Dissipative Effects in the ADM Hamiltonian Formalism for Point-Mass Systems

Arkadiusz Błaut[1] Piotr Jaranowski[2] Gerhard Schäfer[3]

[1]Faculty of Physics and Astronomy, University of Wrocław, Poland

[2]Faculty of Physics, University of Białystok, Poland

[3]Institute of Theoretical Physics, Friedrich Schiller University Jena, Germany

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Background and Outline (1/2)

- We consider a system of two point masses (i.e. monopolar, pointlike bodies), which interact gravitationally according to general relativity theory.
  Spin- and tidal-related effects will not be discussed here (but these effects can be and have been added to the formalism).
- We model point masses by means of Dirac δ distributions.
- We employ the ADM canonical formalism in $D = d + 1$ spacetime dimensions.
- We work in asymptotically flat D-dimensional spacetime
  and use asymptotically Minkowskian reference frame.
- All calculations are done within the post-Newtonian (PN) approximation:
  0th order—Newtonian gravity;
  nPN order—corrections to the Newtonian gravity of order

$$\left(\frac{v}{c}\right)^{2n} \sim \left(\frac{Gm}{rc^2}\right)^{n}.$$

- δ-sources lead to ultraviolet (UV) divergences, i.e., divergences at the locations of the particles. We control them by means of dimensional regularization (DR).
- PN expansion of the retardations in the field functions (which is essentially a near-zone expansion) lead to infrared (IR) divergences.

- We treat differently conservative and dissipative sections of dynamics: to solve (perturbatively) equations for the field degrees of freedom, we use time-symmetric (half-retarded half-advanced) Green's function for conservative dynamics, and retarded Green's function for dissipative dynamics.
- The following conservative Hamiltonians were uniquely calculated: at Newtonian, 1PN, 2PN, 3PN, and 4PN orders (at 3PN and 4PN by Damour/Jaranowski/Schäfer in 2001 and 2014, respectively); dissipative Hamiltonians were computed at 2.5PN and 3.5PN orders (by Jaranowski/Schäfer in 1997).
- For conservative dynamics, near-zone IR divergences show up at the 4PN order, they are linked to nonlocal-in-time tail effects and were analytically regulated using a new (i.e., different from DR-related one ℓ_0, $G_D = G_{\mathrm{N}}\ell_0^{d-3}$) length scale. The result of regularization was ambiguous and the ambiguity was resolved by using a beyond-near-zone information (delivered by gravitational self-force approach).
- For dissipative dynamics, IR divergences are not an issue at 2.5PN, 3.5PN, and 4.5PN orders (but they will be an issue for the higher orders).
- From 2.5PN and 3.5PN dissipative Hamiltonians one can deduce (and it was done) the leading-order (Newtonian) and 1PN formulae for GW luminosities.
- In this report, we propose a new formula for calculating GW luminosities that does not require finding an explicit form of dissipative Hamiltonians.
- Using the new formula we calculated Newtonian and 1PN GW luminosities, calculation of GW luminosity at the 2PN order is in progress (2PN-order GW luminosity can also be derived from 4.5PN dissipative Hamiltonian).

The dimensionless GW strain measured by the laser-interferometric detector, induced by gravitational waves from coalescing compact binary made of nonspinning bodies in circular orbits during the inspiral phase:

$$h(t) = \frac{C}{D}\left[\dot{\phi}(t)\right]^{2/3} \sin\left[2\phi(t) + \alpha\right],$$

where $\phi(t)$ is the orbital phase of the binary [$\dot{\phi}(t) := \frac{\mathrm{d}\phi(t)}{\mathrm{d}t}$ is the angular frequency], D is the luminosity distance of the binary to the Earth (C and α are some constants).

The time evolution of the orbital phase $\phi(t)$ is computed from the balance equation (E—binding energy, $\mathcal{L}$—GW luminosity):

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{L} \quad \Longrightarrow \quad \phi = \phi(t),$$

where both sides are calculated within the PN approximation (the dates: for energies—the first complete and correct derivations of equations of motion at a given PN order; for luminosities—often the dates of the first derivation of formula valid only for circular orbits):

$$E = \overset{1687}{\boxed{E_{\mathrm{N}}}} + \frac{1}{c^2}\overset{1917}{\boxed{E_{\mathrm{1PN}}}} + \frac{1}{c^4}\overset{1982}{\boxed{E_{\mathrm{2PN}}}} + \frac{1}{c^6}\overset{2001}{\boxed{E_{\mathrm{3PN}}}} + \frac{1}{c^8}\overset{2014}{\boxed{E_{\mathrm{4PN}}}} + \mathcal{O}\left((v/c)^{10}\right),$$

