Finite Accretion Disks in the Kerr Space Time

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Kinetic theory

- In galaxies or globular clusters, stars act like gas particles, with collisions between them being so rare that they can be ignored to some extent. These stars interact solely through their gravitational fields and can be described by the Einstein-Vlasov system.

- In the case of a collisionless gas, matter is described by the particle distribution function $\mathcal{F}$ defined on the phase space. The distribution function $\mathcal{F}$ is a non-negative function supported on the mass shell.

$$\mathrm{PM} := \{g_{\alpha\beta}p^{\alpha}p^{\beta} = -m_0^2,\ p^{\alpha} \text{ future pointing}\}$$

Other applications

- Modeling of plasma near black holes (magnetic filed needed).

- Modeling dark matter.

Vlasov equation

- Vlasov gas can be understood as a collection of particles moving along timelike future-directed geodesic lines. For collisionless particles, the Vlasov equation can be expressed as follows:

$$\frac{d\mathcal{F}}{d\tau} = \frac{\partial\mathcal{F}}{\partial x^{\mu}}\frac{dx^{\mu}}{d\tau} + \frac{\partial\mathcal{F}}{\partial p_{\nu}}\frac{dp_{\nu}}{d\tau} = \frac{\partial\mathcal{F}}{\partial x^{\mu}}\frac{\partial H}{\partial p_{\mu}} - \frac{\partial\mathcal{F}}{\partial p_{\nu}}\frac{\partial H}{\partial x^{\nu}} = \{H, \mathcal{F}\} = 0$$

The distribution function $\mathcal{F}$ is not an observable quantity on its own. Our focus is on physical quantities.

Physical quantities

- Let $(\mathcal{M}, g)$ denote the spacetime manifold. The cotangent bundle on $(\mathcal{M}, g)$ is defined as:

$$T^{*}\mathcal{M} = \{(x,p) \colon x \in \mathcal{M},\, p \in T_{x}^{*}\mathcal{M}\}$$

$$P_{x}^{+} = \{p \in T_{x}^{*}\mathcal{M} \colon g^{\mu\nu}p_{\mu}p_{\nu} < 0, p \text{ is future directed}\}$$

Physical quantities

- Let S be a three-dimensional spacelike surface in $\mathcal{M}$. The average number of particle trajectories whose projections on $\mathcal{M}$ intersect S:

$$\mathrm{N(S)} = -\int_{S} \left[\int_{p_x^+} \mathcal{F}(x,p)p_\mu\, s^\mu \mathrm{dvol}_x(p)\right] \eta_S$$

Where s is a unit vector normal to S, η_s denotes the three-dimensional volume element. The dvol_x denotes the volume element in P_x^+ :

$$\mathrm{dvol}_x(p) = \sqrt{-\det g^{\mu\nu}(x)}\, dp_0 dp_1 dp_2 dp_3.$$

- The expression for $N(S)$:

$$\mathrm{N(S)} = -\int_{S} J_\mu s^\mu \eta_S\,.$$

- The energy momentum tensor in the Vlasov model is defined as:

$$T_{\mu\nu}(x) = \int_{P_x^+} \mathcal{F}(x,p)p_\mu p_\nu\, \mathrm{dvol}_x(p).$$

Model

- We consider Bondi-type accretion in the Kerr spacetime, assuming that the gas is confined to the equatorial plane (thin disk).

- In a physical situation, the accretion starts from a finite distance r_0. The boundary conditions are assumed at a finite radius, at which matter is added to the system.

- In horizon-penetrating coordinates the Kerr metric can be written as:

$$g = -dt^2 + dr^2 - 2a\sin^2\vartheta dr d\varphi + (r^2 + a^2)\sin^2\vartheta d\varphi^2 + \rho^2 d\vartheta^2 + \frac{2\,Mr}{\rho^2}(dt + dr - a\sin^2\vartheta d\varphi)^2$$

$$\rho^2 = r^2 + a^2 \sin\vartheta^2$$

- We used distribution functions of the type:

$$\mathcal{F}(x,p) = A\,\delta(m - m_0)\delta\left(\theta - \frac{\pi}{2}\right)\delta(p_\theta)\,f_0(\epsilon,\lambda)$$

Dimensionless Coordinates

$$E = -p_t \qquad l_z = p_\varphi \qquad l = \epsilon_\sigma(p_\varphi + a p_t)$$

- We use the following dimensionless variables

$$r = M\xi \qquad a = M\alpha \qquad E = m\epsilon \qquad l = Mm\lambda \qquad l_z = Mm\,\lambda_z$$

Radial potential and its critical points

- Orbits confined to the equatorial plane are governed by the radial potential $\tilde{R}(\xi)$:

$$\tilde{R}(\xi) = \xi^4[\epsilon - W_-(\xi)][\epsilon - W_+(\xi)] \qquad \tilde{R}(\xi) \geq 0$$

$$W_\pm(\xi) = \frac{\epsilon_\sigma \alpha \lambda}{\xi^2} \pm \frac{\sqrt{\tilde{\Delta}(\xi^2 + \lambda^2)}}{\xi^2}$$

$$\lambda_c = \frac{\xi(\sqrt{\xi} - \epsilon_\sigma \alpha)}{\sqrt{\xi(\xi - 3) + 2\epsilon_\sigma \alpha \sqrt{\xi}}}$$

Critical angular momentum

$$\epsilon_c = \frac{\xi^2 - 2\xi + \epsilon_\sigma \alpha \sqrt{\xi}}{\xi\sqrt{\xi(\xi - 3) + 2\epsilon_\sigma \alpha \sqrt{\xi}}}$$

Critical energy

Minimum energy with respect to λ (angular momentum)

- What is the minimum allowed energy ϵ of a particle that can reach ξ_0?

$$W_{\pm}(\xi) = \frac{\epsilon_\sigma \alpha \lambda}{\xi^2} \pm \frac{\sqrt{\widetilde{\Delta}(\xi^2 + \lambda^2)}}{\xi^2}$$

$$\frac{dW_+}{d\lambda} = \frac{\epsilon_\sigma \alpha \sqrt{\lambda^2 + \xi^2} + \lambda\sqrt{\widetilde{\Delta}}}{\xi^2\sqrt{\lambda^2 + \xi^2}} = 0$$

