Coherent states quantization. The fiducial vector dependence.

Aleksandra Pędrak

National Centre for Nuclear Research, Warsaw, Poland

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Motivation

- The Coherent States (CS) quantization method is a method of construction of a quantum counterpart of a given classical system. It is used effectively to quantize gravitational systems.

Scheme of quantization

1. Every point κ configuration space T is represented by projection operator in some carrier space $\mathcal{H}$:
$$(\kappa) \to |\kappa\rangle\langle\kappa|$$
2. Quantization:
$$f(\kappa) \to \hat{f} = \int_T d\mu(\kappa)|\kappa\rangle f(\kappa)\langle\kappa|$$

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- Because of a direct correlation between spacetime points and appropriate projection operators time operator can be constructed in the same footing as position.
- CS quantization method provides a proper transition to the classical picture. In coherent states expectation value of quantum observable comes from CS quantization is very close to classical value.

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- Because of a direct correlation between spacetime points and appropriate projection operators time operator can be constructed in the same footing as position.
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Construction of $|\kappa\rangle\langle\kappa|$ depends on fiducial vector Φ_0.

- What is the interpretation of a fiducial vector in CS quantization?
- Is the fiducial vector dependence qualitative or quantitative?

CS quantization method-building blocks

Classical model

T is a space of required variables κ,

Observables $f : T \to \mathbb{R}$

Example

- Spherically symmetric models of spacetime:

$$T = \{(t, r) \mid (t, r) \in \mathbb{R} \times \mathbb{R}_+\}$$

where t and r are time and radial coordinate respectively.

- Minkowski spacetime phase space:

$$T = \{(p, x) \mid (p, x) \in \mathbb{R}^4 \times \mathbb{R}^4\}$$

where
x space-time coordinates,
p four-momentum coordinates.

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Group
G is a locally compact group
$\chi : T \xrightarrow{1:1} \mathcal{G}$

- G is a locally compact group;
- $d\mu(g)$ is a left Haar measure on this group;
- $\mathcal{G}$–group manifold (set of group parameters);
- There exist one to one correspondence between space of parameters and group manifold $\chi : T \xrightarrow{1:1} \mathcal{G}$

Example

- Spherically symmetric models of spacetime: $G = \mathrm{Aff}(\mathbb{R})$
- Minkowski spacetime: $G = HW(4)/U(1)$

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Example

- Carrier space in case of Affine group: $\mathcal{H}_x = L^2(\mathbb{R}_+, d\nu(x))$ where $d\nu(x) = \frac{dx}{x}$

  where $(t, r) \in \mathbb{R} \times \mathbb{R}_+$ is configuration space in spherically symmetric models of spacetime;

- Carrier space in case of Heisenberg-Weyl group: $\mathcal{H}_x = L^2(\mathbb{R}^4, d^4\xi)$

  where $(p, x) \in \mathbb{R}^8$ is phase space in Minkowski spacetime.

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Coherent state construction
Representation of group
$\phi_g(x) = \hat{U}(g)\phi(x)$
Fiducial vector $\Phi_0(x) \in L^2(X, d\nu(x))$

- Irreducible unitary representation of G in space $L^2(X, d\nu(x))$
  $\hat{U}(g)\phi(x) = \phi_g(x)$
- Fiducial vector
  $\Phi_0(x) \in L^2(X, d\nu(x))$,
  $\langle x|g\rangle = \hat{U}(g)\Phi_0(x)$
- If the condition is satisfied
  $\int_G d\mu(g)|g\rangle\langle g| = \hat{1}$
  in the weak sense on the Hilbert space $L^2(X, d\nu(x))$, then
  $\langle x|g\rangle$
  are coherent states in space $\mathcal{H}_x$.

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Quantization

- Representation of point in Hilbert space

$$(\kappa) \;\to\; |\chi(\kappa)\rangle\langle\chi(\kappa)| = |g\rangle\langle g|$$

- One can map any observable $f : T \to \mathbb{R}$ into a symmetric operator $\hat{f} : \mathcal{H}_x \to \mathcal{H}_x$ as follows

$$\hat{f} := \int_G d\mu(g)|g\rangle f(g)\langle g|$$

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CS quantization method-building blocks

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Observables $f : T \rightarrow \mathbb{R}$

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CS quantization method- nonlocal inner product

- In CS quantization method quantum observables are defined as a operators on carrier space. It is more convenient to transform the model to the space $L^2(G, d\mu(g))$.