$$\mathcal{L} = \overset{1918/1963}{\boxed{\mathcal{L}_{\mathrm{N}}}} + \frac{1}{c^2}\overset{1976}{\boxed{\mathcal{L}_{\mathrm{1PN}}}} + \frac{1}{c^3}\overset{1992}{\boxed{\mathcal{L}_{\mathrm{1.5PN}}}} + \frac{1}{c^4}\overset{1995}{\boxed{\mathcal{L}_{\mathrm{2PN}}}} + \frac{1}{c^5}\overset{1996}{\boxed{\mathcal{L}_{\mathrm{2.5PN}}}}$$
$$+ \frac{1}{c^6}\overset{2004}{\boxed{\mathcal{L}_{\mathrm{3PN}}}} + \frac{1}{c^7}\overset{1998}{\boxed{\mathcal{L}_{\mathrm{3.5PN}}}} + \frac{1}{c^8}\overset{2023}{\boxed{\mathcal{L}_{\mathrm{4PN}}}} + \frac{1}{c^9}\overset{2023}{\boxed{\mathcal{L}_{\mathrm{4.5PN}}}} + \mathcal{O}\left((v/c)^{10}\right).$$

4.5PN-Accurate Binding Energy in the Center-of-Mass Frame for Circular Orbits

Notation:

masses of the bodies: m_1, m_2, $\quad M := m_1 + m_2$, $\quad \mu := \dfrac{m_1 m_2}{M}$,

$$\nu := \frac{\mu}{M} = \frac{m_1 m_2}{(m_1+m_2)^2}, \quad 0 \le \nu \le \frac{1}{4};$$

$$x := \frac{(GM\dot{\phi})^{2/3}}{c^2} \quad \text{(dimensionless PN parameter for circular orbits).}$$

Binding energy of two-point-mass system in circular orbits:

$$E(x,\nu) = -\frac{\mu c^2 x}{2}\left(1 + e_{1\mathrm{PN}}(\nu)\,x + e_{2\mathrm{PN}}(\nu)\,x^2 + e_{3\mathrm{PN}}(\nu)\,x^3 + \left(e_{4\mathrm{PN}}(\nu) + \frac{448}{15}\nu\ln x\right)x^4 + \mathcal{O}(x^5)\right),$$

$$e_{1\mathrm{PN}}(\nu) = -\frac{3}{4} - \frac{1}{12}\nu, \qquad e_{2\mathrm{PN}}(\nu) = -\frac{27}{8} + \frac{19}{8}\nu - \frac{1}{24}\nu^2,$$

$$e_{3\mathrm{PN}}(\nu) = -\frac{675}{64} + \left(\frac{34445}{576} - \frac{205}{96}\pi^2\right)\nu - \frac{155}{96}\nu^2 - \frac{35}{5184}\nu^3,$$

$$e_{4\mathrm{PN}}(\nu) = -\frac{3969}{128} + \left(-\frac{123671}{5760} + \frac{9037}{1536}\pi^2 + \frac{896}{15}(2\ln 2 + \gamma_{\mathrm{E}})\right)\nu$$
$$+ \left(-\frac{498449}{3456} + \frac{3157}{576}\pi^2\right)\nu^2 + \frac{301}{1728}\nu^3 + \frac{77}{31104}\nu^4$$

(γ_{E} is the Euler's constant).

4.5PN-Accurate GW Luminosity for Circular Orbits

$$\mathcal{L}(x,\nu) = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ 1 + \ell_{1\mathrm{PN}}(\nu)\,x + 4\pi\,x^{3/2} + \ell_{2\mathrm{PN}}(\nu)\,x^2 + \ell_{2.5\mathrm{PN}}(\nu)\,x^{5/2} + \Big(\ell_{3\mathrm{PN}}(\nu) - \frac{856}{105}\ln(16x)\Big)\,x^3$$
$$+ \ell_{3.5\mathrm{PN}}(\nu)\,x^{7/2} + \bigg(\ell_{4\mathrm{PN}}(\nu) + \Big(\frac{232597}{8820} + \frac{20739}{245}\nu\Big)\ln x\bigg)\,x^4$$
$$+ \Big(\ell_{4.5\mathrm{PN}}(\nu) - \frac{3424}{105}\pi\ln(16\,x)\Big)\,x^{9/2} + \mathcal{O}(x^5)\Bigg\},$$

$$\ell_{1\mathrm{PN}}(\nu) = -\frac{1247}{336} - \frac{35}{12}\nu, \qquad \ell_{2\mathrm{PN}}(\nu) = -\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2, \qquad \ell_{2.5\mathrm{PN}}(\nu) = \Big(-\frac{8191}{672} - \frac{535}{24}\nu\Big)\pi,$$

$$\ell_{3\mathrm{PN}}(\nu) = \frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_\mathrm{E} + \Big(-\frac{134543}{7776} + \frac{41}{48}\pi^2\Big)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3,$$

$$\ell_{3.5\mathrm{PN}}(\nu) = \Big(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\Big)\pi,$$

