# On heteroclinic networks and Einstein-Rosen gravitational waves

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20-09-2024 POTOR2024

# Setup

Einstein-Rosen gravitational wave (Einstein & Rosen 1937):

$$ds^2 = e^{2(\gamma-\psi)}(-dt^2 + d\rho^2) + \rho^2 e^{-2\psi} d\phi^2 + e^{2\psi} dz^2 \tag{1}$$

$$\rho > 0;\ t, z \in (-\infty, \infty);\ \phi \in [0, 2\pi)$$

Specific solution of standing cylindrical wave:

$$\psi(t,\rho) = A\frac{1}{2}J_0\left(\frac{\rho}{\lambda}\right)\cos\left(\frac{t}{\lambda}\right)$$

$$\gamma(t,\rho) = A^2\frac{\rho^2}{8\lambda^2}\left[J_0^2\left(\frac{\rho}{\lambda}\right) + J_1^2\left(\frac{\rho}{\lambda}\right) - 2\frac{\lambda}{\rho}J_0\left(\frac{\rho}{\lambda}\right)J_1\left(\frac{\rho}{\lambda}\right)\cos^2\left(\frac{t}{\lambda}\right)\right]$$

# Szybka & Naqvi (2023) - chaotic behaviour of the geodesics

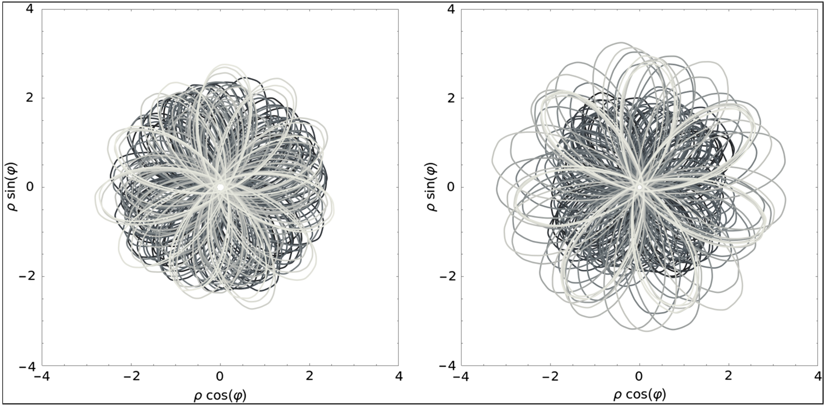
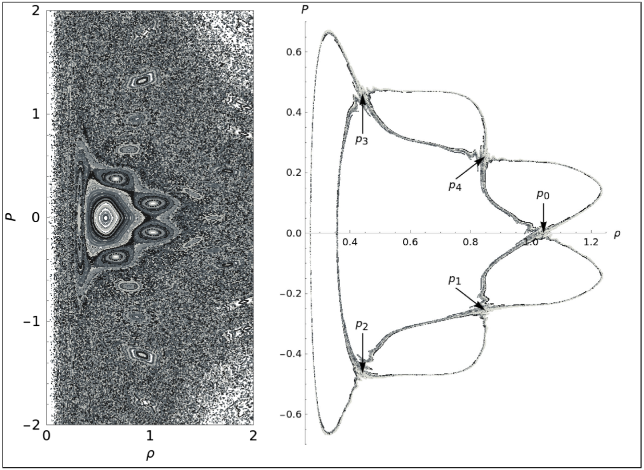