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- Quantum space of state is $L^2(X,\, d\nu(x))$

$$\hat{f}_X\psi(x) = \int_X d\nu(y) \left[\int_G d\mu(g)\langle x|g\rangle \; f(g) \; \langle g|y\rangle \right] \psi(y)$$

- Quantum space of state is $L^2(G,\, d\mu(g))/N$

$$\hat{f}_G\psi(g) = \int_G d\mu(g'') \left[\int_G d\mu(g')\langle g|g'\rangle \; f(g') \; \langle g'|g''\rangle \right] \psi(g'')$$

$$N = \{\psi \in L^2(G, d\mu(g)) : \int d\mu(g')\langle g|g'\rangle\psi(g') = 0\}$$

CS quantization method- nonlocal inner product

One can construct isometry between spaces $L^2(X, d\nu(x))$ and $L^2(G, d\mu(g))/N$

$$\Psi(x) = \int_G d\mu(g)\ \langle x|g\rangle\ \Psi(g)$$
$$\Phi(g) = \int_X d\nu(x)\ \langle g|x\rangle\ \Phi(x)$$

Representations $\hat{f}_G$ on space $L^2(G, d\mu(g))/N$ and $\hat{f}_X$ on space $L^2(X, d\nu(x))$ are equivalent.

CS quantization method- nonlocal inner product

On the other hand space $L^2(G, d\mu(g))/N$ can be interpret as a Hilbert $\mathcal{H}$ space of functions $f : G \to \mathbb{C}$ with nonlocal inner product:

$$\langle\psi|\phi\rangle = \int_G d\mu(g) \int_G d\mu(g')\, \psi(g)^\star \, \langle g|g'\rangle \, \phi(g'), \qquad \psi, \phi \in \mathcal{H}$$

- CS quantization method leads to quantum Hilbert space with nonlocal inner product. The fiducial vectors shapes nonlocality in this model.
- $\langle g|g'\rangle$ denotes transition amplitude from coherent state $|g'\rangle$ to $|g\rangle$.

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CS quantization of Minkowski spacetime

Building blocks for Heisenberg-Weyl group CS quantization

- The group $G = HW(4)/\mathrm{U}(1)$
  it is parametrized by 8 parameters $p_\mu, x^\mu \in \mathbb{R}$ $\mu = 0, \ldots, 3$.
  The multiplication law for the group reads

$$g(p,x)g(\tilde{p},\tilde{x}) = \exp\left(-\frac{i}{2\hbar}(x^\mu \tilde{p}_\mu - p_\mu \tilde{x}^\mu)\hat{I}\right) g(p+\tilde{p}, x+\tilde{x})\,.$$

- Carrier space is $\mathcal{H}_x = L^2(\mathbb{R}^4, d^4\xi)$.
- The unitary irreducible representation

$$\hat{\mathcal{U}}(p,x)\psi(\xi) = \exp\left(\frac{-ip_\mu x^\mu}{2\hbar}\right)\exp\left(\frac{ip_\mu \xi^\mu}{\hbar}\right)\psi(\xi - x)\,,$$

- The coherent states are defined as follows

$$|p,x\rangle = \hat{\mathcal{U}}(p,x)|\Phi_0\rangle,$$

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Examples

Extreme fiducial vectors

- Case 1

$$\Phi_0(\xi) = \frac{1}{(2\pi\hbar)}$$

$$\langle p'', x'' | p', x' \rangle = \exp\left[i \frac{p''(x'' - x')}{2\hbar} \right] \delta^4(p'' - p')$$

If the fiducial vector is constant function the integrating kernel is orthogonal in momenta and x'', x' gives only phase shift.

- Case 2

$$\Phi_0(\xi) = \delta^4(\xi)$$

$$\langle p'', x'' | p', x' \rangle = \exp\left[-i \frac{(p'' - p')x'}{2\hbar} \right] \delta^4(x'' - x')$$

If the fiducial vector is a Dirac delta of x the integrating kernel is orthogonal in position and p'', p' gives the phase shift.

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Examples

4D harmonic oscillator ground state

It is convenient to use the fiducial vector in the form of the 4D harmonic oscillator ground state function

$$\Phi_0(\xi) = \prod_{\mu=0}^{3} \left(\frac{\lambda_\mu}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{(\xi^\mu)^2}{2\hbar\frac{1}{\lambda_\mu}}\right)$$

$$\begin{aligned} \langle p'', x''|p', x'\rangle &= \exp\left(i\frac{p''x'' - p'x'}{\hbar}\right)\exp\left(i\frac{p''x' - p'x''}{2\hbar}\right) \\ &\prod_{\mu=0}^{3} \exp\left(-\frac{(x''^\mu - x'^\mu)^2}{4\hbar\frac{1}{\lambda_\mu}} - \frac{(p''_\mu - p'_\mu)^2}{4\hbar\lambda_\mu}\right) \end{aligned}$$

The transition amplitude is given by Gaussian distribution where the variance is equal to $\hbar\lambda_\mu$ for momentum and $\hbar/\lambda_\mu$ for position coordinate.