$$\ell_{4\mathrm{PN}}(\nu) = -\frac{323105549467}{3178375200} + \frac{232597}{4410}\gamma_\mathrm{E} - \frac{1369}{126}\pi^2 + \frac{39931}{294}\ln 2 - \frac{47385}{1568}\ln 3$$
$$+ \Big(-\frac{1452202403629}{1466942400} + \frac{41478}{245}\gamma_\mathrm{E} - \frac{267127}{4608}\pi^2 + \frac{479062}{2205}\ln 2 + \frac{47385}{392}\ln 3\Big)\nu$$
$$+ \Big(\frac{1607125}{6804} - \frac{3157}{384}\pi^2\Big)\nu^2 + \frac{6875}{504}\nu^3 + \frac{5}{6}\nu^4,$$

$$\ell_{4.5\mathrm{PN}}(\nu) = \bigg(\frac{265978667519}{745113600} - \frac{6848}{105}\gamma_\mathrm{E} + \Big(\frac{2062241}{22176} + \frac{41}{12}\pi^2\Big)\nu - \frac{133112905}{290304}\nu^2 - \frac{3719141}{38016}\nu^3\bigg)\pi.$$

Project's Goals

Short-Term Goals

- Higher-PN-order perturbative solutions of two-body problem are complicated, both from computational and from conceptual point of view. Therefore it is highly desired to have more than one independent derivation of any analytical result: making independent derivation within the ADM Hamiltonian approach of GW luminosities of two-point-mass system at the N, 1PN, 1.5PN, 2PN, 2.5PN, 3PN, 3.5PN, 4PN, and 4.5PN order.
- Recompute and regularize IR divergences in the 4PN two-point-mass ADM Hamiltonian, without usage of gravitational self-force results.

Long-Term Goals

- Completion of computations of 5PN, 5.5PN, 6PN, ... EOM of two-point-mass systems together with computation of GW luminosities at 5PN, 5.5PN, 6PN, ... orders, and construction of $\geq$5PN-accurate templates for inspiralling compact binaries.
- Computation, within the PN framework, higher-order spin-dependent effects and, in the case of binaries containing neutron stars, higher-order tidal corrections, both in conservative dynamics and in GW luminosities.

Notation and Units

Spacetime coordinates: $x^0 = c\,t, \quad \mathbf{x} = (x^1, \dots, x^d)$.

Particle labels: $a, b \in \{1, 2\}$,

masses of the particles: m_a,

position vectors of the particles: $\mathbf{x}_a = (x_a^1, \dots, x_a^d)$,

linear momentum vectors of the particles: $\mathbf{p}_a = (p_{a1}, \dots, p_{ad})$.

For any d-vectors $\mathbf{v} = (v^1, \dots, v^d)$ and $\mathbf{w} = (w^1, \dots, w^d)$:

$$\mathbf{v} \cdot \mathbf{w} := \delta_{ij} v^i w^j, \quad |\mathbf{v}| := \sqrt{\mathbf{v} \cdot \mathbf{v}}.$$

$$\mathbf{r}_a := \mathbf{x} - \mathbf{x}_a, \quad r_a := |\mathbf{r}_a|, \quad \mathbf{n}_a := \mathbf{r}_a / r_a;$$

$$\text{for } a \neq b\text{:} \quad \mathbf{r}_{ab} := \mathbf{x}_a - \mathbf{x}_b, \quad r_{ab} := |\mathbf{r}_{ab}|, \quad \mathbf{n}_{ab} := \mathbf{r}_{ab} / r_{ab}.$$

Units: quite often $c = 1$ and $G_D = 1/(16\pi)$.

A $(d+1)$-Splitting of Spacetime Metric $g_{\mu\nu}$

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -(N\,\mathrm{d}t)^2 + \gamma_{ij}(\mathrm{d}x^i + N^i\,\mathrm{d}t)(\mathrm{d}x^j + N^j\,\mathrm{d}t),$$

where N and N^i are respectively lapse and shift functions,

$$\gamma_{ij} := g_{ij}, \quad N := (-g^{00})^{-1/2}, \quad N^i = \gamma^{ij}N_j \quad \text{with} \quad N_i := g_{0i},$$

here γ^{ij} is the metric inverse to γ_{jk} $(\gamma^{ij}\gamma_{jk} = \delta^i_k)$,

$$\gamma := \det(\gamma_{ij});$$

lowering and raising of spatial indices is with γ_{ij}.

Canonical Matter+Field Variables

Canonical matter variables:

$$\mathbf{x}_a = (x_a^1, \dots, x_a^d), \qquad \mathbf{p}_a = (p_{a1}, \dots, p_{ad}), \qquad a = 1, 2.$$

Canonical field variables:

$$\gamma_{ij} := g_{ij},$$

$$\pi^{ij} := \sqrt{\gamma}(K^{ij} - \gamma^{ij}\gamma^{kl}K_{kl}),$$

K_{ij} is the extrinsic curvature of the hypersurface $t = const$.

