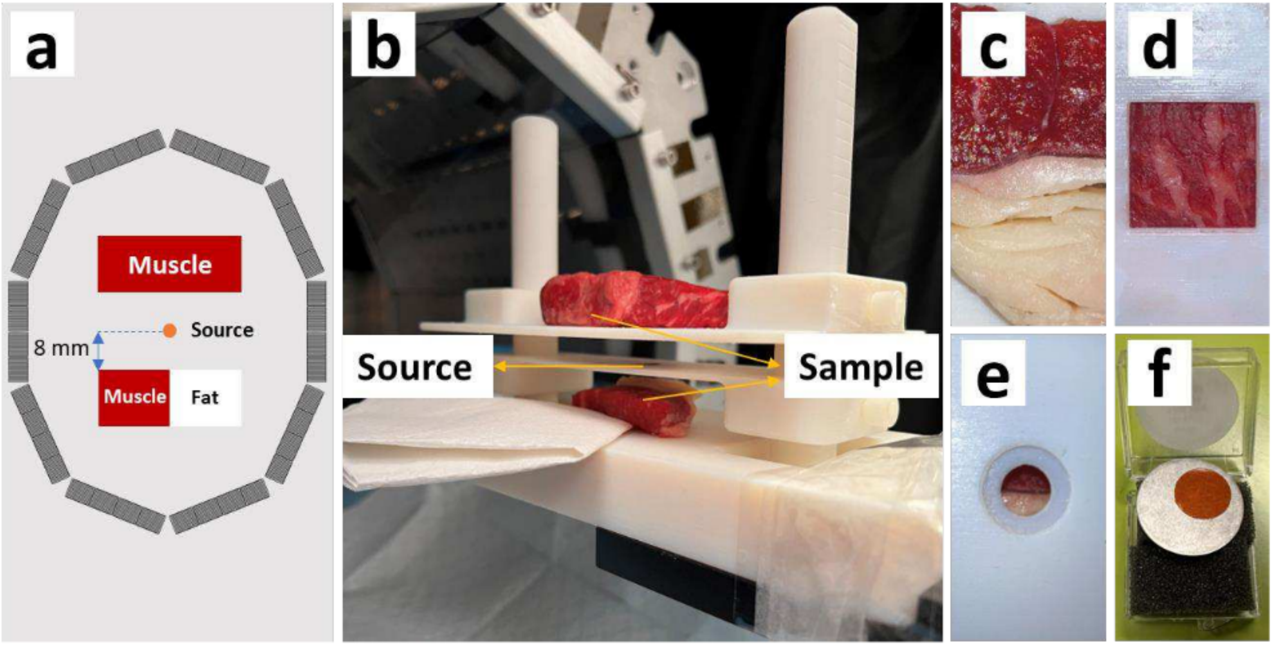


Fig. 2. (a) Schematic illustration of the experimental setup of the source and biological tissue. The point source is suspended over the sample with an 8-mm clearance. (b) The 3D-printed holder with the source and sample. (c) The bottom beef sample consists of muscle and fat tissue. (d) The top beef sample. (e) The tray for the ^{22}Na source looked from above. Positrons pass through the hole and irradiate the bottom sample. (f) The ^{22}Na point source (orange disk).

The SIMPLE (experiment)

