# Highly relativistic solutions of the Einstein-Vlasov system and a comparison to solutions of the Einstein-Dirac system

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# Plan of the talk

- I will introduce the Einstein-Vlasov (EV) system and I will outline how to construct spherically symmetric static solutions of the EV system. I will present some old results of mine on compact solutions in the massive case.
- I will discuss the existence of static solutions of the *massless* EV system and the relation to *geons*. This is joint work with David Fajman and Maximilian Thaller.
- I will present a more recent result on massless solutions surrounding a Schwarzschild black hole. As a consequence a new class of compact solutions is obtianed.
- I will present results from an ongoing study on the comparison between spherically symmetric static solutions of the EV system and the Einstein-Dirac system. This is a joint work with Joakim Blomqvist.

# The Einstein-Vlasov system

This system describes a collisionless ensemble of particles, where the particles *typically* are stars, galaxies or clusters of galaxies, which interact through the gravitational field created collectively. In the *massless* case the particles could be photons.

This system has rich dynamics:

- dispersion for small data
- formation of black holes for large data
- steady states exist (both stable and unstable)
- time periodic oscillations
- serves as a good model in cosmology
- static solutions of the Einstein-Dirac system are very similar (as we will see)

# The Einstein-Vlasov system

Let $(x^\alpha, p^\alpha)$ be local coordinates on the tangent bundle of the spacetime $(M, g)$.

The mass shell

$$PM = \{g_{\alpha\beta}p^\alpha p^\beta = -m^2,\ p^\alpha \text{ is future pointing}\} \subset TM,$$

is invariant under geodesic flow

$$\dot{x}^\alpha = p^\alpha,\ \dot{p}^\alpha = -\Gamma^\alpha_{\beta\gamma}p^\beta p^\gamma.$$

In the massive case we normalize the rest mass so that $m = 1$, and in the massless case $m = 0$.

Typically $p^0$ can be expressed in terms of $p^a,\ a = 1, 2, 3$ by the mass shell condition.

On PM we thus use coordinates $(t, x^a, p^a),\ a = 1, 2, 3.$

# The Einstein-Vlasov system

The Vlasov equation for $f = f(t, x^a, p^a)$ on PM reads

$$\partial_t f + \frac{p^a}{p^0}\partial_{x^a} f - \frac{1}{p^0}\Gamma^a_{\beta\gamma} p^\beta p^\gamma \partial_{p^a} f = 0.$$

Define the energy momentum tensor by

$$T_{\alpha\beta} := \sqrt{|g|}\int \frac{p_\alpha p_\beta}{-p_0} f \; dp^1 dp^2 dp^3.$$

The Einstein-Vlasov system reads (with $\Lambda = 0$)

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = 8\pi T_{\alpha\beta}.$$

It has nice mathematical properties!

# The static spherically symmetric EV system

The metric takes the following form in Schwarzschild coordinates

$$ds^2 = -e^{2\mu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

where $r \geq 0$, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi]$.

Asymptotic flatness is expressed by the boundary conditions

$$\lim_{r\to\infty} \lambda(r) = \lim_{r\to\infty} \mu(r) = 0,$$

and a regular center requires

$$\lambda(0) = 0.$$

# The Einstein equations

The Einstein equations read

$$e^{-2\lambda}(2r\lambda_r - 1) + 1 = 8\pi r^2 \rho,$$
$$e^{-2\lambda}(2r\mu_r + 1) - 1 = 8\pi r^2 p,$$
$$e^{-2\lambda}(\mu_{rr} + (\mu_r - \lambda_r)(\mu_r + \frac{1}{r})) = 8\pi p_T.$$

The two first equations, together with the Vlasov equation, imply the last equation.

Here $\rho, p$ and $p_T$ denote the energy density, the radial pressure and the tangential pressure respectively.

# The Vlasov equation

By symmetry $f = f(r, w, L),\ w \in \mathbb{R},\ L \geq 0$.

The variables $w$ and $L$ can be thought of as the radial momentum and the square of the angular momentum respectively.