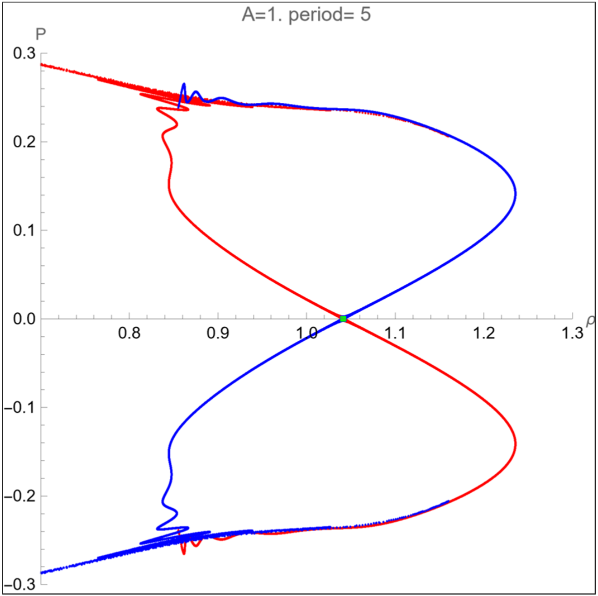

Figure: Szybka & Naqvi (2023)

# Szybka & Naqvi (2023) - chaotic heteroclinic network

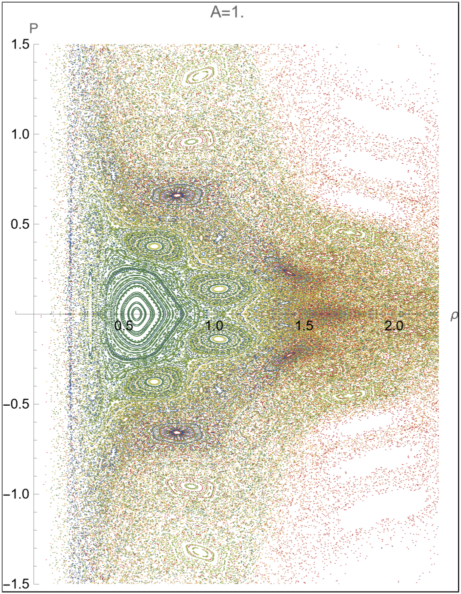



Figures: Szybka & Naqvi (2023)

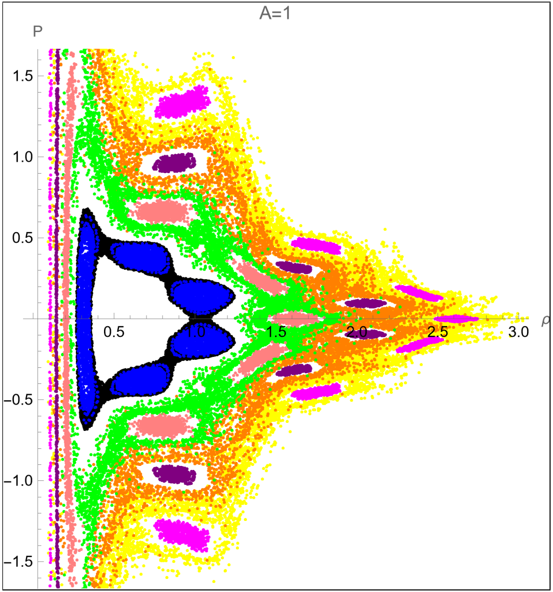
# Chaotic heteroclinic network - (un)stable manifolds



# Deeper look into the networks



# Multiple heteroclinic networks



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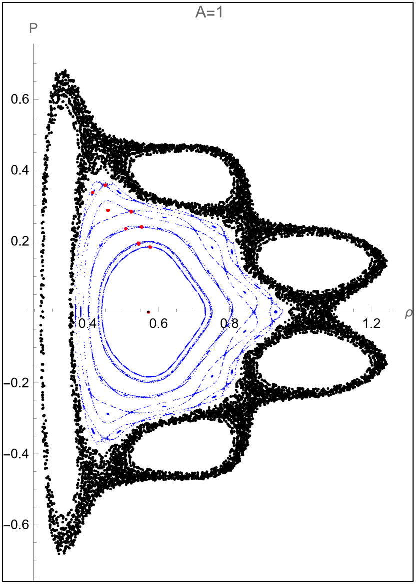