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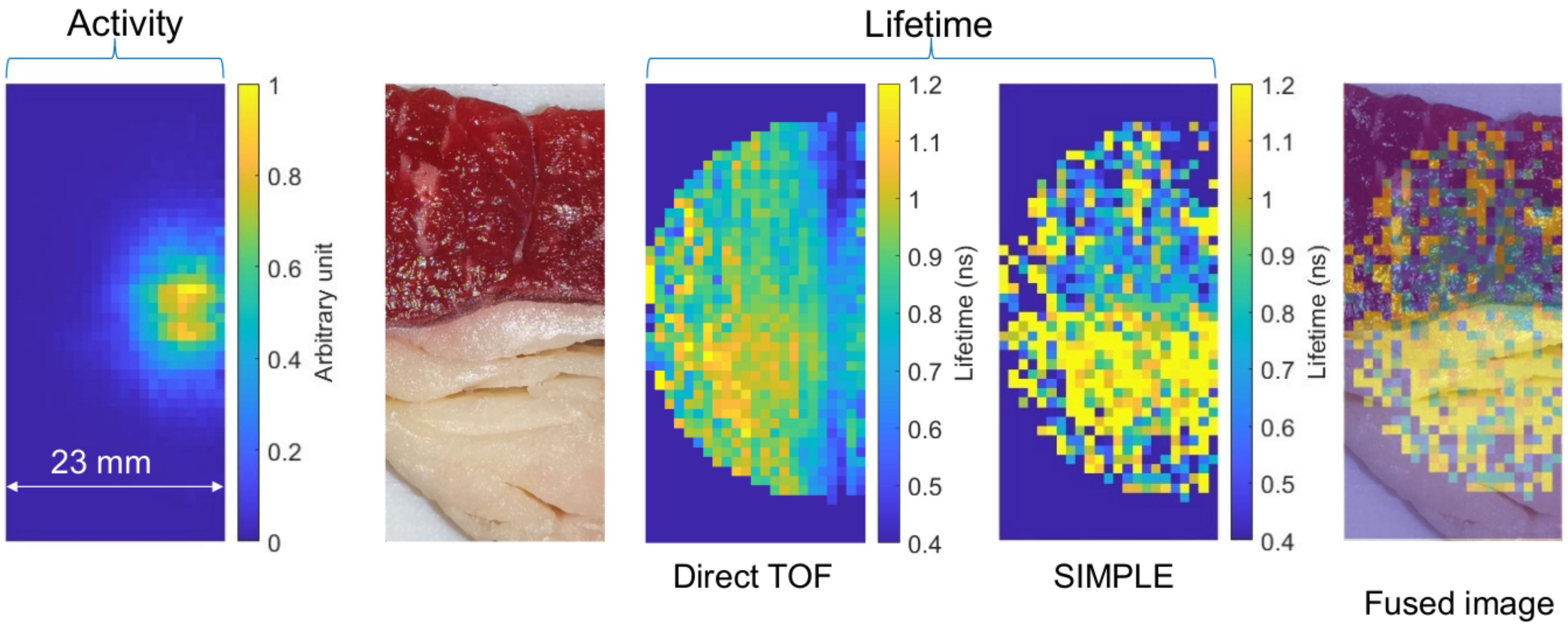
From J.Qi presentation (Jagiellonian symposiun 2024)

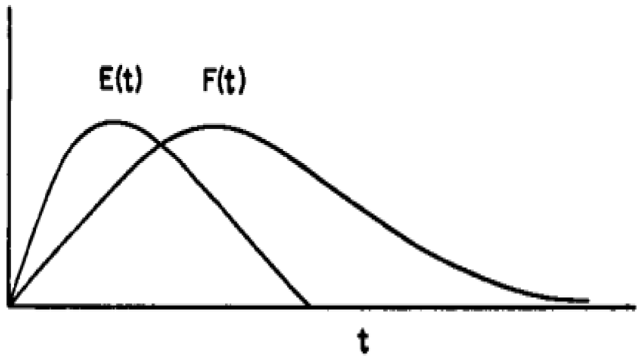


- The lifetime image reconstructed by our SIMPLE method shows clear contrast between fat and muscle

Huang et al, 2023 IEEE MIC

Method of moments

Isenberg I and Dyson RD (1969) Biophys. J.



$$f(t) = \sum_{n=1}^{N} \alpha_n e^{-\lambda_n t}$$

Let $E(t)$ be the intensity of the exciting lamp and $F(t)$ the emission of the sample when excited by $E(t)$.

$E(t)$ and $F(t)$ are shown, schematically, in Fig. 1.

$F(t)$ is given by the convolution

$$F(t) = \int_0^t E(t-u)f(u)\,du. \qquad (2)$$

$E(t)$ and $F(t)$ constitute, of course, the experimentally determined data.

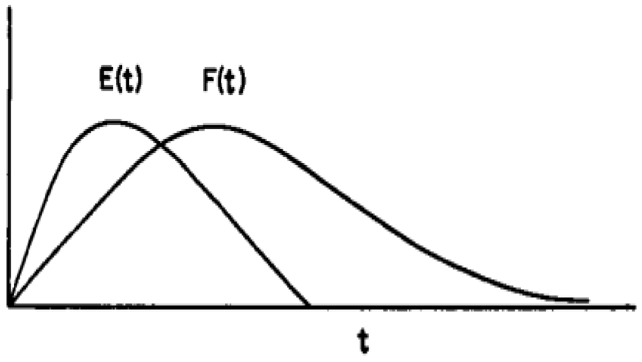
Let us define the k^{th} moments of F and E by