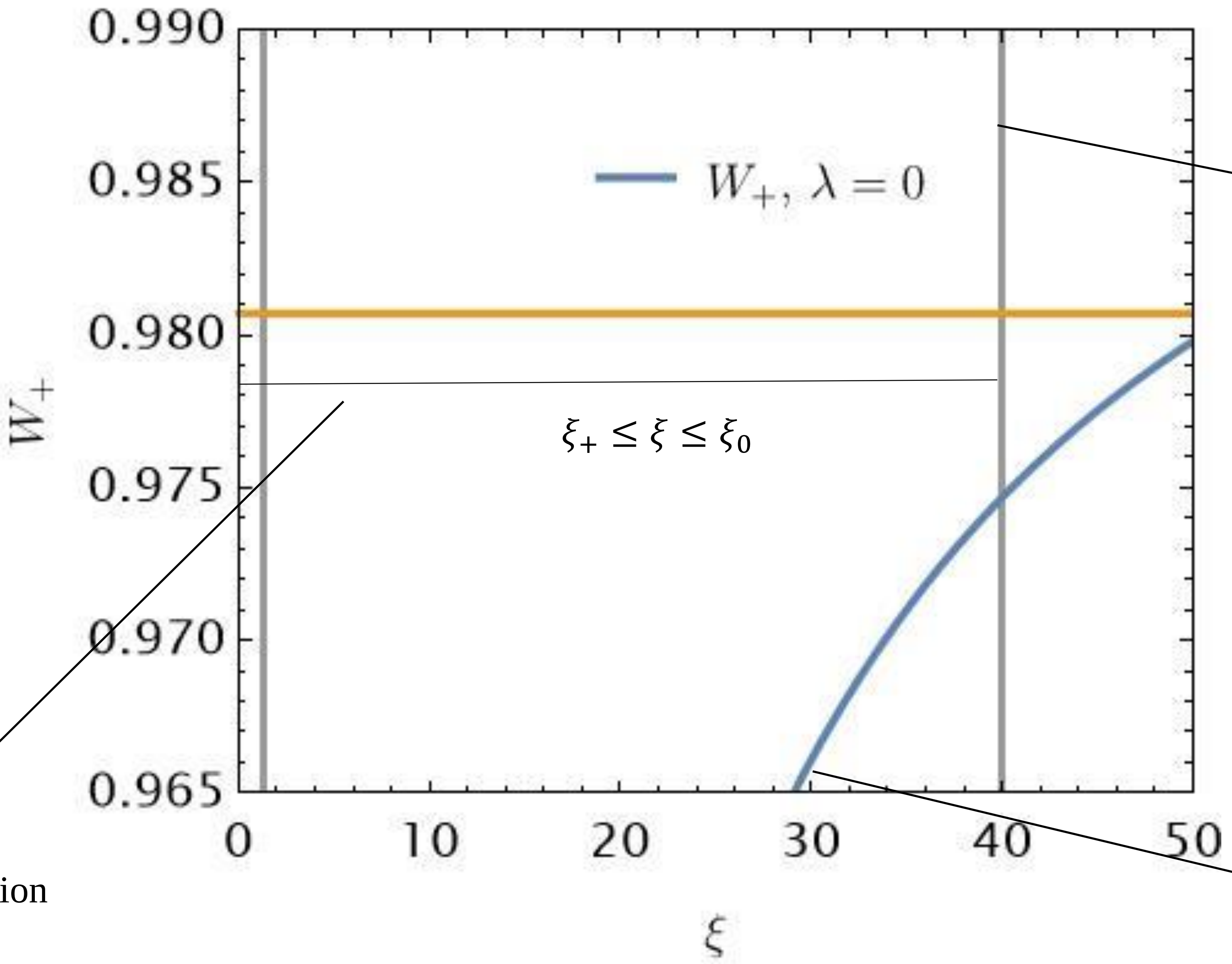
- If $\epsilon_\sigma \alpha < 0$ and $\xi > 2$

$$W_{min}(\xi) = W_+(\xi)|_{\lambda = -\epsilon_\sigma \alpha\sqrt{\xi}/\sqrt{\xi - 2}} = \sqrt{1 - \frac{2}{\xi}}$$

- If $\epsilon_\sigma \alpha > 0$ and $\xi > 2$, the value of λ becomes negative, which is physically meaningless in this context. Therefore, we set $\lambda = 0$ in

$$W_{min}(\xi) = W_+(\xi)|_{\lambda = 0} = \sqrt{1 - \frac{2}{\xi} + \frac{\alpha^2}{\xi^2}}$$