*Remark:* Note that each particle can carry angular momentum although the total angular momentum is zero due to spherical symmetry.

The Vlasov equation for $f = f(r, w, L)$ is given by

$$\frac{w}{\mathcal{E}}\partial_r f - (\mu_r \mathcal{E} - \frac{L}{r^3\mathcal{E}})\partial_w f = 0,$$

where

$$\mathcal{E} = \mathcal{E}(r, w, L) = \sqrt{1 + w^2 + L/r^2}.$$

# The matter quantities

The matter quantities are given by

$$\rho(r) = \frac{\pi}{r^2}\int_{-\infty}^{\infty}\int_0^{\infty}\mathcal{E}f(r,w,L)\,dwdL,$$

$$p(r) = \frac{\pi}{r^2}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{w^2}{\mathcal{E}}f(r,w,L)\,dwdL,$$

$$p_T(r) = \frac{\pi}{r^4}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{L}{\mathcal{E}}f(r,w,L)\,dwdL.$$

# The ansatz

In addition to the angular momentum $L$ the quantity

$$E := e^{\mu(r)}\sqrt{1 + w^2 + L/r^2} = e^{\mu(r)}\mathcal{E},$$

is conserved along characteristics.

The ansatz

$$f(r, w, L) = \Phi(E, L),$$

for some function $\Phi$, then satisfies the Vlasov equation and constitutes an efficient way to construct static solutions of the EV system.

*Remark:* Jeans' theorem states that for the spherically symmetric Vlasov-Poisson system (i.e. the Newtonian case), all solutions are obtained in this way. This is not true for the Einstein-Vlasov system [Schaeffer '99].

# Choice of $\Phi$

The following form of $\Phi$ will be used (the polytropic ansatz)

$$\Phi(E, L) = (E_0 - E)_+^k (L - L_0)_+^l,$$

where $l \geq -1/2,\ k \geq 0,\ L_0 \geq 0,\ E_0 > 0$, and $x_+ := \max\{x, 0\}$.

With this ansatz $\rho$ takes the form

$$\rho(r) \;=\; \frac{2\pi}{r^2} \int_{\sqrt{1+\frac{L_0}{r^2}}}^{E_0 e^{-\mu(r)}} \int_{L_0}^{r^2(s^2-1)} \frac{(E_0 - e^{\mu} s)^k (L - L_0)^l \, s^2}{\sqrt{s^2 - 1 - \frac{L}{r^2}}} dL ds.$$

Similar expressions for $p$ and $p_T$.

# To summarize

The metric is

$$ds^2 = -e^{2\mu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

The static Einstein-Vlasov system takes the form

$$e^{-2\lambda}(2r\lambda_r - 1) + 1 = 8\pi r^2\rho,$$
$$e^{-2\lambda}(2r\mu_r + 1) - 1 = 8\pi r^2 p,$$

with boundary conditions $\lambda(0) = \lambda(\infty) = \mu(\infty) = 0$, where

$$\rho(r) = \frac{2\pi}{r^2}\int_{\sqrt{1+\frac{L_0}{r^2}}}^{E_0 e^{-\mu(r)}}\int_{L_0}^{r^2(s^2-1)} \frac{(E_0 - e^{\mu}s)^k(L-L_0)^l\, s^2}{\sqrt{s^2 - 1 - \frac{L}{r^2}}}dLds,$$

$$p(r) = \frac{2\pi}{r^2}\int_{\sqrt{1+\frac{L_0}{r^2}}}^{E_0 e^{-\mu(r)}}\int_{L_0}^{r^2(s^2-1)} (E_0 - e^{\mu}s)^k(L-L_0)^l\sqrt{s^2 - 1 - \frac{L}{r^2}}dLds.$$

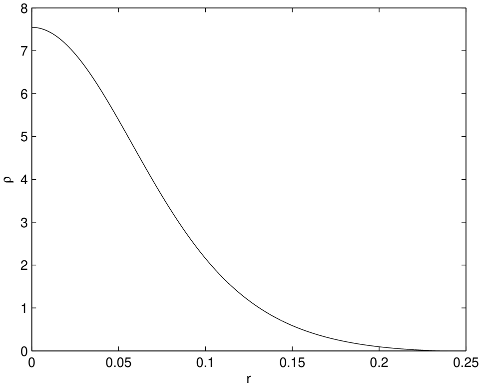
# Existence and numerics

Rein and Rendall have shown existence (1995-1999) of solutions. Main difficulty to show finite extension.