$$\mu_k = \int_0^\infty t^k F(t)\,dt \qquad (3)$$

and

$$m_k = \int_0^\infty t^k E(t)\,dt. \qquad (4)$$

Method of moments



$$f(t) = \sum_{n=1}^{N} \alpha_n e^{-\lambda_n t}$$

Taking the k^{th} moment of F, we obtain

$$\mu_k = \int_0^\infty t^k \int_0^t E(t-u) f(u)\, du\, dt. \quad (5)$$

The right hand side of equation 5 is the double integral of $t^k E(t-u) f(u)$, in the u, t plane, taken over the area bounded by the lines $u = 0$ and $u = t$. By reversing the order of integration, one obtains

$$\mu_k = \int_0^\infty f(u) \int_0^\infty (u+v)^k E(v)\, dv\, du \quad (6)$$

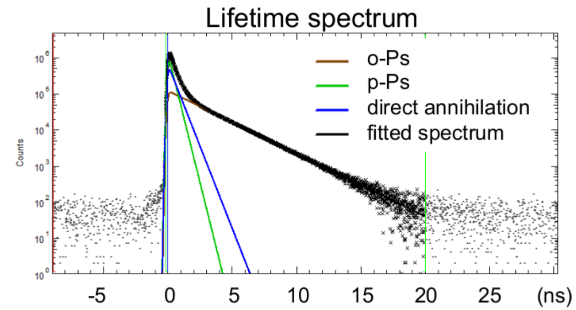
$$= k! \sum_{s=1}^{k+1} G_s\,(m_{k+1-s})/(k+1-s)! \quad (7)$$

$$G_s = \sum_{n=1}^{N} \alpha_n/\lambda_n^{\,s}$$

$$= \sum_{n=1}^{N} \alpha_n \tau_n^{\,s}.$$

Isenberg I and Dyson RD (1969) Biophys. J.

SIMPLE-Moment



$$f(\tau|A_{l,j},\lambda_{l,j},l\in\{o,p,d\}) = g(\tau) * \sum_{l\in\{\mathrm{o,p,d}\}} A_{l,j}\lambda_{l,j}\exp\left(-\lambda_{l,j}\tau\right)u(\tau)$$

The n-th moment of the lifetime $m_j^n = E[(\tau)^n] = \int_{-\infty}^{\infty}(\tau)^n f(\tau|A_{l,j},\lambda_{l,j},l\in\{\mathrm{o,p,d}\})$

$$m_j^n = \sum_{s=0}^{n}\frac{n!}{(n-s)!}\mu_{n-s}G_{s,j}$$

$\mu_k = E_{g(\tau)}\left[\tau^k\right]$ is the kth moment of the Gaussian distribution $g(\tau)$

$$G_{s,j} = \sum_{l\in\{\mathrm{o,p,d}\}}\frac{A_{l,j}}{\lambda_{l,j}^s}$$

$$\frac{A_\mathrm{o}}{A_\mathrm{p}} = 3 \Rightarrow \begin{cases} G_0 = 1 \\ G_1 = m^1 - \mu_1 \\ G_2 = \frac{1}{2}\left(m^2 - \mu_2 - 2\mu_1 G_1\right) \end{cases}$$

$$\lambda_\mathrm{o} = \frac{-B+\sqrt{B^2-4AC}}{2A}$$

$$\begin{cases} A = G_0\lambda_\mathrm{d} - G_1\lambda_\mathrm{d}^2 - G_0\lambda_\mathrm{p} + G_2\lambda_\mathrm{p}\lambda_\mathrm{d}^2 + 4G_1\lambda_\mathrm{p}^2 - 4G_2\lambda_\mathrm{p}^2\lambda_\mathrm{d}, \\ B = -3G_0\lambda_\mathrm{p}^2 + 3G_2\lambda_\mathrm{p}^2\lambda_\mathrm{d}^2, \\ C = 3G_0\lambda_\mathrm{p}^2\lambda_\mathrm{d} - 3G_1\lambda_\mathrm{p}^2\lambda_\mathrm{d}^2. \end{cases}$$

B Huang and J Qi (2024) PMB

How to reconstruct moments?