ADM Hamiltonian

- The full Einstein field equations in D dimensions in an asymptotically flat space-time and in an asymptotically Minkowskian coordinate system are derivable from the Hamiltonian

$$H[\mathbf{x}_a, \mathbf{p}_a, \gamma_{ij}, \pi^{ij}, N, N^i] = \int \mathrm{d}^d x \, (N\mathcal{H} - N^i \mathcal{H}_i) + \oint_{i^0} \mathrm{d}^{d-1} S_i \, \partial_j (\gamma_{ij} - \delta_{ij}\gamma_{kk}),$$

  i^0 denotes spacelike infinity and $\mathrm{d}^{d-1}S_i$ is the $(d-1)$-dimensional out-pointing surface element there.

- The super-Hamiltonian $\mathcal{H}$ and super-momentum $\mathcal{H}_i$ are defined as follows:

$$\mathcal{H}(\mathbf{x}_a, \mathbf{p}_a, \gamma_{ij}, \pi^{ij}) := \sqrt{\gamma} N^2 \left(T^{00} - 2G^{00} \right),$$
$$\mathcal{H}_i(\mathbf{x}_a, \mathbf{p}_a, \gamma_{ij}, \pi^{ij}) := \sqrt{\gamma} N \left(T_i^0 - 2G_i^0 \right).$$

  where $T^{\mu\nu}$ and $G^{\mu\nu}$ denote the energy-momentum and the Einstein tensor, respectively,

Constraint Equations

The lapse and shift functions are Lagrangian multipliers and deliver the Hamiltonian and momentum constraint equations of the Einstein theory,

$$\mathcal{H} = 0, \quad \mathcal{H}_i = 0.$$

2-Point-Mass Energy-Momentum Tensor

Source terms for the constraint equations are derived from the 2-point-mass energy-momentum tensor

$$T^{\alpha\beta}(x^\mu) := \sum_{a=1}^{2} m_a \int_{-\infty}^{+\infty} \frac{u_a^\alpha u_a^\beta}{\sqrt{-\det(g_{\mu\nu})}} \delta^{d+1}\big(x^\mu - \xi_a^\mu(\tau_a)\big)\, \mathrm{d}\tau_a,$$

τ_a is the proper time along the world line $x^\mu = \xi_a^\mu(\tau_a)$ of the ath particle, and $u_a^\alpha := \mathrm{d}\xi_a^\alpha/\mathrm{d}\tau_a$.

Constraint Equations for 2-Point-Mass Systems

- The constraint equations:

$$\sqrt{\gamma}\, R - \frac{1}{\sqrt{\gamma}} \left(\gamma_{ik}\, \gamma_{j\ell}\, \pi^{ij}\, \pi^{k\ell} - \frac{(\gamma_{ij}\, \pi^{ij})^2}{d-1} \right) = \sum_{a=1}^{2} \sqrt{\gamma_a^{ij} p_{ai} p_{aj} + m_a^2}\, \delta^d(\mathbf{x} - \mathbf{x}_a),$$

$$-2D_j \pi^{ij} = \sum_{a=1}^{2} \gamma_a^{ij} p_{aj}\, \delta^d(\mathbf{x} - \mathbf{x}_a),$$

R is the spatial scalar curvature of the hypersurface $t = const$,
D_j is the spatial d-dimensional covariant derivative
(acting on a tensor density of weight one),
$\gamma_a^{ij} := \gamma_{\text{reg}}^{ij}(\mathbf{x}_a)$ is perturbatively unambigously defined and finite
(at least up to the 4.5PN order).

Fixing the Gauge: ADMTT Gauge (1/2)

- The ADM Transverse-Traceless (TT) gauge:

$$\gamma_{ij} = \left(1 + \frac{d-2}{4(d-1)}\phi\right)^{4/(d-2)} \delta_{ij} + h_{ij}^{\mathrm{TT}}, \quad \pi^{ii} = 0,$$

where $h_{ii}^{\mathrm{TT}} = 0$ and $\partial_j h_{ij}^{\mathrm{TT}} = 0$.

- Splitting of the field momentum:

$$\pi^{ij} = \tilde{\pi}^{ij}(V^k) + \pi_{\mathrm{TT}}^{ij},$$

$$\tilde{\pi}^{ij}(V^k) = \partial_i V^j + \partial_j V^i - \frac{2}{d}\delta^{ij}\partial_k V^k,$$

where $\pi_{\mathrm{TT}}^{ii} = 0$ and $\partial_j \pi_{\mathrm{TT}}^{ij} = 0$.