Critical energy, bound on λ



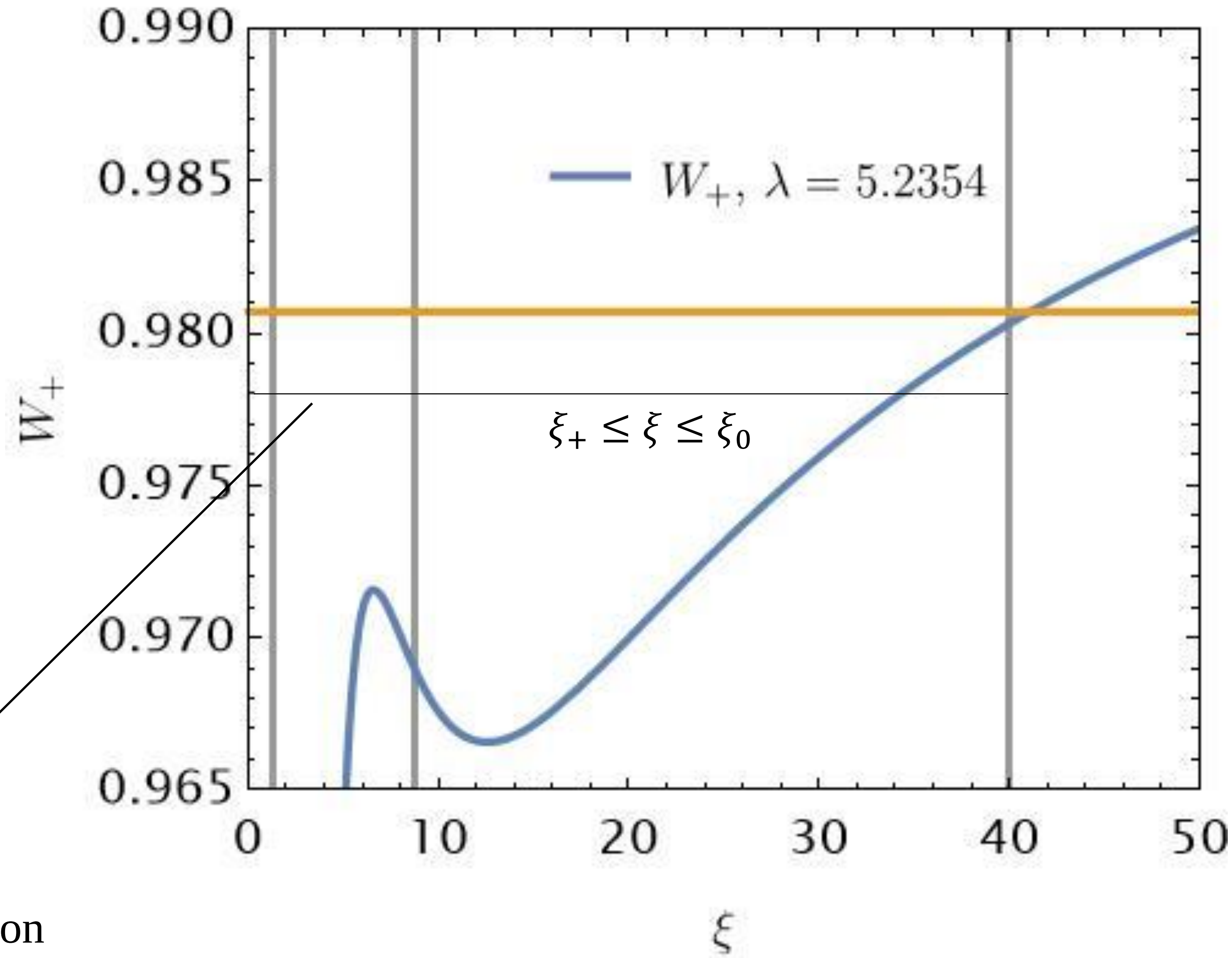
ξ_0

$\xi_+ \leq \xi \leq \xi_0$

λ_{min}

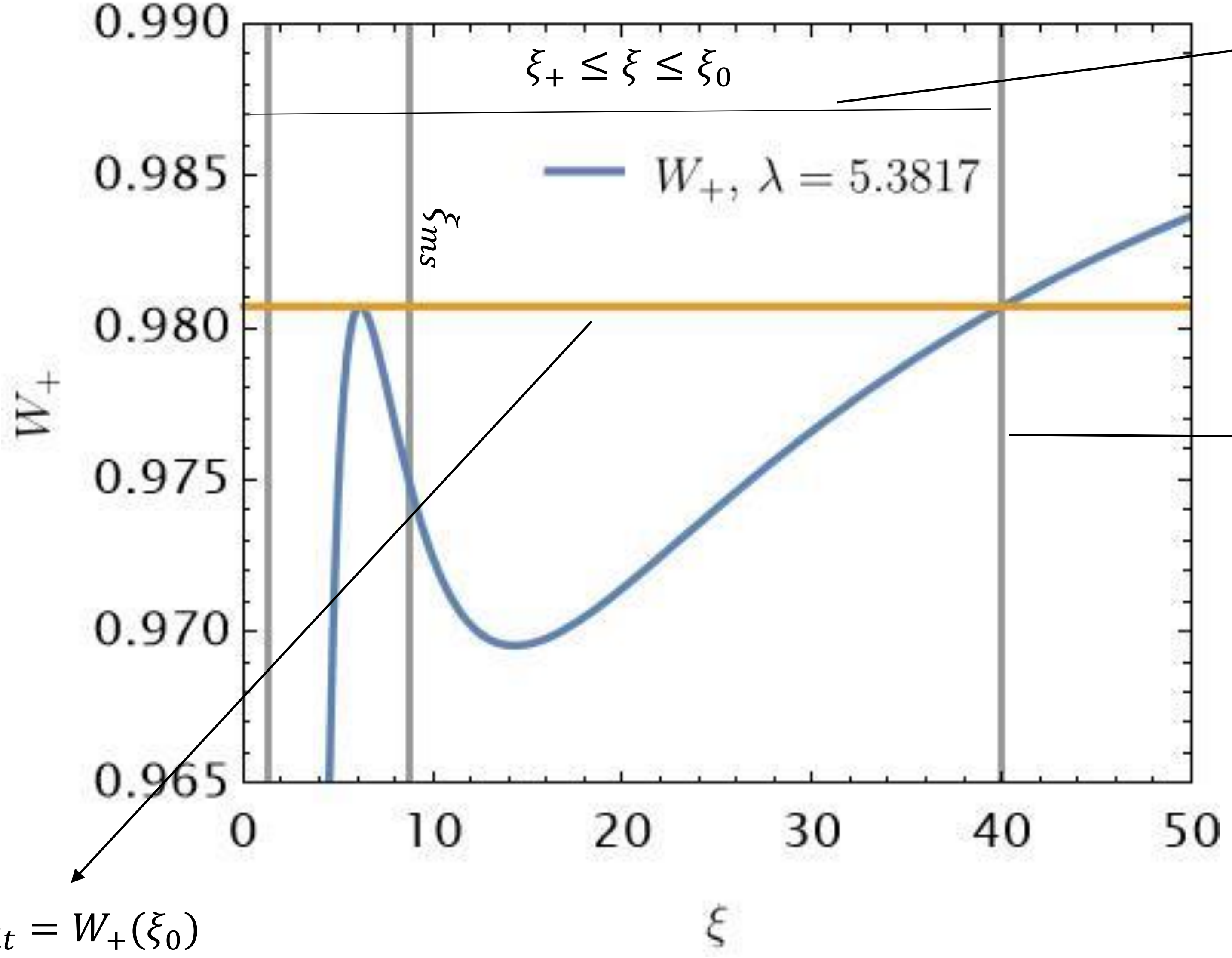
Region allowed for motion

Critical energy, bound on λ



Region allowed for motion

Critical energy, bound on λ



Region allowed for motion

ξ_0

$\epsilon_{crit} = W_+(\xi_0)$

ξ_+ = Horizon

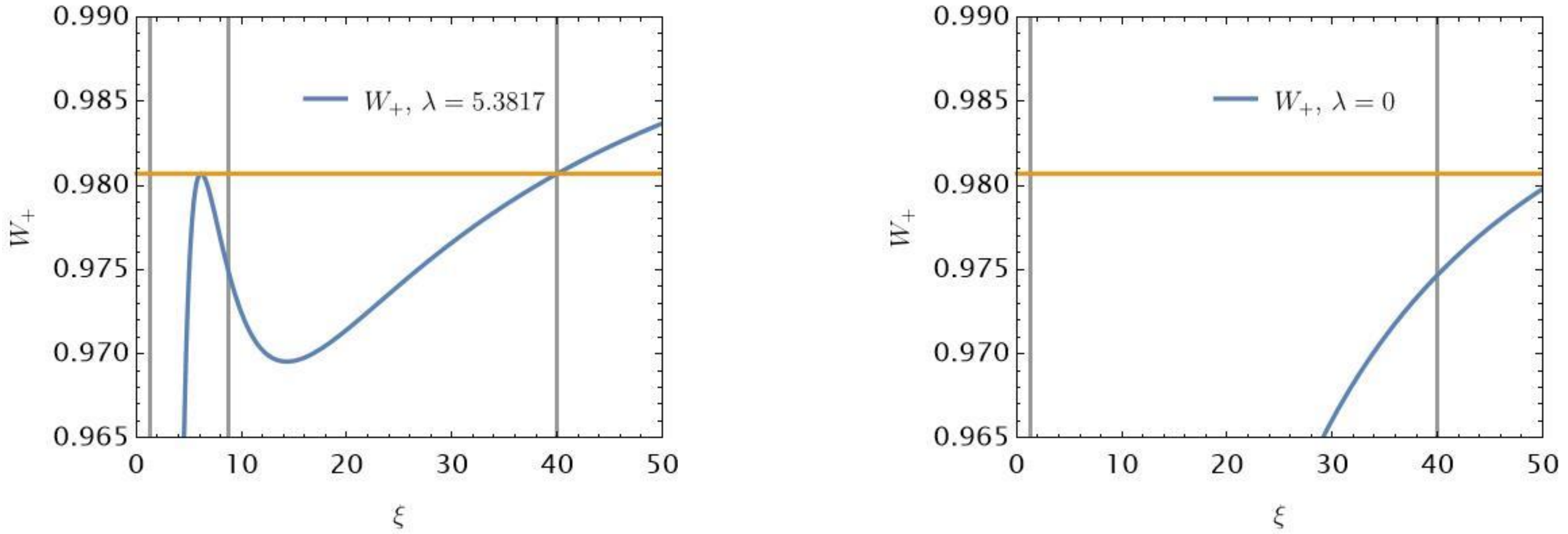
ϵ_{crit} = Critical energy

λ_{crit} = Critical angular momentum

ξ_{ms} = Marginally stable orbit

ξ_0 = Finite radius

Phase-space parameters governing accretion from a finite radius



- Absorbed Orbits

$$W_{\min}(\xi_0) \le \epsilon \le \epsilon_{\mathrm{crit}}(\xi_0) \qquad \text{and} \qquad 0 \le \lambda \le \lambda_{\max}(\xi_0, \epsilon, \epsilon_\sigma)$$

$$\epsilon_{\mathrm{crit}}(\xi_0) < \epsilon < +\infty \qquad \text{and} \qquad 0 \le \lambda \le \lambda_c(\epsilon)$$

- Scattered Orbits

$$\epsilon_{\min}(\xi) < \epsilon < +\infty \qquad \text{and} \qquad \lambda_c(\epsilon) < \lambda \le \lambda_{\max}(\xi, \epsilon, \epsilon_\sigma)$$

Particle current surface density

- Component of particle current surface density