Reconstruct an activity-weighted moment image: $w_j^n = q_j x_j m_j^n$

(x_j is the radioactivity in voxel j, q_j is the detection probability of a prompt gamma emitted from voxel j)

A time-delay weighted projection data: $$z_i^n = \sum_{k \in K_i} (\tau_k)^n$$

Its expectation: $$E(z_i^n) = E\left[\sum_{k \in K_i} (\tau_k)^n\right] = \sum_j E\left[\sum_{k \in K_{ij}} (\tau_k)^n\right] \qquad K_i = \bigcup_j K_{ij}$$

Show that the expectation ~ forward projection of $\boldsymbol{w}$:

$$E\left[\sum_{k \in K_{ij}} (\tau_k)^n\right] = E\left(\left|K_{ij}\right|\right) m_j^n \qquad E\left(\left|K_{ij}\right|\right) = H_{ij} q_j x_j \qquad E(z_i^n) = \sum_j E\left[\sum_{k \in K_{ij}} (\tau_k)^n\right] = \sum_j H_{ij} q_j x_j m_j^n = \sum_j H_{ij} w_j^n$$

$\left|K_{ij}\right|$ is equal to the number of tri-coincidence events originated in voxel j and detected in LOR i

B Huang and J Qi (2024) PMB

How to reconstruct moments?

Reconstruct an activity-weighted moment image: $w_j^n = q_j x_j m_j^n$

(x_j is the radioactivity in voxel j, q_j is the detection probability of a prompt gamma emitted from voxel j)

$$E(z_i^n) = \sum_j E\left[\sum_{k \in K_{ij}} (\tau_k)^n\right] = \sum_j H_{ij} q_j x_j m_j^n = \sum_j H_{ij} w_j^n$$

Update (with the expectation of background events):

$$w_j^{n(l+1)} = \frac{w_j^{n(l)}}{\sum_i H_{ij}} \sum_{k \in S_l} \frac{H_{i_k j} (\tau_k)^n}{\sum_j H_{i_k j} w_j^{n(l)} + \bar{r}_{i_k}^n}$$

Three reconstructions with $n = 0, 1, 2$ (2 – for o-Ps).

$$m_j^n = \frac{w_j^n}{w_j^0}$$

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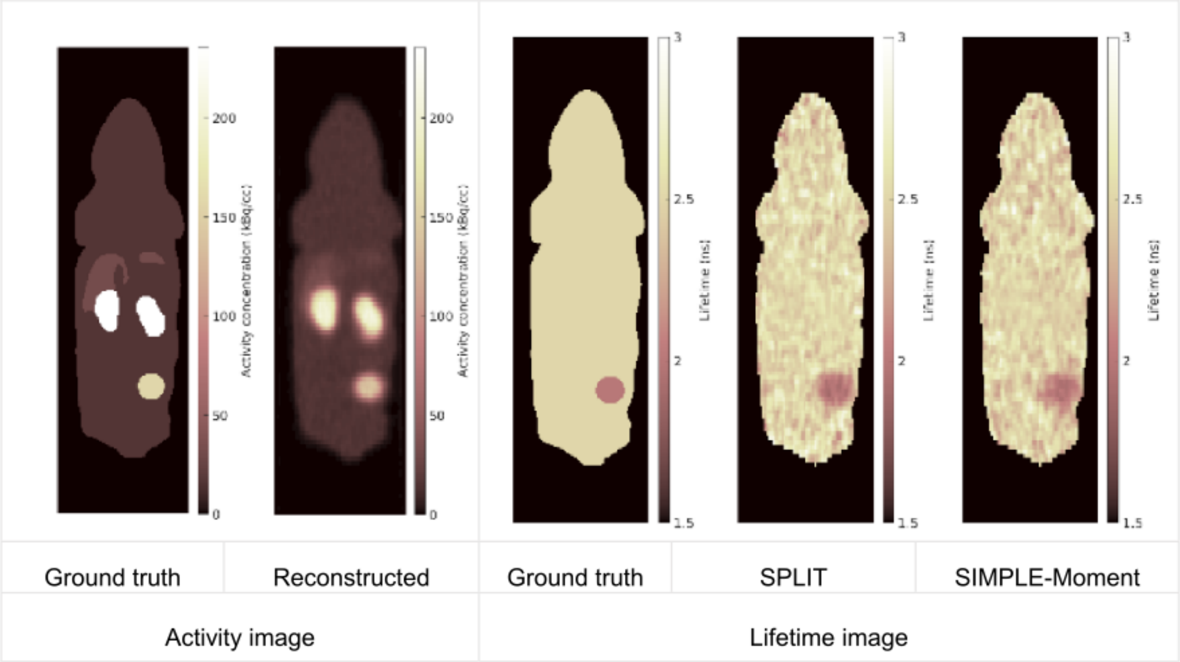
SIMPLE-Moment

Simulated in GATE for Neuro-EXPLORER, (R = 52 cm, L = 49 cm), ^{44}Sc 20kBq/mL, 30 min, CRT = 250 ps.