The super/subscript TT denotes the application of the d-dimensional (spatially nonlocal) TT-projection operator:

$$f_{ij}^{\mathrm{TT}} := \delta_{ij}^{\mathrm{TT}kl} f_{kl},$$

where

$$\delta_{ij}^{\mathrm{TT}kl} := \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{d-1}\delta_{ij}\delta_{kl} - \frac{1}{2}(\delta_{ik}\partial_j\partial_l + \delta_{jl}\partial_i\partial_k + \delta_{il}\partial_j\partial_k + \delta_{jk}\partial_i\partial_l)\Delta^{-1} + \frac{1}{d-1}(\delta_{ij}\partial_k\partial_l + \delta_{kl}\partial_i\partial_j)\Delta^{-1} + \frac{d-2}{d-1}\partial_i\partial_j\partial_k\partial_l\Delta^{-2}.$$

Fixing the Gauge: ADMTT Gauge (2/2)

Asymptotic behavior for $r \to \infty$:

$$\phi \sim \frac{1}{r^{d-2}}, \quad V^i \sim \frac{1}{r^{d-2}}, \quad h_{ij}^{\mathrm{TT}} \sim \frac{1}{r^{d-2}}, \quad \pi_{\mathrm{TT}}^{ij} \sim \frac{1}{r^{d-1}}.$$

A Perturbative Solving of the Constraints

- ϕ and V^i are expressed in terms of $(\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\mathrm{TT}}, \pi_{\mathrm{TT}}^{ij})$ by a perturbative solving of the constraint equations—this is done by the PN expansion of the constraints together with the PN expansion of the functions ϕ and V^i
  (the numbers within parentheses denote the order in the inverse velocity of light, e.g. $\phi_{(2)} \sim \mathcal{O}(c^{-2})$):

$$\phi = \phi_{(2)} + \phi_{(4)} + \phi_{(6)} + \cdots, \qquad V^i = V^i_{(3)} + V^i_{(5)} + \cdots.$$

Reduced Matter+Field ADM Hamiltonian

- If the constraint equations and the gauge conditions are both satisfied, the total matter+field Hamiltonian can be written in its reduced form:

$$H_{\text{red}}\left[\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \pi_{\text{TT}}^{ij}\right] = -\sum_{n=2}^{\infty} \int \mathrm{d}^d x \, \Delta\phi_{(n)}\left[\mathbf{x}_a, \mathbf{p}_a, h_{ij}^{\text{TT}}, \pi_{\text{TT}}^{ij}\right]$$

- The equations of motion for the particles:

$$\dot{\mathbf{p}}_a = -\frac{\delta H_{\text{red}}}{\delta \mathbf{x}_a}, \quad \dot{\mathbf{x}}_a = \frac{\delta H_{\text{red}}}{\delta \mathbf{p}_a} \quad (a = 1, 2).$$

  Evolution equations for the field degrees of freedom:

$$\frac{\partial}{\partial t} h_{ij}^{\text{TT}} = \delta_{ij}^{\text{TT}kl} \frac{\delta H_{\text{red}}}{\delta \pi_{\text{TT}}^{kl}}, \quad \frac{\partial}{\partial t} \pi_{\text{TT}}^{ij} = -\delta_{kl}^{\text{TT}ij} \frac{\delta H_{\text{red}}}{\delta h_{kl}^{\text{TT}}}.$$

- There is no involvement of lapse and shift functions in the equations of motion and in the field equations for the independent degrees of freedom.

Field Equations (1/3)

- For computing 2PN-order GW luminosity one needs to use field equations which follow from the 4.5PN-accurate reduced Hamiltonian:

$$H^{\rm red}_{\leq 4.5{\rm PN}}[\mathbf{x}_a, \mathbf{p}_a, h^{\rm TT}_{ij}, \pi^{ij}_{\rm TT}] = \int \mathrm{d}^d x\, \mathfrak{h}_{\leq 4.5{\rm PN}}[\mathbf{x} - \mathbf{x}_a, \mathbf{p}_a, h^{\rm TT}_{ij}, \pi^{ij}_{\rm TT}],$$

where

$$\mathfrak{h}_{\leq 4.5{\rm PN}}[\mathbf{x} - \mathbf{x}_a, \mathbf{p}_a, h^{\rm TT}_{ij}, \pi^{ij}_{\rm TT}] = \sum_a m_a \delta^d(\mathbf{x} - \mathbf{x}_a) + \mathfrak{h}_{(4)}\big(\mathbf{x} - \mathbf{x}_a, \mathbf{p}_a\big)$$
$$+ \mathfrak{h}_{(6)}\big(\mathbf{x} - \mathbf{x}_a, \mathbf{p}_a\big) + \mathfrak{h}_{(8)}\big(\mathbf{x} - \mathbf{x}_a, \mathbf{p}_a, h^{\rm TT}_{ij}\big) + \mathfrak{h}_{(10)}\big(\mathbf{x} - \mathbf{x}_a, \mathbf{p}_a, h^{\rm TT}_{ij}, \pi^{ij}_{\rm TT}\big)$$
$$+ \mathfrak{h}_{(12)}[\mathbf{x} - \mathbf{x}_a, \mathbf{p}_a, h^{\rm TT}_{ij}, \pi^{ij}_{\rm TT}].$$