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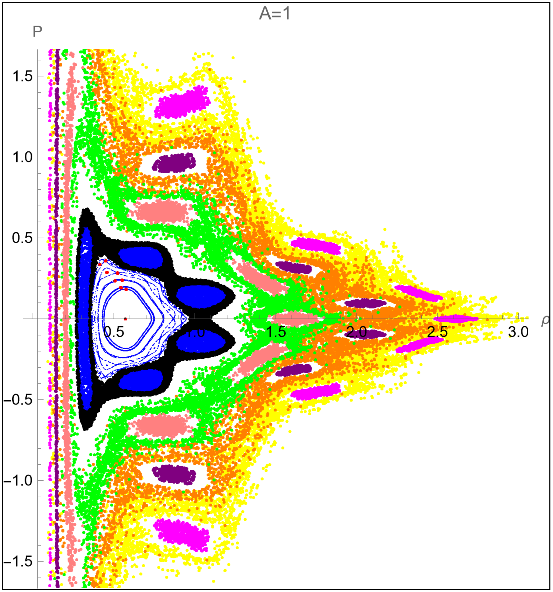
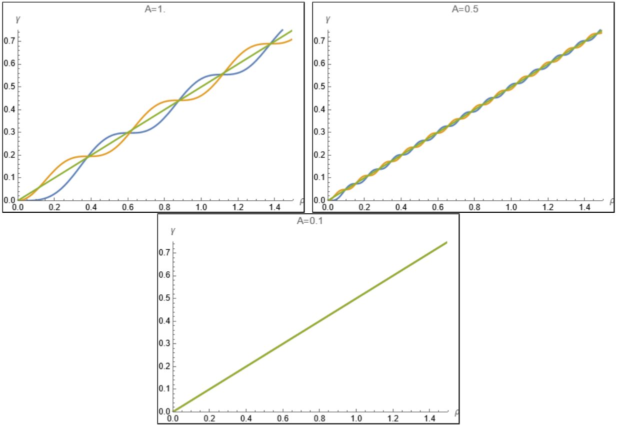


# High frequency limit

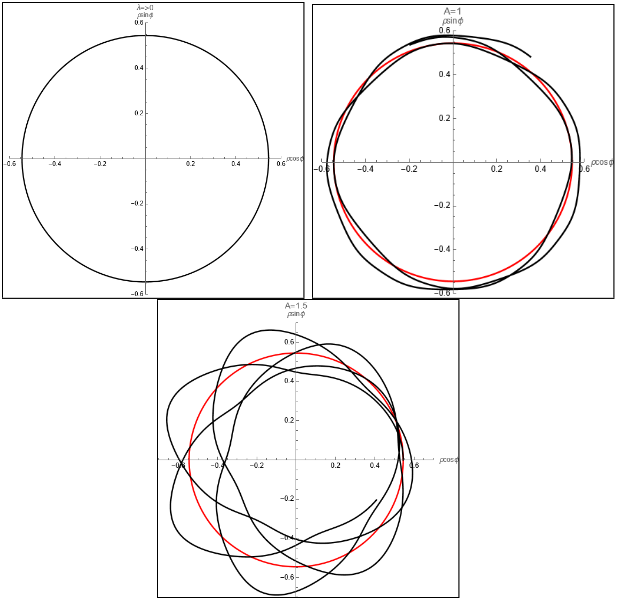
We go to high frequency limit $\lambda \to 0$ by setting $\lambda = A^2 \lambda_0$.