The results are compared with the SPLIT ML-based algorithm (more precise, more time-consuming)

Huang B et al. (2024) IEEE Trans. Med. Imaging



. The biases (standard deviations) of reconstructed lifetimes in the lesion, kidney, liver and body background (in ns)

	Lesion	Kidney	Liver	BG
True value	2.0	2.5	2.5	2.5
SPLIT	0.007(54)	0.028(59)	0.035(91)	0.014(136)
SIMPLE-Moment	0.015(56)	−0.006(55)	−0.002(87)	−0.003(132)

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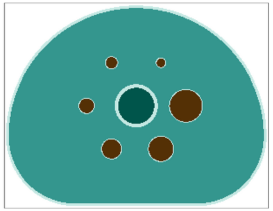
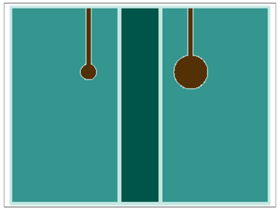
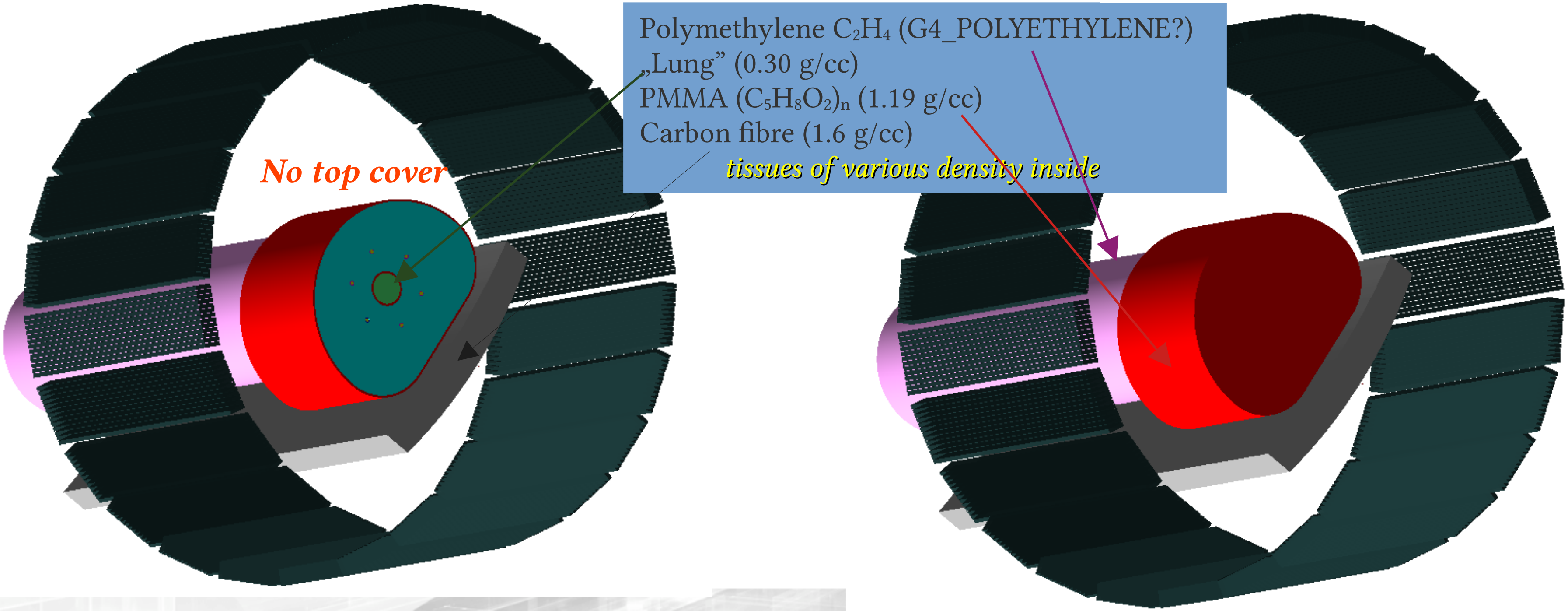
Geant4 simulation (Kamil Dulski)