$$J = J_\mu^{scat} + J_\mu^{abs}$$

For a stationary planar model

$$J_t = -Am_0^3\xi \sum_{\epsilon_\sigma=\pm1} \int \frac{f_0(\epsilon,\lambda)\epsilon \, d\epsilon \, d\lambda}{\sqrt{\widetilde{R}(\xi)}}$$

$$J^r = \frac{Am_0^3}{\xi} \sum_{\epsilon_\sigma=\pm1} \int \epsilon_r f_0(\epsilon,\lambda)\epsilon \, d\epsilon d\lambda$$

$$J_\varphi = Am_0^3 M \, \xi \sum_{\epsilon_\sigma=\pm1} \int \frac{f_0(\epsilon,\ \lambda)\ (\epsilon_\sigma\lambda + \alpha\ \epsilon)\ d\epsilon d\lambda}{\sqrt{\widetilde{R}(\xi)}}$$

- Energy-momentum surface density (needed in the computation of accretion rates):

$$T_t^r = -\frac{Am_0^4}{\xi} \sum_{\epsilon_\sigma=\pm1} \int \epsilon_r f_0(\epsilon,\lambda)\epsilon \, d\epsilon \, d\lambda$$

$$T_\varphi^r = \frac{AMm_0^4}{\xi} \sum_{\epsilon_\sigma=\pm1} \int \epsilon_r f_0(\epsilon,\lambda)(\epsilon_\varphi\lambda + \alpha\epsilon) d\epsilon d\lambda$$

Particle current surface density with integral limits

- We divide the particle current surface density into two parts, corresponding to absorbed trajectories (abs) and scattered trajectories (scat)