Examples

Uncertainty principle

By using variance one can also construct the uncertainty principle [a]

$$\mathrm{var}(\hat{A};\psi)\mathrm{var}(\hat{B};\psi) \geq \frac{1}{4}\left|\langle\psi|[\hat{A},\hat{B}]|\psi\rangle\right|^2$$

[a] H. P. Robertson, "The Uncertainty Principle", Phys. Rev. **34**, 163 (1929).

For opeartors $\hat{p}_\mu, \hat{x}^\nu$ one gets $(\Phi_0(\xi) \in \mathbb{R})$

$$\frac{1}{4}\left|\langle p,x|[\hat{p}_\mu,\hat{x}^\nu]|p,x\rangle\right|^2 = \hbar^2\left|\int_{\mathbb{R}^4} d^4\xi\, \xi^\nu \Phi_0(\xi)\frac{d}{d\xi^\mu}\Phi_0(\xi)\right|^2$$

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Incase of $\Phi_0(\xi)$ taken as 4D harmonic oscillator ground state one gets

$$\frac{1}{4}\left|\langle p,x|[\hat{p}_\mu,\hat{x}^\nu]|p,x\rangle\right|^2 = \begin{cases} 0 & \text{for} \quad \nu \neq \mu \\ \frac{\hbar^2}{4} & \text{for} \quad \nu = \mu \end{cases}$$

Nonlocality analysis

It is interesting to analyze the influence of non-locality generated by fiducial vector in this Minkowski space-time model to elementary observables

- qualitative analysis of eigenproblems of elementary observables;
- analysis of transition probability of test particle.

Fiducial vector dependence

Eigensolution of momentum operator

It is easy to show that states

$$\eta_k(\xi) = \langle \xi | \eta_k \rangle := \left(\frac{1}{\sqrt{2\pi\hbar}} \right)^4 \exp\left(i \frac{k\,\xi}{\hbar} \right) ,$$

where $x\,\xi := k_\mu \xi^\mu$, with $\mu = 0, 1, 2, 3,$ are generalized eigenstates of $\hat{p}_\mu$

$$\langle \eta_k | \hat{p}_\mu | \eta_{k'} \rangle = \left[k_\mu + \int d^4 p |\tilde{\Phi}_0(p)|^2 p_\mu \right] \delta^4(k - k')$$

where $\tilde{\Phi}_0(p)$ is the Fourier transform of the fiducial vector.

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Fiducial vector dependence

Eigensolution of position operator

It is easy to show that states

$$\kappa_q(\xi) = \delta^4(\xi - q)$$

are generalized eigenstates of $\hat{x}^\mu$

$$\langle \kappa_q | \hat{x}^\mu | \kappa_{q'} \rangle = (2\pi)^2 \left[q^\mu - \int d^4x |\Phi_0(x)|^2 x^\mu \right] \delta^4(q - q')$$

Fiducial vector dependence

Eigensolution of position operator

It is easy to show that states

$$\kappa_q(\xi) = \delta^4(\xi - q)$$

are generalized eigenstates of $\hat{x}^\mu$

$$\langle \kappa_q | \hat{x}^\mu | \kappa_{q'} \rangle = (2\pi)^2 \left[q^\mu - \int d^4x |\Phi_0(x)|^2 x^\mu \right] \delta^4(q - q')$$

- In both cases the fiducial vector the eigenvectors does not depend on fiducial vector. A dependence appears only as shift of eigenvalues.
- One can lead the shift to zero by properly choose of Φ_0. It is sufficient to take Φ_0 as an even or odd function of each of its variables to provide this property.