Replicated geometry of the modular Jagiellonian PET + NEMA phantom

^{22}Na, no positron range. 7.8E+6 3-photon events (2.1E+6 - true)







Preliminary results

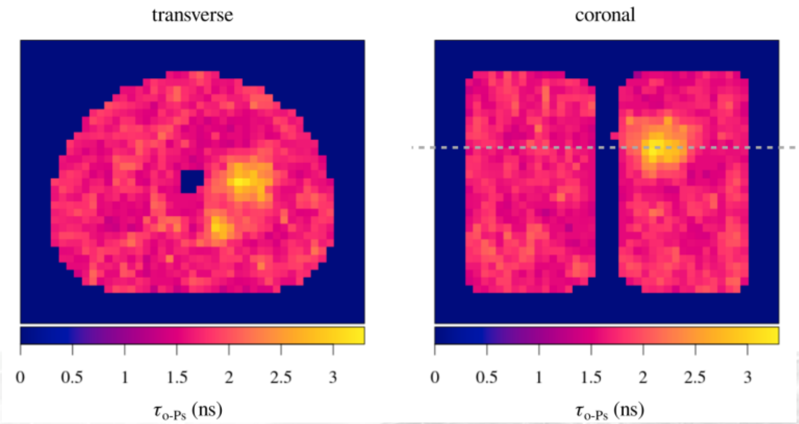
2.1E+6 true events

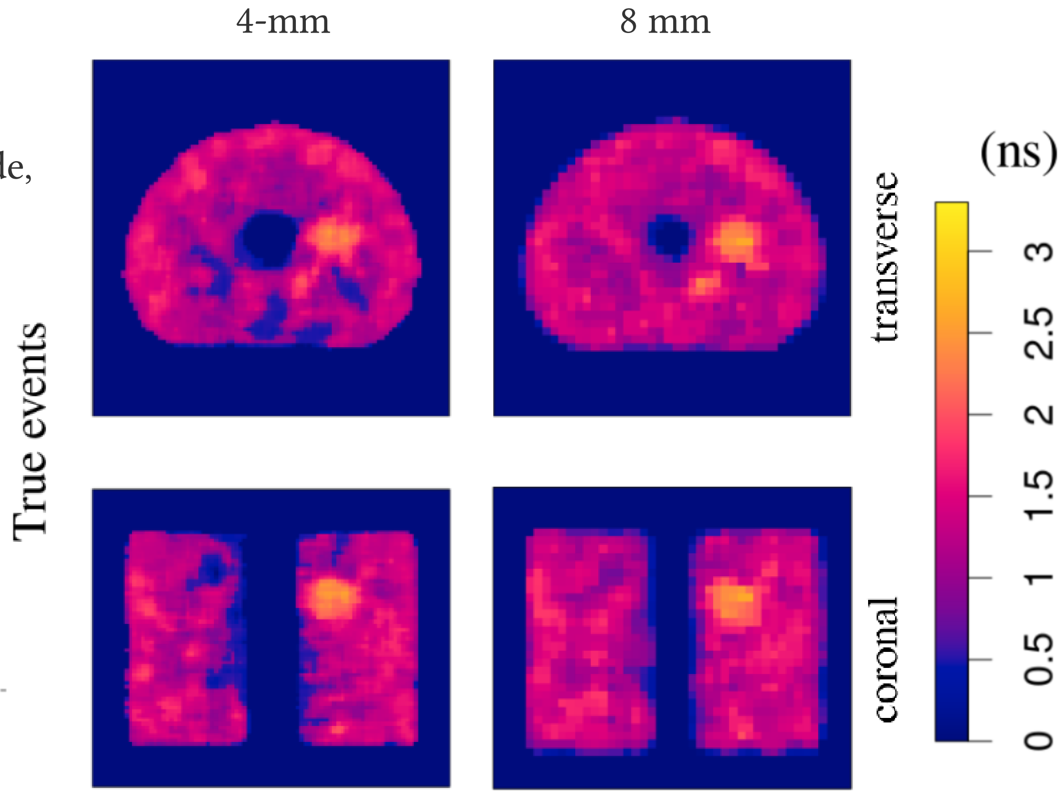
Original granularity – cubic voxels of 4-mm side,

Also downscaled to 8-mm

Median post-filter (MPF) applied
with a cubic mask of the radius R = 8 mm.

Reference reconstruction (no MPF)
(original PI algorithm), voxel side – 8 mm





SIMPLE-Moment

Thank You for Your attention!