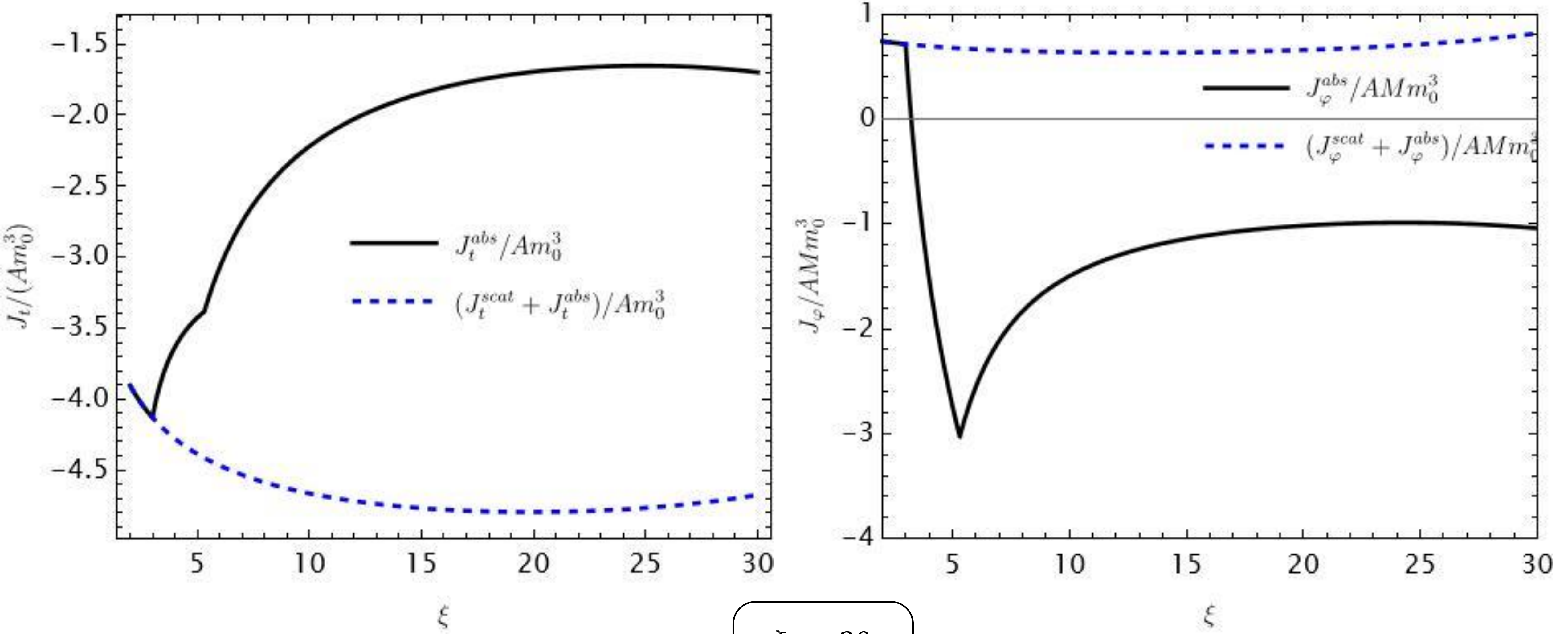
$$J_t^{abs} = -A\, m_0^3\, \xi \sum_{\epsilon_\sigma=\pm 1}\left[\int_{W_{min}(\xi_0,\,\epsilon_\sigma)}^{\epsilon_{crit}(\xi_0,\,\epsilon_\sigma)} d\epsilon \int_0^{\lambda_{max}(\xi_0,\epsilon,\epsilon_\sigma)} d\lambda\, \frac{f_0(\epsilon,\lambda)\epsilon}{\sqrt{\tilde{R}(\xi)}} + \int_{\varepsilon_{crit}(\xi_0,\,\epsilon_\sigma)}^{\infty} d\epsilon \int_0^{\lambda_c(\epsilon,\,\epsilon_\sigma)} d\lambda\, \frac{f_0(\epsilon,\lambda)\epsilon}{\sqrt{\tilde{R}(\xi)}}\right]$$

$$J_t^{scat} = -2\, A m_0^3\, \xi \sum_{\epsilon_\sigma=\pm 1}\left[\int_{\epsilon_{min}(\xi,\,\epsilon_\sigma)}^{\infty} d\epsilon \int_{\lambda_c(\epsilon,\,\epsilon_\sigma)}^{\lambda_{max}(\xi,\epsilon,\,\epsilon_\sigma)} d\lambda\, \frac{f_0(\epsilon,\lambda)\epsilon}{\sqrt{\tilde{R}(\xi)}}\right]$$

- Similar formulas for J_φ and J^r

Examples

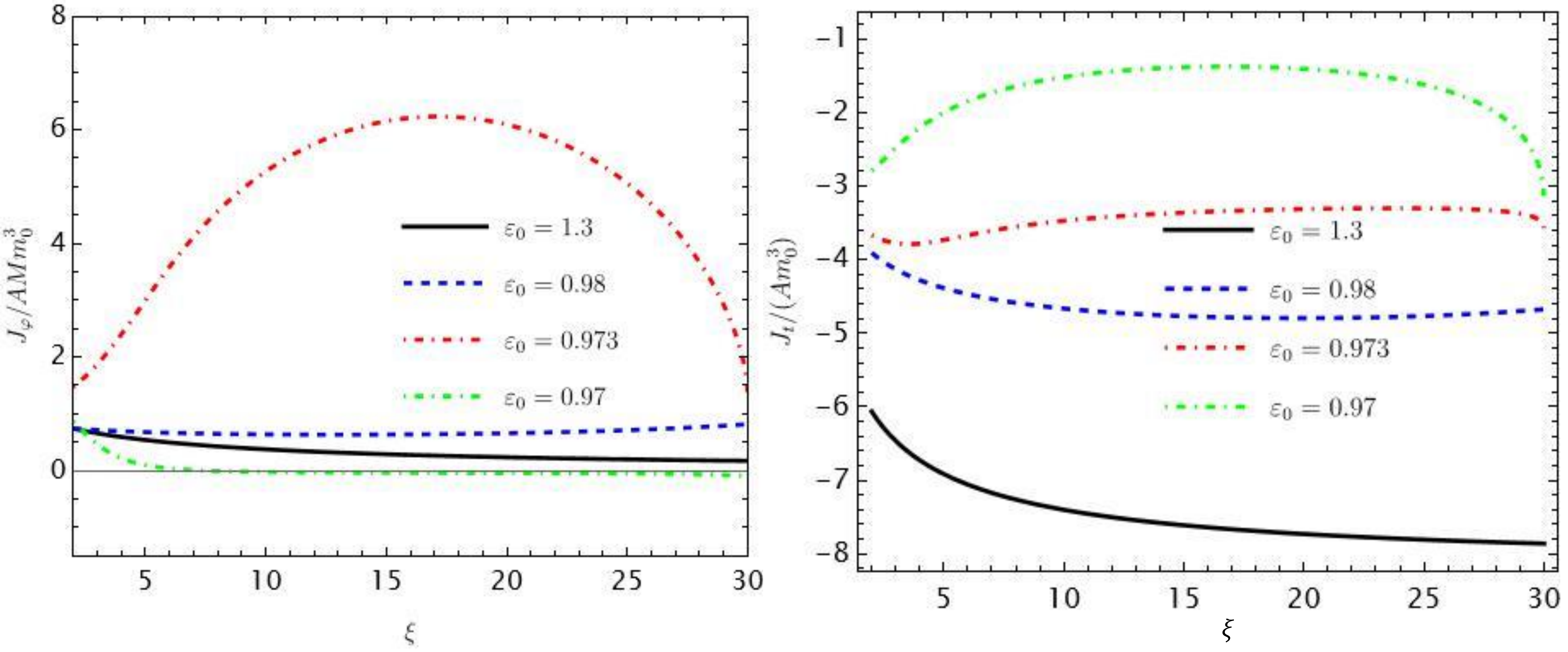
Monoenergetic distribution $f_0 = \delta(\epsilon - \epsilon_0)$