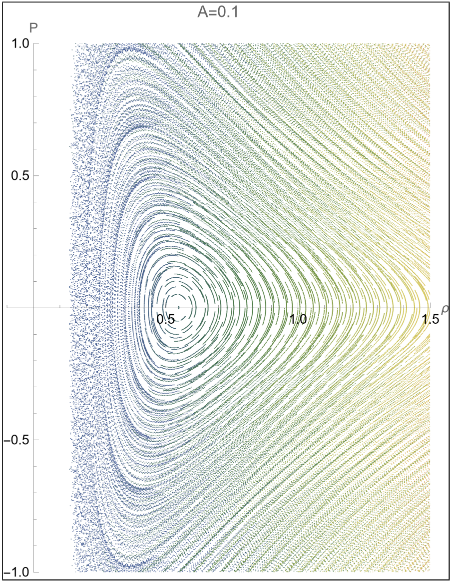

For this scaling:

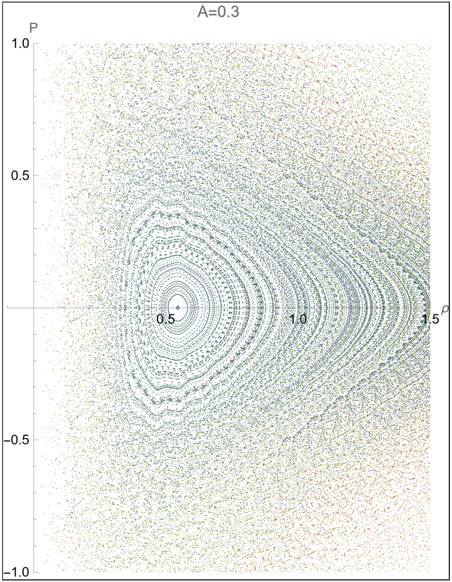
$$\lim_{A \to 0} \gamma(t, \rho) = \frac{\rho}{4\pi\lambda_0}$$

# High frequency limit - perturbations of circular orbits



# Poincaré sections for different amplitudes and frequencies



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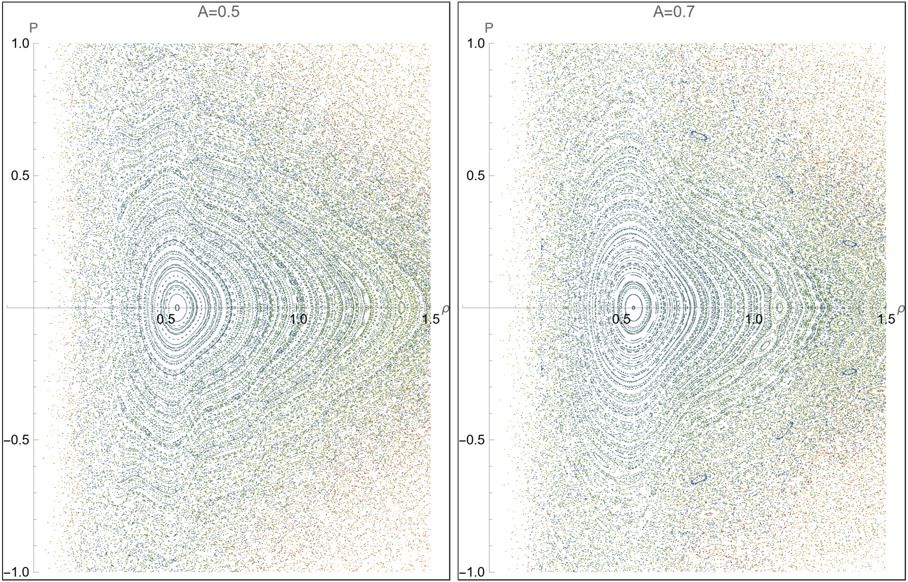