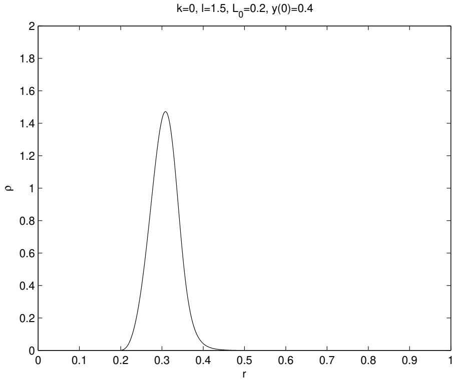
In a numerical project with Gerhard Rein (2005) we studied some features of solutions of the static Einstein-Vlasov system. We found

- Multi-peaks
- Arbitrarily thin shells (these are not Einstein clusters)
- Spirals in the M-R diagram
- The Buchdahl inequality holds (although the assumptions by Buchdahl are violated)

# A ball configuration, isotropic case ($L_0 = 0$ and $l = 0$)



# A single shell ($L_0 = 0.2$, $k = 0$, $l = 1.5$ and $y(0) = 0.4$)




$$y(0) = \frac{e^{\mu(0)}}{E_0}$$

# A shell with two peaks ($y(0) = 0.12$)

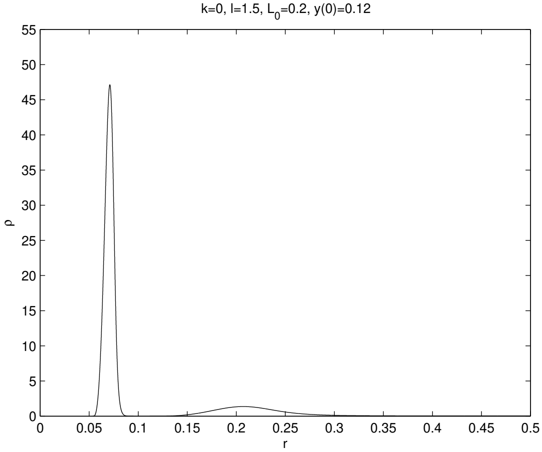
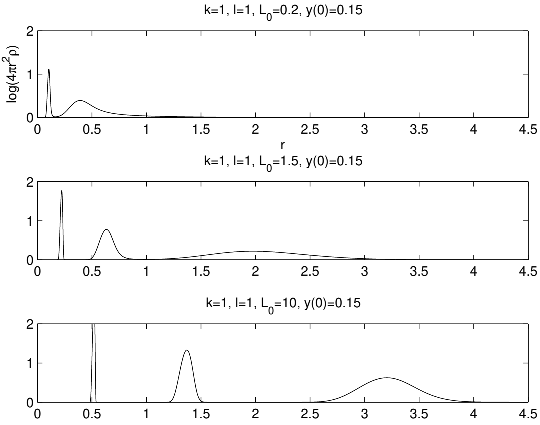