$\xi_0 = 30$
$\varepsilon = 0.98$
$\alpha = 0.5$

Examples

Monoenergetic distribution $f_0 = \delta(\epsilon - \epsilon_0)$



Accretion rates

$$\dot{M} = -2\,\pi r_1 m_0\, J^r$$ Mass accretion rate

$$\dot{\mathcal{E}} = 2\pi r_1 T^r_t$$ Energy accretion rate

$$\dot{\mathcal{L}} = -2\pi r_1 T^r_\varphi$$ Angular momentum accretion rate

Accretion rates with integral limits

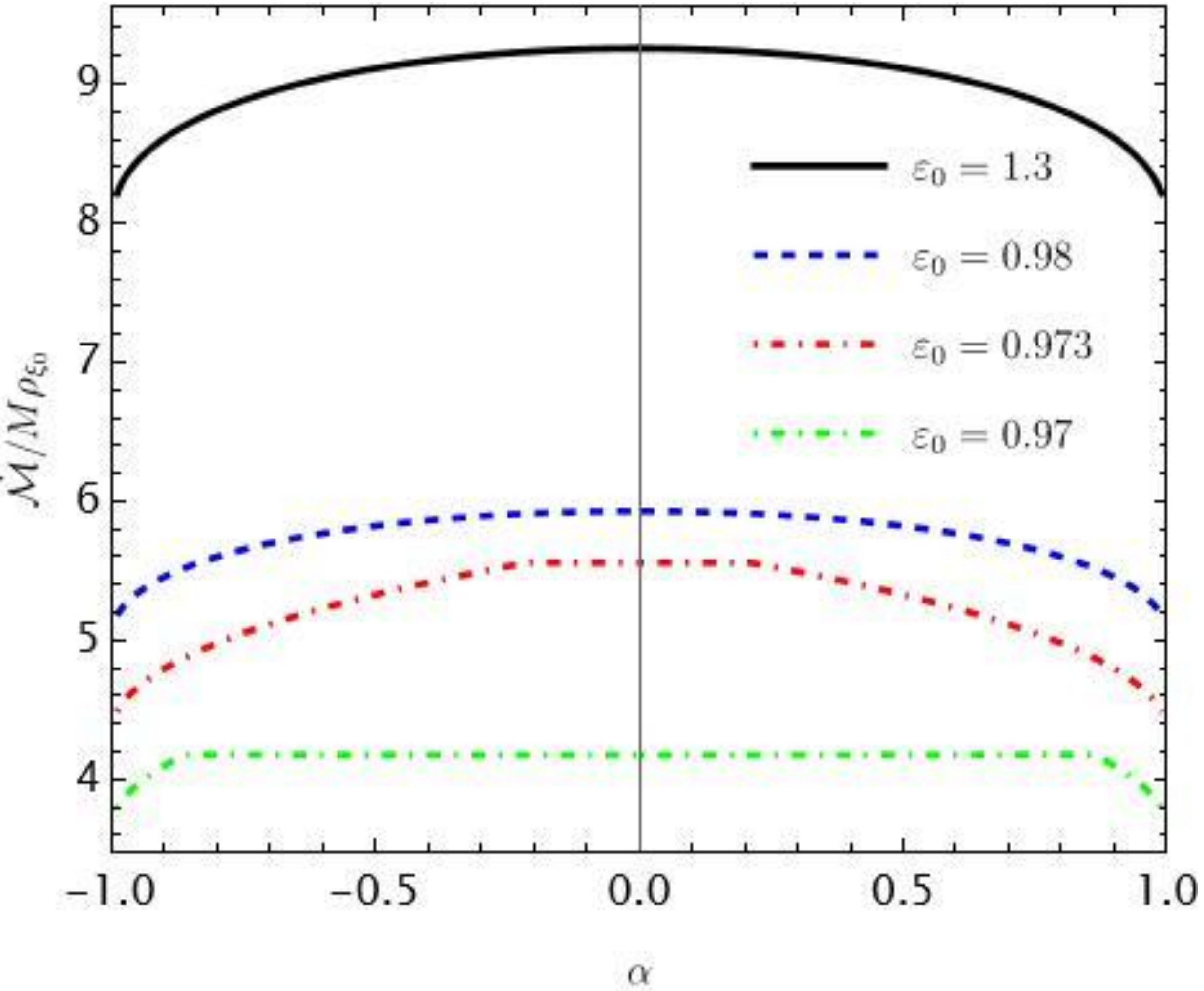
$$\dot{M} = 2\pi A m_0^4 \sum_{\epsilon_\sigma=\pm1} \left[\int_{W_{min}(\xi_0,\,\epsilon_\sigma)}^{\epsilon_{crit}(\xi_0,\epsilon_\sigma)} d\epsilon \int_0^{\lambda_{max}(\xi_0,\epsilon,\epsilon_\sigma)} d\lambda f_0(\epsilon,\lambda) + \int_{\epsilon_{crit}(\xi_0,\epsilon_\sigma)}^{\infty} d\epsilon \int_0^{\lambda_c(\epsilon,\,\epsilon_\sigma)} d\lambda\, f_0(\epsilon,\lambda)\right]$$