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Fiducial vector dependence

Eigensolution of Hamiltonian operator

$$\hat{H} = (2\pi\hbar)^{-4} \int_{\mathbb{R}^8} d^4p\, d^4x\, |p, x\rangle \frac{1}{2} g^{\mu\nu} p_\mu p_\nu \langle p, x|$$

The eigenvectors of Hamiltonian are the same as in momentum operator

$$\langle \eta_{k'} | \hat{H} | \eta_k \rangle = \tfrac{1}{2} g^{\alpha\beta} \Big[k_\alpha k_\beta + +2k_\alpha \int_{\mathbb{R}^4} d^4p\, p_\beta |\tilde{\Phi}_0(p)|^2 + + \int_{\mathbb{R}^4} d^4p\, p_\alpha p_\beta |\tilde{\Phi}_0(p)|^2 \Big] \delta^4(k' - k)$$

Taking Φ_0 as a harmonic oscillator ground state with $\lambda_0/3 = \lambda_1 = \lambda_2 = \lambda_3$ the second and third terms vanish.

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Transition probability of a test particle

The mass layer of thickness ϵ for a test particle $m \geq 0$

$$\mathcal{J}_{m,\epsilon} = \left\{ p \ : \ -\sqrt{m^2 + \vec{p}^2 + \epsilon} \leq p_0 \leq -\sqrt{m^2 + \vec{p}^2}, \quad \vec{p} \in \mathbb{R}^3 \right\} ,$$

The operator which projecting onto the mass layer

$$P_{\mathcal{J}_{m,\epsilon}} = \int_{\mathbb{R}^4} d^4 p \, |\eta_p\rangle \chi(p \in \mathcal{J}_{m,\epsilon}) \langle \eta_p| \, ,$$

where

$$\chi(p \in Q) = \begin{cases} 1 & \text{if} \quad p \in Q \\ 0 & \text{if} \quad p \notin Q \end{cases}$$

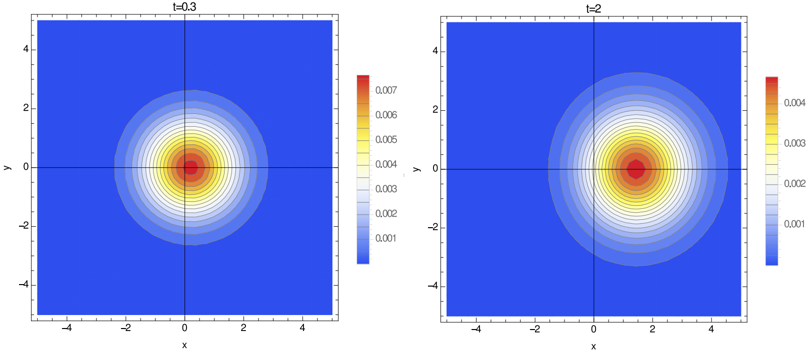
Transition amplitude

The transition amplitude of the particle of mass m from the state $\langle p', x'|$ to the state $\langle p'', x''|$ is given as follows

$$\mathcal{A}_{m,\epsilon} = \langle p'', x''| P_{\mathcal{J}_{m,\epsilon}} |p', x'\rangle = \int_{\mathbb{R}^4} d^4 p \, \langle p'', x''|\eta_p\rangle \chi(p \in \mathcal{J}_{m,\epsilon}) \langle \eta_p|p', x'\rangle$$

Transition probability of a test particle–Examples $\lambda_0 = 3\lambda_3 = 3$

$$\Phi_0(\xi) = \frac{\sqrt[4]{\lambda_0 \lambda_3^3}}{\pi \hbar} \exp\left(-\frac{\lambda_0 (\xi^0)^2}{2\hbar}\right) \exp\left(-\frac{\lambda_3 \vec{\xi}^2}{2\hbar}\right)$$



Plot of $|\mathcal{A}_{m,\epsilon}(\vec{x}'')/\epsilon|^2$, as a function of $\vec{x}''$, in the xy-plane
$m = 1$, $\vec{p}\,' = (1, 0, 0)$, $\vec{x}\,' = (0, 0, 0)$, $p_0' = -\sqrt{m^2 + (\vec{p}\,')^2}$
$p' = p''$.