Figure: A vacuum region before the tail

# Multi-peaks of shells ($L_0 > 0$)



# ... more peaks

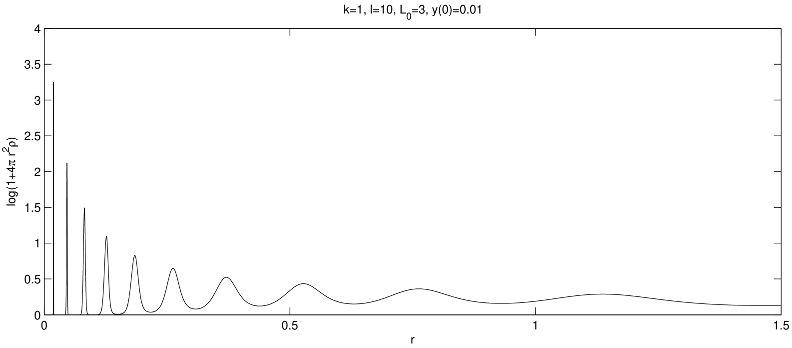



Figure: Multi-peaks of shells

Not possible for the Vlasov-Poisson system, it is a purely relativistic feature!

# Spirals in the $(R, M)$ diagram

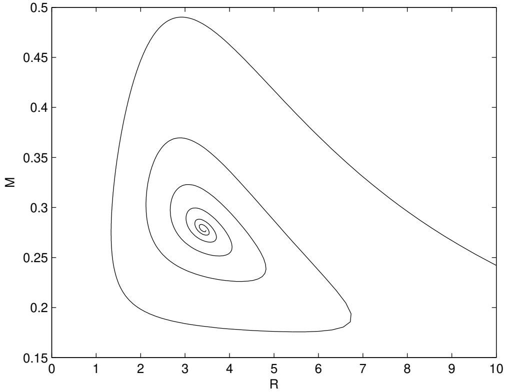



Figure: $k = 0,\ l = 10.5,\ 0.01 \leq y(0) \leq 0.99$

In astrophysics conclusions about stability are often drawn from the "Poincaré turning point principle" based on a spiral diagram.

# How large can $2m/r$ possibly be?

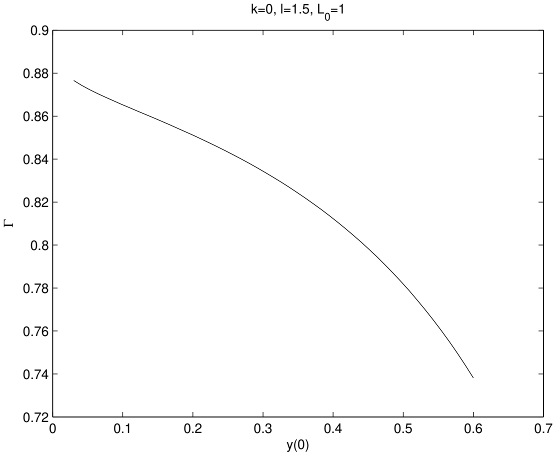



Figure: $\Gamma := \sup 2m/r$ versus $y(0)$

# The Schwarzschild solution

Consider a spherically symmetric object with mass $M$ and radius $R$. For $r > R$ there is vacuum and the Einstein equations can be solved explicitly by the Schwarzschild solution:

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{1}{1 - \frac{2M}{r}}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2).$$

Note that the Schwarzschild solution is singular when $r = 2M$.

Schwarzschild asked already in 1916 the question:

How large can $2M/R$ possibly be for a static solution?

He gave the answer

$$2M/R \leq 8/9$$

in the special case of the Schwarzschild interior solution which has constant energy density and isotropic pressure.

# The Buchdahl inequality

In 1959 Buchdahl extended the result by Schwarzschild; namely under the hypotheses that

i) the energy density is non-increasing outwards
ii) the pressure is isotropic ($p = p_T$)

Buchdahl showed that

$$2M/R \leq 8/9.$$

# Restrictions

- The assumptions made by Buchdahl are very restrictive: they are not satisfied by any known stable field configuration and they are not satisfied for most of the static solutions of the Einstein-Vlasov system.
- The satuarting solution is the constant energy density solution found by Schwarzschild where the pressure becomes infinite. A consequence is that the **dominant energy condition is violated**. Hence, it is non-physical.

# A general inequality

Let

$$m(r) = 4\pi \int_0^r s^2 \rho(s)\, ds,$$

so that $m(R) = M$, where $R$ is the outer boundary of the object.