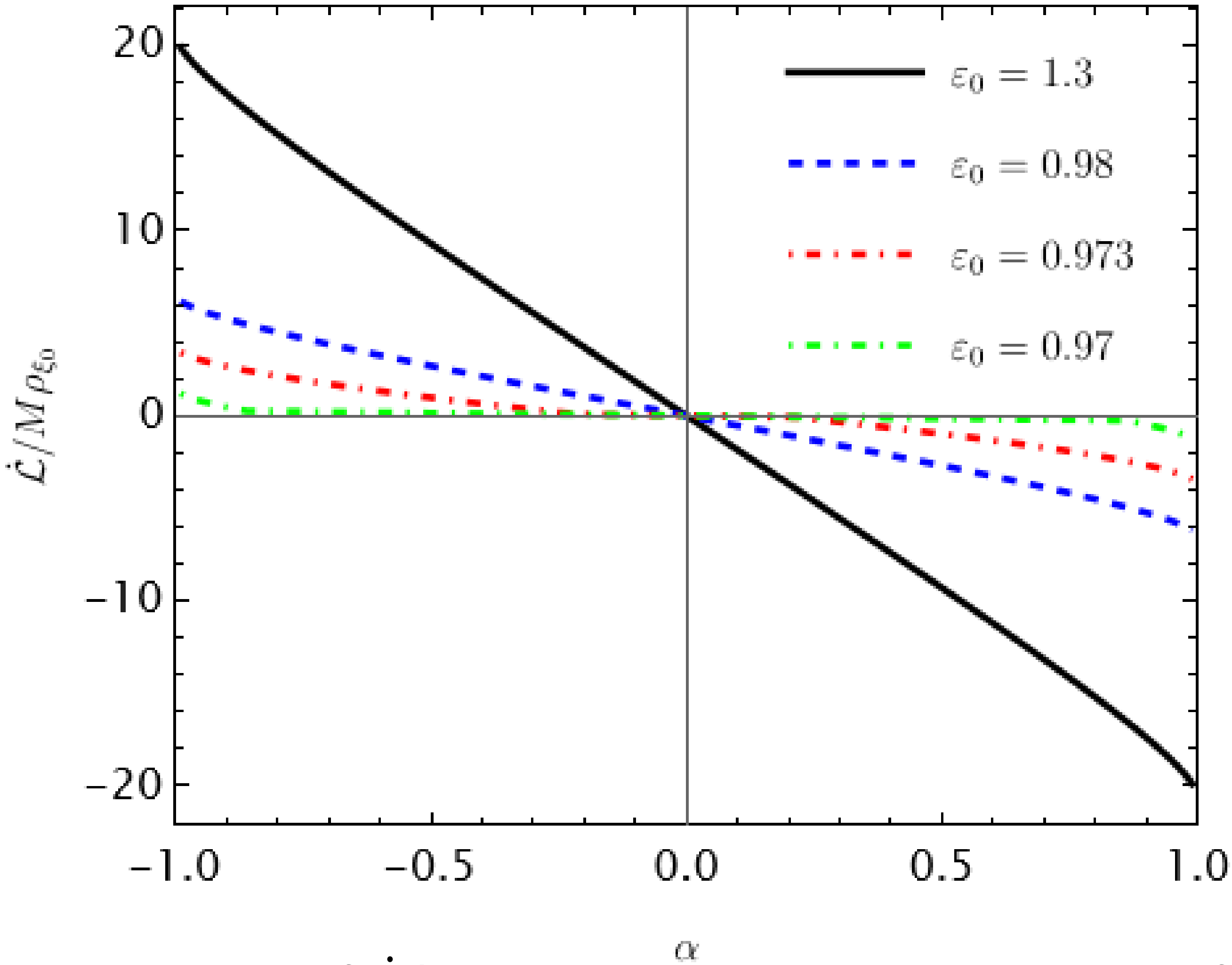
$$\dot{\mathcal{E}} = 2\pi A\, m_0^4 \sum_{\epsilon_\sigma=\pm1} \left[\int_{W_{min}(\xi_0,\,\epsilon_\sigma)}^{\epsilon_{crit}(\xi_0,\,\epsilon_\sigma)} d\epsilon \int_0^{\lambda_{max}(\xi_0,\epsilon,\epsilon_\sigma)} d\lambda\, f_0(\epsilon,\lambda) + \int_{\epsilon_{crit}(\xi_0,\epsilon_\sigma)}^{\infty} d\epsilon\, \epsilon \int_0^{\lambda_c(\epsilon,\,\epsilon_\sigma)} d\lambda\, f_0(\epsilon,\lambda)\right]$$

$$\dot{\mathcal{L}} = 2\pi A M m_0^4 \sum_{\epsilon_\sigma=\pm1} \left[\int_{W_{min}(\xi_0,\,\epsilon_\sigma)}^{\epsilon_{crit}(\xi_0,\,\epsilon_\sigma)} d\epsilon \int_0^{\lambda_{max}(\xi_0,\epsilon,\,\epsilon_\sigma)} d\lambda\, f_0(\epsilon,\lambda)\,(\epsilon_\sigma\lambda + \alpha\epsilon) + \int_{\epsilon_{crit}(\xi_0,\,\epsilon_\sigma)}^{\infty} d\epsilon \int_0^{\lambda_c(\epsilon,\,\epsilon_\sigma)} d\lambda\, f_0(\epsilon,\lambda)\,(\epsilon_\sigma\lambda + \alpha\epsilon)\right]$$