- For this Hamiltonian the field equations take the form

$$\dot{h}^{\rm TT}_{ij} = \delta^{{\rm TT}kl}_{ij} \frac{\partial \mathfrak{h}_{\leq 4.5{\rm PN}}}{\partial \pi^{kl}_{\rm TT}} + \mathcal{O}(c^{-9}),$$
$$\dot{\pi}^{ij}_{\rm TT} = -\delta^{{\rm TT}ij}_{kl} \left\{ \frac{\delta \mathfrak{h}_{\leq 4.5{\rm PN}}}{\delta h^{\rm TT}_{kl}} - \left(\frac{\partial \mathfrak{h}_{\leq 4.5{\rm PN}}}{\partial h^{\rm TT}_{kl,m}} \right)_{,m} + \left(\frac{\partial \mathfrak{h}_{\leq 4.5{\rm PN}}}{\partial h^{\rm TT}_{kl,mn}} \right)_{,mn} \right\} + \mathcal{O}(c^{-10}).$$

- More explicitly (we display only leading-order and next-to-leading-order terms),

$$\dot{h}_{ij}^{\mathrm{TT}} = \delta_{ij}^{\mathrm{TT}kl}\left\{2\pi_{\mathrm{TT}}^{kl} - \frac{2(d-2)}{d-1}\phi_{(2)}\tilde{\pi}_{(3)}^{kl}\right\} + \mathcal{O}(c^{-7}),$$

$$\dot{\pi}_{\mathrm{TT}}^{ij} = -\delta_{ij}^{\mathrm{TT}kl}\left\{\frac{1}{2}S_{(4)kl} - \frac{1}{2}\Delta h_{kl}^{\mathrm{TT}} + B_{(6)kl} + \frac{1}{2(d-1)}\left(\phi_{(2)}\Delta h_{kl}^{\mathrm{TT}} + \Delta\left(\phi_{(2)}h_{kl}^{\mathrm{TT}}\right)\right)\right\} + \mathcal{O}(c^{-8}).$$

- By combining these two equations one gets the equation for h_{ij}^{TT},

$$\Box h_{ij}^{\mathrm{TT}} = S_{ij}^{\mathrm{TT}}, \quad \Box := -\partial_t^2 + \Delta,$$

where the source term is

$$S_{ij}^{\mathrm{TT}} = \delta_{ij}^{\mathrm{TT}kl}\left\{S_{(4)kl} + 2B_{(6)kl} + \frac{2(d-2)}{d-1}\partial_t\left(\phi_{(2)}\tilde{\pi}_{(3)}^{kl}\right) + \frac{1}{d-1}\left(\phi_{(2)}\Delta h_{kl}^{\mathrm{TT}} + \Delta\left(\phi_{(2)}h_{kl}^{\mathrm{TT}}\right)\right)\right\} + \mathcal{O}(c^{-8}).$$

- After solving field equation for h_{ij}^{TT} one can obtain π_{TT}^{ij}:

$$\pi_{\mathrm{TT}}^{ij} = \frac{1}{2}\dot{h}_{ij}^{\mathrm{TT}} + \frac{d-2}{d-1}\delta_{ij}^{\mathrm{TT}kl}\left(\phi_{(2)}\tilde{\pi}_{(3)}^{kl}\right) + \mathcal{O}(c^{-7}).$$

Field Equations (3/3)

- After making the PN expansion of the formal retarded solution of the field equation, one gets (conservative dynamics does not depend on the functions marked in red)

$$h_{ij}^{\mathrm{TT}} = h_{(4)ij}^{\mathrm{TT}} + h_{(5)ij}^{\mathrm{TT}} + h_{(6)ij}^{\mathrm{TT}} + h_{(7)ij}^{\mathrm{TT}} + \mathcal{O}(c^{-8}),$$
$$\pi_{\mathrm{TT}}^{ij} = \pi_{\mathrm{TT}}^{(5)ij} + \pi_{\mathrm{TT}}^{(6)ij} + \mathcal{O}(c^{-7}).$$

- The PN expansion of the retardations in the field functions leads to new functions which diverge for $r \to \infty$:

$$h_{ij}^{\mathrm{TT}}(t, \mathbf{n}r) = \frac{h_{ij}(t - r, \mathbf{n})}{r^{d-2}} + \mathcal{O}\Big(\frac{1}{r^{d-1}}\Big)$$
$$= \frac{h_{ij}(t, \mathbf{n})}{r^{d-2}} - \dot{h}_{ij}(t, \mathbf{n}) r^{3-d} + \frac{1}{2}\ddot{h}_{ij}(t, \mathbf{n}) r^{4-d} + \cdots + \mathcal{O}\Big(\frac{1}{r^{d-1}}\Big).$$

This is the source of infrared near-zone divergences.