**Theorem (A. (2007))**

*Assume that $p + 2p_T \leq \Omega\rho$, where $p$ and $\rho$ are non-negative, and $\Omega > 0$ is a constant. Then*

$$\sup \frac{2m(r)}{r} \leq \frac{(1+2\Omega)^2 - 1}{(1+2\Omega)^2},$$

*and the inequality is sharp. The saturating solution is given by an infinitely thin shell solution.*

*Remark:* Note that $\Omega = 1$ holds for Vlasov matter and that it gives $2m/r \leq \frac{8}{9}$.

# Arbitrarily thin shells of the EV system do exist

**Theorem (A. (2006))**

*Given $\epsilon > 0$, there exists static solutions of the spherically symmetric Einstein-Vlasov system such that the density function $f$ is positive on $(R_0, R_1)$, where*

$$\frac{R_1}{R_0} \leq 1 + \epsilon,$$

*and vanishes for $r \leq R_0$ and for $r \geq R_1$.*

As a consequence, for any $\epsilon > 0$, there is a solution to the Einstein-Vlasov system such that

$$\frac{8}{9} - \epsilon < \frac{2M}{R} < \frac{8}{9}.$$

*Remark:* The thin shells are not Einstein clusters.

# The charged case

Let us now consider the charged case. The Schwarzschild solution is now replaced by the Reissner-Nordström solution

$$1 - \frac{2M}{r} \to 1 - \frac{2M}{r} + \frac{Q^2}{r^2}.$$

The problem of finding an upper bound on $M/R$ for a given total charge $Q$ were considered in a number of papers but neither a transparent nor a general inequality similar to the case without charge, i.e.,

$$\frac{2M}{R} \leq \frac{8}{9},$$

were found.

# A general inequality with charge

Dedicated to the memory of my father
Dan Andréasson (1933-2008).

**Theorem (A. (2008))**

*Assume that* $p + 2p_T \leq \rho$, *where* $p$ *and* $\rho$ *are non-negative and assume that* $Q \leq M$. *Then*

$$\sqrt{M} \leq \frac{\sqrt{R}}{3} + \sqrt{\frac{R}{9} + \frac{Q^2}{3R}},$$

*and the inequality is sharp.*

**Remark:** i) Note that the inequality is saturated in the extreme case $M = Q = R$. In particular there is no gap to a transition to a black hole!

# Geons

- Wheeler introduced the concept of a geon 1955
- Wheeler studied numerically idealized spherically symmetric geons
- Static thin shell solutions where found with the property that $2m/r \approx 8/9$
- The massless EV system models a photon gas
- The massless EV system thus provides an alternative model for a geon

# Existence of massless solutions of the EV system

**Theorem (A., Fajman and Thaller (2016))**

*There exist static, spherically symmetric, asymptotically flat solutions to the massless Einstein-Vlasov system, with compactly supported matter quantities. These solutions have the property that*

$$\frac{4}{5} < \sup_{r \in [0,\infty)} \frac{2m(r)}{r} < \frac{8}{9}.$$

# Existence of massless solutions surrounding a black hole

Motivated by the recent proof of stability of Kerr I became interested in massless solutions surrounding a black hole.

**Theorem (A. (2021))**

*Let $M_0 \geq 0$ be the ADM mass of a Schwarzschild black hole. Then there exist static solutions with finite ADM mass to the massless spherically symmetric Einstein-Vlasov system surrounding the black hole. The matter components are supported on a finite interval $[R_0, R_1]$, where $R_0 > 3M_0$, and spacetime is asymptotically flat.*

# New class of solutions with $\frac{2M}{R} \to \frac{8}{9}$

Note that the result holds also in the case when $M_0 = 0$ (and in the massive case) but that the family of solutions is different from the family of solutions discussed above. In the present situation we require the inner radius $R_0$ to be large whereas in the previous case $R_0$ is required to be small.