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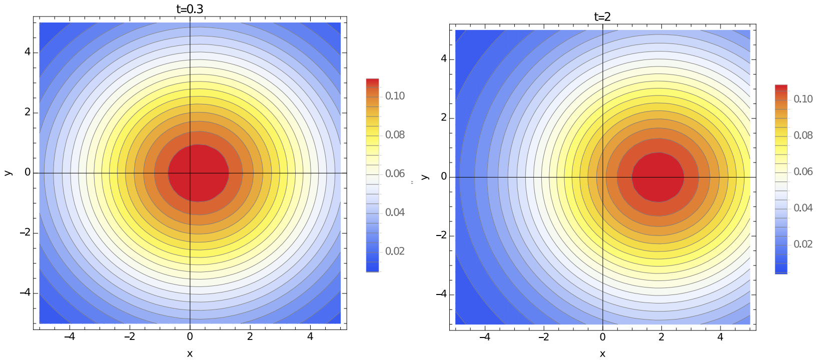
Transition probability of a test particle–Examples $\lambda_0 = 3\lambda_3 = 0.3$

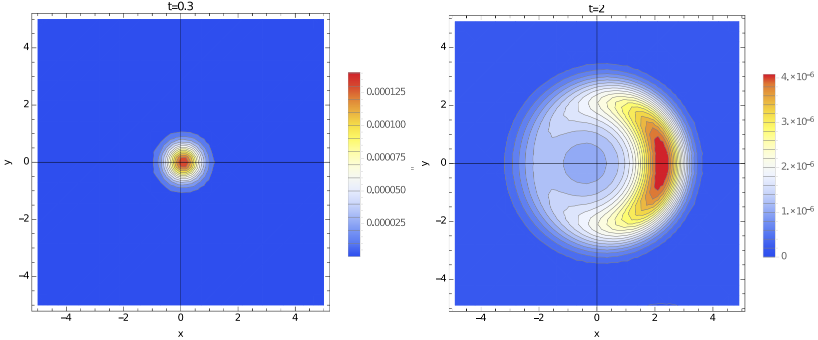
$$\Phi_0(\xi) = \frac{\sqrt[4]{\lambda_0 \lambda_3^3}}{\pi \hbar} \exp\left(-\frac{\lambda_0 (\xi^0)^2}{2\hbar}\right) \exp\left(-\frac{\lambda_3 \vec{\xi}^2}{2\hbar}\right)$$



Plot of $|\mathcal{A}_{m,\epsilon}(\vec{x}'')/\epsilon|^2$, as a function of $\vec{x}''$, in the xy-plane $m = 1$, $\vec{p}_0 = (1, 0, 0)$, $\vec{x}' = (0, 0, 0)$, $p'_0 = -\sqrt{m^2 + (\vec{p}')^2}$ $p' = p''$.

Transition probability of a test particle–Examples $\lambda_0 = 3\lambda_3 = 30$

$$\Phi_0(\xi) = \frac{\sqrt[4]{\lambda_0 \lambda_3^3}}{\pi \hbar} \exp\left(-\frac{\lambda_0 (\xi^0)^2}{2\hbar}\right) \exp\left(-\frac{\lambda_3 \vec{\xi}^2}{2\hbar}\right)$$



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Expectation value

Expectation vale of space position x^n, $n = 1, 2, 3$ with probability distribution given by transition amplitude

$$P(\vec{x}' \to \vec{x}'') = \frac{1}{\mathcal{N}} \left| \langle p'', x'' | P_{\mathcal{J}_{m,\epsilon}} | p', x' \rangle \right|^2$$

under condition $p', p'', x', x''^0 = const.$

$$E(x^n) = \frac{1}{\mathcal{N}} \int_{\mathbb{R}^2} d^3\vec{x}'' \, x''^n \left| \langle p'', x'' | P_{\mathcal{J}_{m,\epsilon}} | p', x' \rangle \right|^2 = \\ = x'^n + (x''^0 - x'^0) \langle p_n \rangle$$

where

$$\langle p_n \rangle = \frac{(2\pi\hbar)^2}{\mathcal{N}} \int_{\mathbb{R}^3} d^3\vec{p} \, \frac{p_n}{\sqrt{m^2 + \vec{p}^2}} \left| \mathbb{F}(\vec{p}) \right|^2$$

$$\mathbb{F}(\vec{p}) = \tilde{\Phi}_0(p'' - p)^\star \tilde{\Phi}_0(p' - p) \Big|_{p_0 = -\sqrt{m^2 + \vec{p}^2}}$$

$$\tilde{\Phi}_0(p) \in \mathbb{R}$$

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Summary

- The CS quantization leads to quantum Hilbert space with nonlocal inner product where nonlocality is shaped by a fiducial vector.
- In the quantum Minkowski spacetime model based on $HW(4)/U(1)$ CS quantization method the position, momentum, and Hamiltonian observables have eigenfunctions independent of the fiducial vector. The fiducial vector shifts the eigenvalues but it can be reduced by the selection of required properties.
- The expectation value of the position of a test particle keeps the classical behavior of the test particle. The fiducial vector scales the value of the "momentum" in expectation value dependence.

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