3.5PN-Accurate Dissipative Matter Hamiltonian (1/2)

- More detailed and refined treatment can be found in Section 3 of 2024 Schäfer/Jaranowski *Living Reviews in Relativity* article.
- The split of the total reduced Hamiltonian:

$$H^{\text{red}}_{\leq 3.5\text{PN}}[\mathbf{x}_a, \mathbf{p}_a, h^{\text{TT}}_{ij}, \pi^{ij}_{\text{TT}}] = H^{\text{matter}}_{\leq 3.5\text{PN}}(\mathbf{x}_a, \mathbf{p}_a) + H^{\text{int}}_{\leq 3.5\text{PN}}[\mathbf{x}_a, \mathbf{p}_a, h^{\text{TT}}_{ij}, \pi^{ij}_{\text{TT}}] + H^{\text{field}}_{\leq 3.5\text{PN}}[h^{\text{TT}}_{ij}, \pi^{ij}_{\text{TT}}].$$

- We define a new Hamiltonian $\widetilde{H}_{\leq 3.5\text{PN}}$, which is the Hamiltonian $H_{\leq 3.5\text{PN}}$ after dropping its field part,

$$\widetilde{H}_{\leq 3.5\text{PN}} := H^{\text{matter}}_{\leq 3.5\text{PN}} + H^{\text{int}}_{\leq 3.5\text{PN}}.$$

3.5PN-Accurate Dissipative Matter Hamiltonian (2/2)

- The Hamiltonian $\widetilde{H}_{\leq 3.5\mathrm{PN}}$ can be decomposed into conservative and dissipative parts:

$$\widetilde{H}_{\leq 3.5\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a, t) = H^{\mathsf{con}}_{\leq 3\mathsf{PN}}(\mathbf{x}_a, \mathbf{p}_a, t) + H^{\mathsf{diss}}_{\leq 3.5\mathsf{PN}}(\mathbf{x}_a, \mathbf{p}_a, t),$$

where

$$\begin{aligned} H^{\mathsf{con}}_{\leq 3\mathsf{PN}}(\mathbf{x}_a, \mathbf{p}_a, t) := H^{\mathsf{matter}}_{\mathrm{N}}(\mathbf{x}_a, \mathbf{p}_a) &+ H^{\mathsf{matter}}_{1\mathrm{PN}}(\mathbf{x}_a, \mathbf{p}_a) \\ &+ H^{\mathsf{matter}}_{2\mathsf{PN}}(\mathbf{x}_a, \mathbf{p}_a) + H^{\mathsf{int}}_{2\mathsf{PN}}[\mathbf{x}_a, \mathbf{p}_a, h^{\mathsf{TT}}_{ij}(t)] \\ &+ H^{\mathsf{matter}}_{3\mathsf{PN}}(\mathbf{x}_a, \mathbf{p}_a) + H^{\mathsf{int}}_{3\mathsf{PN}}[\mathbf{x}_a, \mathbf{p}_a, h^{\mathsf{TT}}_{ij}(t), \pi^{ij}_{\mathsf{TT}}(t)], \end{aligned}$$

$$\begin{aligned} H^{\mathsf{diss}}_{\leq 3.5\mathsf{PN}}(\mathbf{x}_a, \mathbf{p}_a, t) := H^{\mathsf{int}}_{2.5\mathrm{PN}}&[\mathbf{x}_a, \mathbf{p}_a, h^{\mathsf{TT}}_{ij}(t), \pi^{ij}_{\mathsf{TT}}(t)] \\ &+ H^{\mathsf{int}}_{3.5\mathsf{PN}}[\mathbf{x}_a, \mathbf{p}_a, h^{\mathsf{TT}}_{ij}(t), \pi^{ij}_{\mathsf{TT}}(t)]. \end{aligned}$$

1PN-Accurate GW Luminosity

- The instantaneous energy loss of the matter system due to the GW emission is defined as

$$\mathcal{L}^{\text{inst}}_{\leq 3.5\text{PN}}(t) := -\frac{\partial}{\partial t} H^{\text{diss}}_{\leq 3.5\text{PN}}\left(\mathbf{x}_a, \mathbf{p}_a, t\right).$$

- 2PN and 3PN interaction Hamiltonians do not contribute to dissipation, because one can show that

$$\frac{\partial}{\partial t} H^{\text{int}}_{2\text{PN}}(\mathbf{x}_a, \mathbf{p}_a, t) = \text{total time derivative},$$
$$\frac{\partial}{\partial t} H^{\text{int}}_{3\text{PN}}(\mathbf{x}_a, \mathbf{p}_a, t) = \text{total time derivative}.$$

- GW luminosity of the matter system is the time average of the instantaneous energy loss:

$$\mathcal{L}_{\leq 3.5\text{PN}} := \left\langle \mathcal{L}^{\text{inst}}_{\leq 3.5\text{PN}}(t) \right\rangle = -\left\langle \frac{\partial}{\partial t} H^{\text{diss}}_{\leq 3.5\text{PN}}(\mathbf{x}_a, \mathbf{p}_a, t) \right\rangle,$$

  where $\langle \cdots \rangle$ denotes time averaging over one period of the motion.