Both families share the property that $\Gamma \to 8/9$ in the extreme limits: $R_0 \to 0$ or when $R_0 \to \infty$. The crucial thing being that

$$\frac{R_1}{R_0} \to 1 \text{ in the limit,}$$

where matter is supported in $[R_0, R_1]$. For instance:

- if $R_1 = R_0 + R_0^{3/2}$ then $R_1/R_0 \to 1$ as $R_0 \to 0$
- if $R_1 = R_0 + R_0^{1/2}$ then $R_1/R_0 \to 1$ as $R_0 \to \infty$

# Static solutions of the Einstein-Dirac system versus solutions of the EV system

- In 1998, Finster, Smoller and Yau, were able to generate spherically symmetric static solutions to the Einstein-Dirac system. The configuration they study consists of two uncharged fermions with opposite spin in order to yield a spherically symmetric system.
- A few years ago, Leith, Hooley, Horne and Dritschel, generalized this study to an even number of fermions, $\kappa$. I noticed that the solutions they obtain have striking similarties with solutions of the Einstein-Vlasov system as $\kappa$ increases.
- This gives rise to a nice opportunity to study similarties of solutions with a quantum signature and solutions of a classical system, for a small number of particles.

# The Einstein-Dirac system

The Einstein-Dirac (ED) system reads

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu},$$

with Dirac's equation

$$(i \not{D} - m)\Psi = 0.$$

Here $\not{D}$ is the Dirac operator in curved space time and $T_{\mu\nu}$ is obtained from $\Psi$ and the metric.

We assume that a collection of fermions occupy a single shell configuration with $j_{tot} = 0$.

The number of fermions is given by $\kappa := 2j + 1$ where $j = (2n+1)/2, \; n \in \mathbb{N}$.

# Deriving the Einstein-Dirac equations

The overall wave function can be written, using the Hartree-Fock formalism, as $\Psi = \psi_{j,k=-j} \wedge \psi_{j,k=-j+1} \wedge \ldots \wedge \psi_{j,k=j}$, where $\psi_{j,k}$ is the wave function of an individual fermion with angular momentum component in the $z$-direction equal to $k$.

The following ansatz for $\psi_{j,k}$ is used

$$\psi_{jk} = \begin{bmatrix} \psi_{jk}^{(1)} \\ \psi_{jk}^{(2)} \\ \psi_{jk}^{(3)} \\ \psi_{jk}^{(4)} \end{bmatrix} = e^{-i\omega t} \frac{\sqrt{T(r)}}{r} \begin{bmatrix} \chi^k_{j-1/2}\alpha(r) \\ i\chi^k_{j+1/2}\beta(r) \end{bmatrix}.$$

In the ansatz, $\chi^k_{j\mp1/2}(\theta,\varphi)$ is a linear combination of spherical harmonics functions $Y^k_{j\mp1/2}$ and the basis $e_1 = [1,0]^\top$, $e_2 = [0,1]^\top$.

# The Einstein-Dirac system

Let the metric be given by

$$ds^2 = -T^{-2}(r)dt^2 + A^{-1}(r)dr^2 + r^2 d\Omega^2,$$

then the Einstein-Dirac system takes the form:

$$\begin{aligned}
\sqrt{A}\alpha' &= \frac{\boldsymbol{\kappa}}{2r}\alpha - (\boldsymbol{\omega} T + m)\beta, \\
\sqrt{A}\beta' &= (\boldsymbol{\omega} T - m)\alpha - \frac{\boldsymbol{\kappa}}{2r}\beta, \\
rA' &= 1 - A - 8\pi\boldsymbol{\kappa}\boldsymbol{\omega} T^2(\alpha^2 + \beta^2), \\
2rA\frac{T'}{T} &= A - 1 - 8\pi\boldsymbol{\kappa}\boldsymbol{\omega} T^2(\alpha^2 + \beta^2) + 8\pi\frac{\boldsymbol{\kappa}^2}{r}T\alpha\beta + 8\pi\boldsymbol{\kappa} m T(\alpha^2 - \beta^2).
\end{aligned}$$

Here $\omega$ is the fermion energy (we only consider the ground state) determined by the shooting algorithm.