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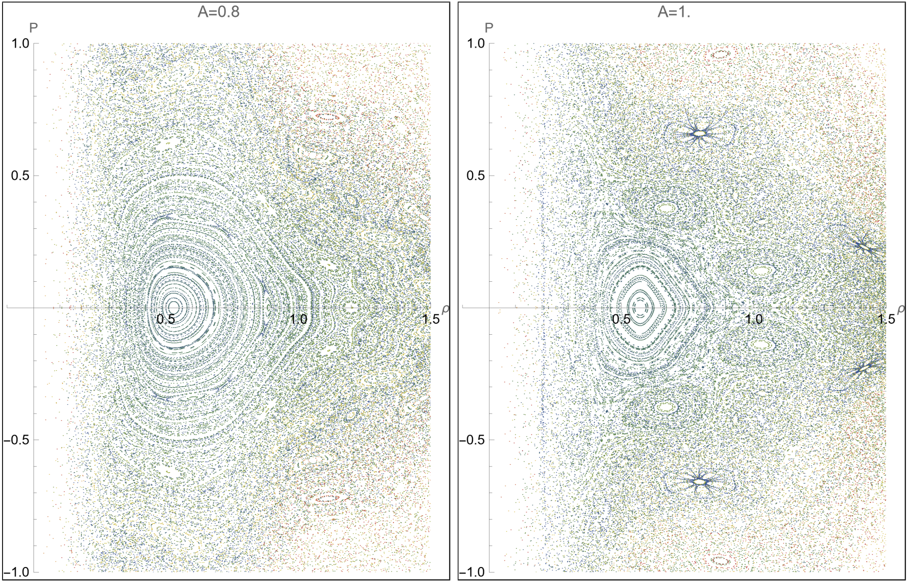

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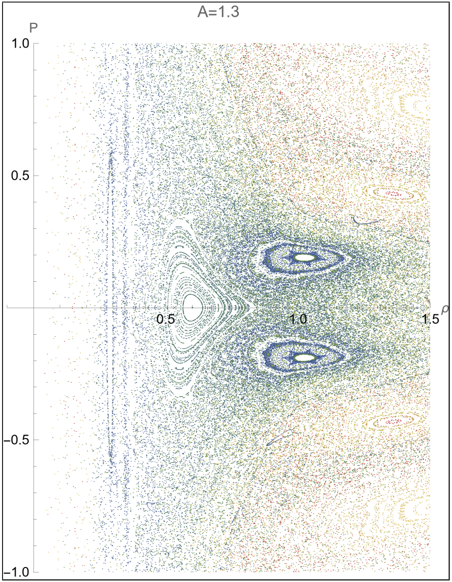
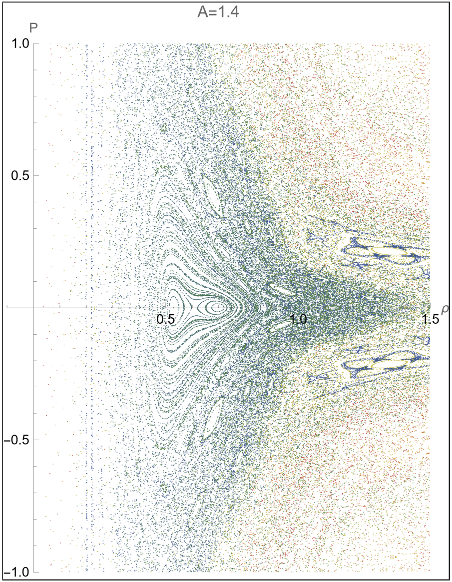
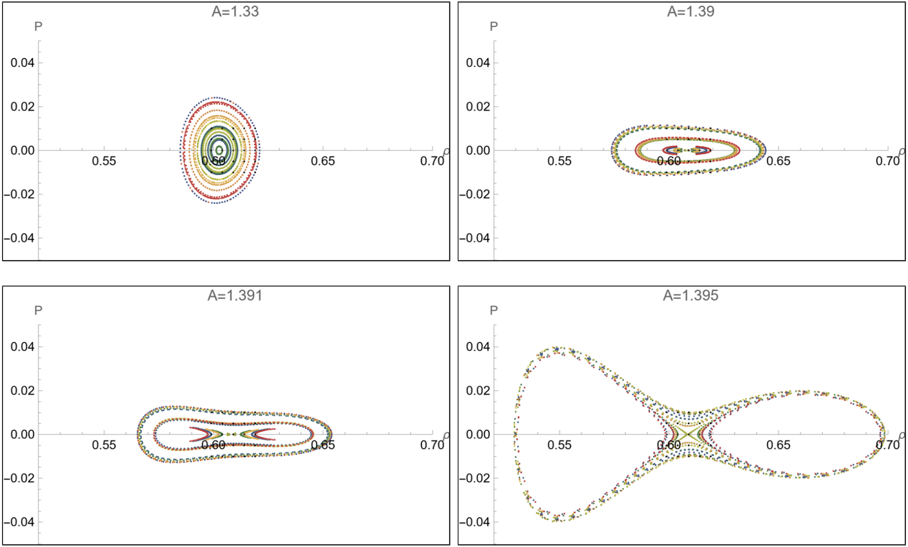