Examples

Monoenergetic distribution $f_0 = \delta(\epsilon - \epsilon_0)$



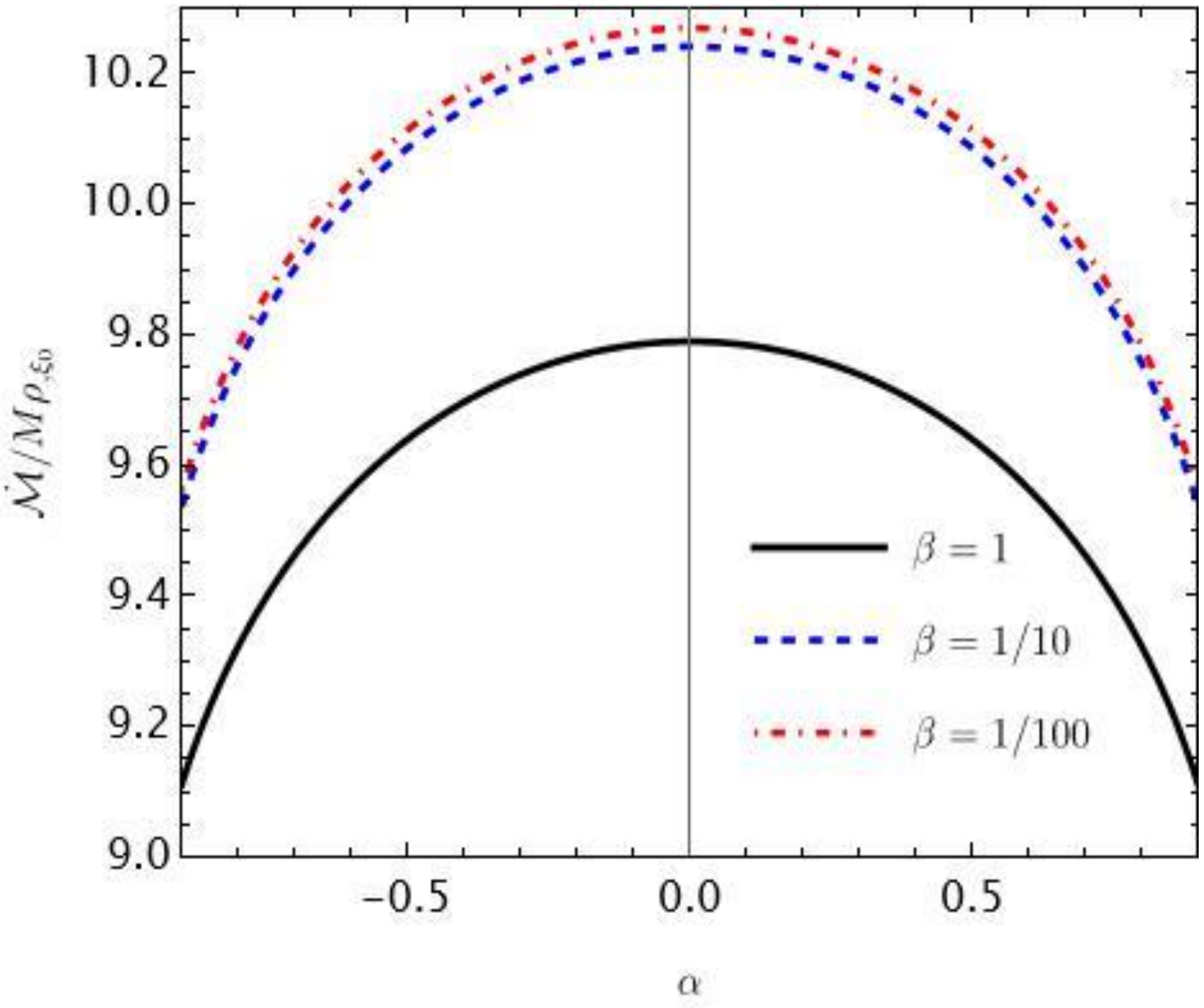


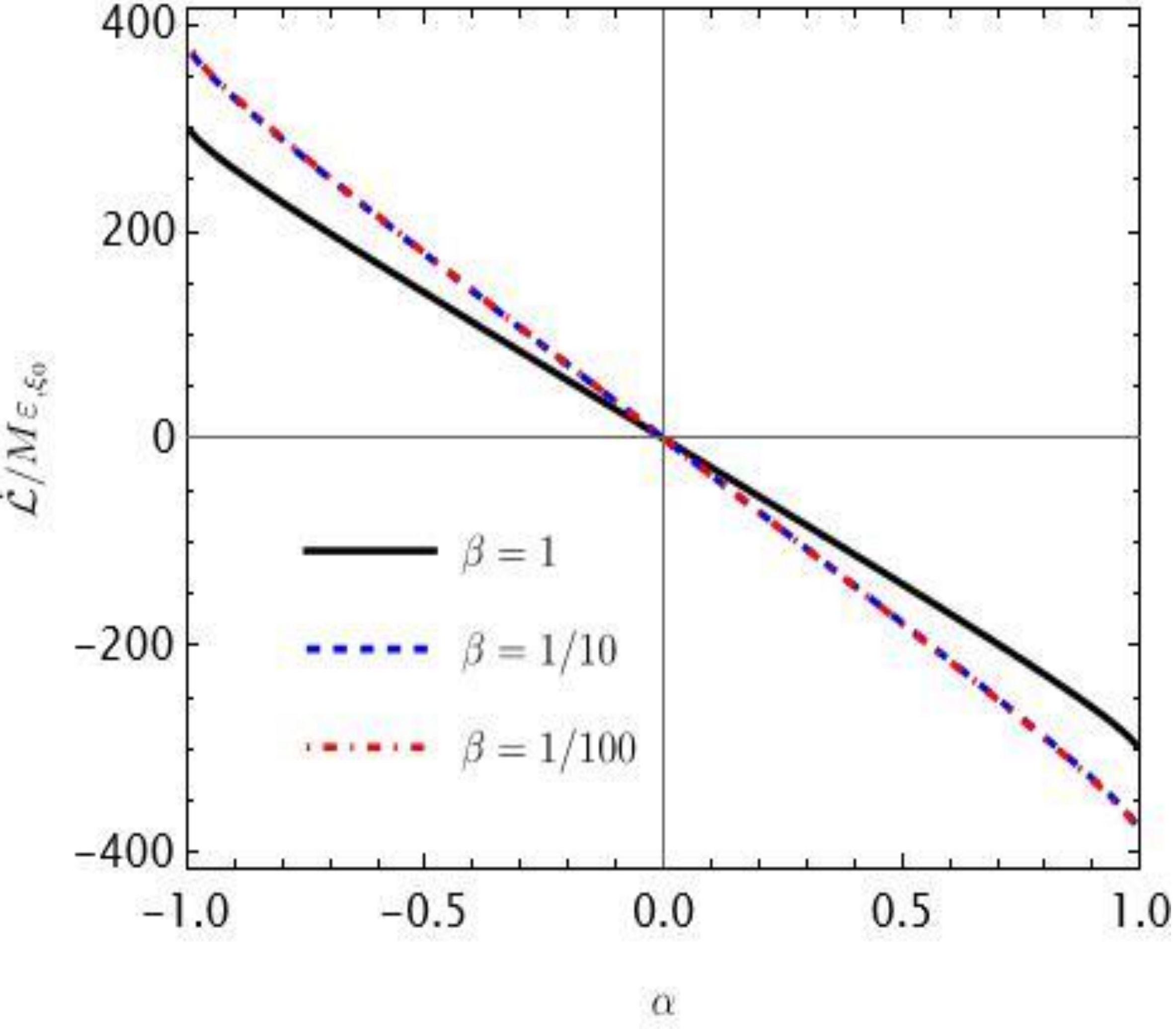
- Eventually $\dot{\mathcal{M}}/M\rho_{\xi_0}$ decreases with α.
- High energy values correspond to high accretion rates.

- The sign of $\dot{\mathcal{L}}/M\rho_{\xi_0}$ is always opposite to the sign of the black hole spin parameter α, indicating that the accretion reduces the black hole rotation.

Examples

Maxwell-Juttner distribution $f_0 = \exp(-\beta\epsilon)$





- Accretion rates decrease with increasing β

Summary

- We employ the general relativistic kinetic theory and restrict ourselves to a collisionless Vlasov gas.
- We compute stationary solutions corresponding to at finite disk in the equatorial plane of the Kerr spacetime.
- In general the mass accretion rates decreases with black hole spin parameter.
- Accretions slows down black hole rotation.
- Potential applications in context of dark matter accretion.