# Energy density and pressure

The energy density $\rho$, radial pressure $p_r$ and the tangential pressure $p_\perp$ are given by:

$$\rho(r) = \kappa\omega\frac{T^2(r)}{r^2}(\alpha^2(r)+\beta^2(r)),$$

$$p_r(r) = \kappa\frac{T(r)}{r^2}[\omega T(r)(\alpha^2(r)+\beta^2(r)) - m(\alpha^2(r)-\beta^2(r)) - \kappa\frac{\alpha(r)\beta(r)}{r}],$$

$$p_\perp(r) = \frac{\kappa^2}{2r^3}T(r)\alpha(r)\beta(r).$$

# Comparison 1

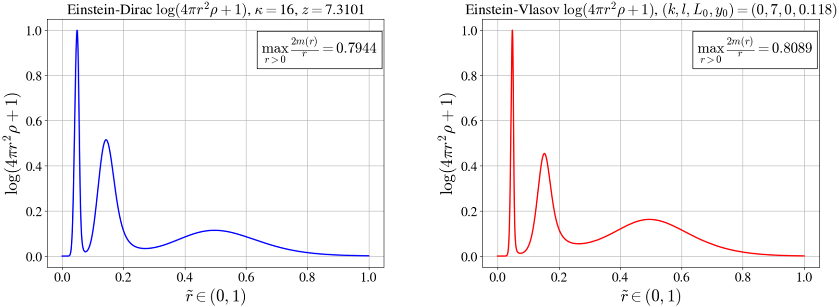



Figure: A graphical comparison of a Einstein-Dirac energy density function and a very similar result for the Einstein-Vlasov system.

# Comparison 2

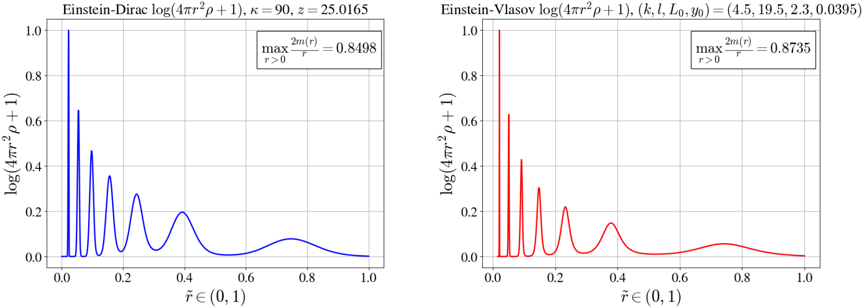



Figure: A graphical comparison of a Einstein-Dirac energy density function and a very similar result for the Einstein-Vlasov system.

# Properties of highly relativistic solutions

Let us recall the assumptions required for the result on the bound of $\sup \frac{2m}{r}$ discussed above. If

$$\boxed{p_r \geq 0 \text{ and } p_r + 2p_\perp \leq \Omega\,\rho} \tag{1}$$

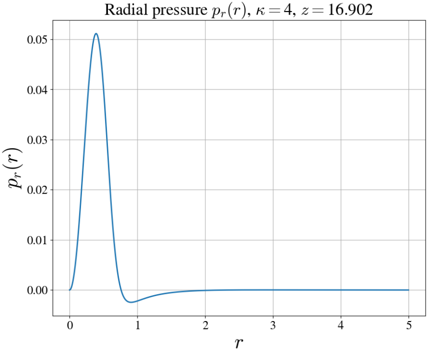
then

$$\sup_{r>0} \frac{2m(r)}{r} \leq \frac{(1+2\Omega)^2 - 1}{(1+2\Omega)^2}$$

The question we ask is whether or not the conditions (1) are satisfied for solutions of the ED system.

# Sign of the radial pressure $p_r$

Solutions to the ED system may have negative pressure, at least in some region. This is a quantum phenomenon since classically the pressure is non-negative.