# Bifurcating fixed point



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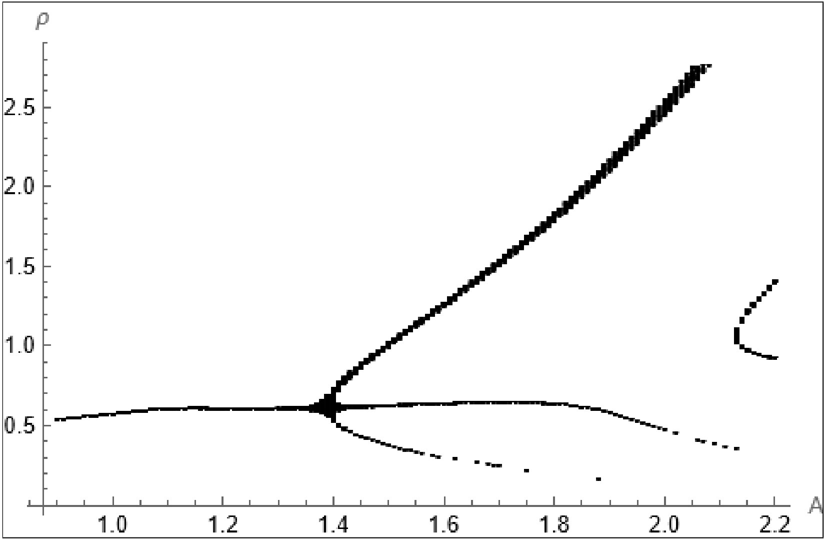

# Summary

- Trajectories of massive particles in Einstein-Rosen standing gravitational wave spacetimes exhibit chaotic behaviour.
- Poincaré sections reveal complex structure of fixed points connected into layered series chaotic heteroclinic networks.
- Fixed points revealed by Poincaré Sections bifurcate and switch between elliptic and hyperbolic types with continuous change of system parameters.
- High frequency/low amplitude limit leads to seemingly non chaotic orbits, but it is hard to pinpoint when chaotic behaviour first manifests.

The End
...questions?

Mentioned literature:

Bondi, H., 2004, Proceedings of the Royal Society of London Series A, 460, 463

Einstein, A. Rosen, N., 1937, Journal of The Franklin Institute, 223, 43

Stephani, H., 2003, General Relativity and Gravitation, 35, 467

Szybka, S. J. Naqvi, S. U., 2023, Phys. Rev. D, 108, L081501

Thorne, K. S., 1965, Phys. Rev., 138, B251

# Appendix

EFE are equivalent to having $\psi(t,\rho)$ satisfy cylindrical wave equation:

$$\psi'' + \frac{1}{\rho}\psi' - \ddot{\psi} = 0$$

Where ' is $\partial_t$ and dot is $\partial_t$. Remaining EFEs give $\gamma$ from $\psi$ by quadratures:

$$\dot{\gamma} = 2\rho\dot{\psi}\psi'$$

$$\gamma' = \rho(\ddot{\psi}^2 + \psi'^2)$$

So amplitude factor $A$ in $\psi$ leads to $A^2$ in $\gamma$.
Thorne (1965) C-energy is equal to $\gamma/4$.

# Appendix

Bondi (2004) Stephani (2003)