# Radial pressure results

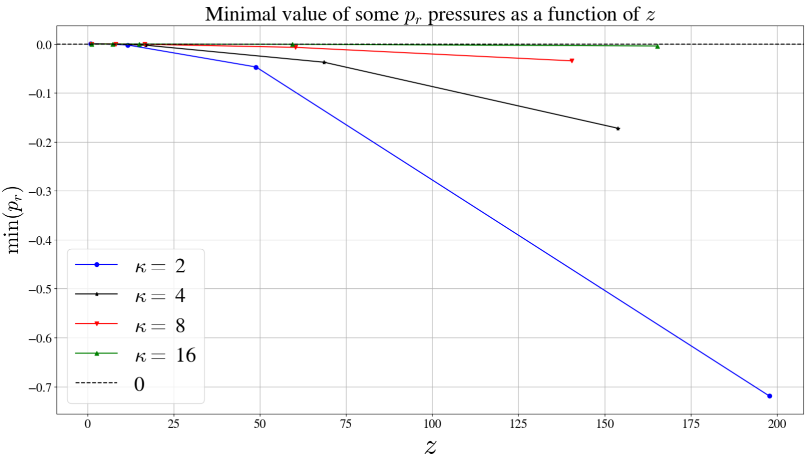
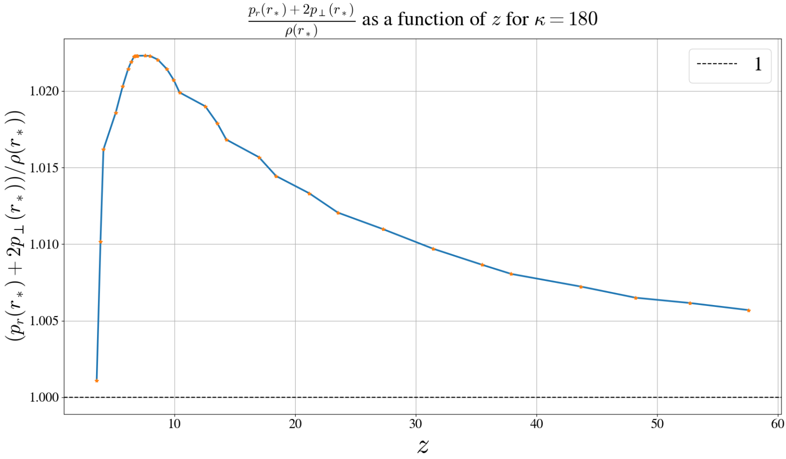
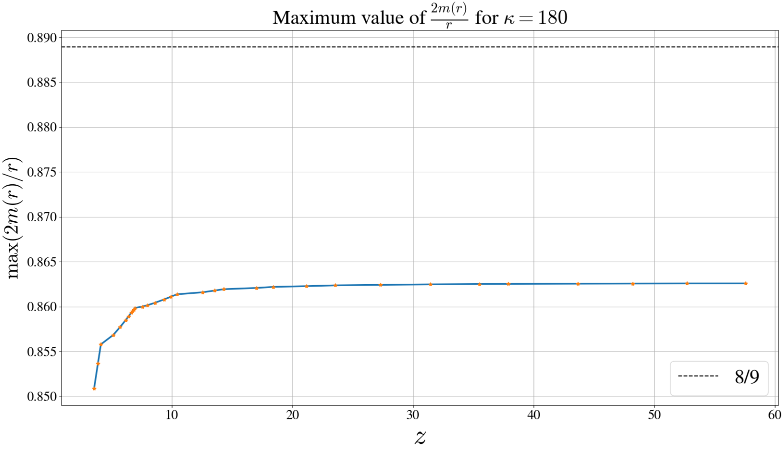



Figure: Einstein-Dirac states for $\kappa > 16$ seems to display classical properties.

# $p_r + 2p_T/\rho$ at the radius with maximum compactness



# Maximum compactness of $2m/r$ in the case $\kappa = 180$



# Thank you!