- This formula was applied to derive, at the leading (Newtonian) and 1PN orders, to derive GW luminosity of the two-point-mass system in quasi-elliptical motion. This was a direct derivation of the leading-order/next-to-leading-order GW luminosities (not assuming that the balance equation holds).

An Auxiliary Formula for GW Luminosity (1/2)

$$\left\langle \frac{\partial}{\partial t} H^{\text{int}}\left[\mathbf{x}_a, \mathbf{p}_a, h^{\text{TT}}_{ij}(t), \pi^{ij}_{\text{TT}}(t)\right] \right\rangle = \frac{1}{2} \left\langle \int \mathrm{d}^d x\, \dot{h}^{\text{TT}}_{ij} S^{\text{TT}}_{ij} \right\rangle$$

Proof (we mostly omit indices, arguments and integration measures).

1. $$H_{\text{red}} = \int \left[\frac{1}{4}(h^{\text{TT}}_{ij,k})^2 + (\pi^{ij}_{\text{TT}})^2\right] + I(h^{\text{TT}}_{ij}, \pi^{ij}_{\text{TT}}) + H^{\text{int}}(\mathbf{x}_a, \mathbf{p}_a, h^{\text{TT}}_{ij}, \pi^{ij}_{\text{TT}}) + H^{\text{matter}}(\mathbf{x}_a, \mathbf{p}_a);$$

2. $$\dot{h}^{\text{TT}} = \left(\frac{\delta H_{\text{red}}}{\delta \pi_{\text{TT}}}\right)^{\text{TT}} = 2\pi_{\text{TT}} + \left(\frac{\delta I}{\delta \pi_{\text{TT}}}\right)^{\text{TT}} + \left(\frac{\delta H^{\text{int}}}{\delta \pi_{\text{TT}}}\right)^{\text{TT}},$$
$$\dot{\pi}_{\text{TT}} = -\left(\frac{\delta H_{\text{red}}}{\delta h^{\text{TT}}}\right)^{\text{TT}} = \frac{1}{2}\Delta h^{\text{TT}} - \left(\frac{\delta I}{\delta h^{\text{TT}}}\right)^{\text{TT}} - \left(\frac{\delta H^{\text{int}}}{\delta h^{\text{TT}}}\right)^{\text{TT}};$$

3. $$\ddot{h}^{\text{TT}} = 2\dot{\pi}_{\text{TT}} + \left[\left(\frac{\delta I}{\delta \pi_{\text{TT}}}\right)^{\cdot}\right]^{\text{TT}} + \left[\left(\frac{\delta H^{\text{int}}}{\delta \pi_{\text{TT}}}\right)^{\cdot}\right]^{\text{TT}}$$
$$= \Delta h^{\text{TT}} - 2\left(\frac{\delta I}{\delta h^{\text{TT}}}\right)^{\text{TT}} - 2\left(\frac{\delta H^{\text{int}}}{\delta h^{\text{TT}}}\right)^{\text{TT}} + \left[\left(\frac{\delta I}{\delta \pi_{\text{TT}}}\right)^{\cdot}\right]^{\text{TT}} + \left[\left(\frac{\delta H^{\text{int}}}{\delta \pi_{\text{TT}}}\right)^{\cdot}\right]^{\text{TT}};$$

4. $$\Box h^{\text{TT}} = -\ddot{h}^{\text{TT}} + \Delta h^{\text{TT}} = 2\left(\frac{\delta I}{\delta h^{\text{TT}}}\right)^{\text{TT}} + 2\left(\frac{\delta H^{\text{int}}}{\delta h^{\text{TT}}}\right)^{\text{TT}} - \left[\left(\frac{\delta I}{\delta \pi_{\text{TT}}}\right)^{\cdot}\right]^{\text{TT}} - \left[\left(\frac{\delta H^{\text{int}}}{\delta \pi_{\text{TT}}}\right)^{\cdot}\right]^{\text{TT}} =: S^{\text{TT}};$$

An Auxiliary Formula for GW Luminosity (2/2)

Proof (contd).

5.

$$\frac{1}{2}\dot{h}^{TT}S^{TT} = \dot{h}^{TT}\left(\frac{\delta I}{\delta h^{TT}}\right)^{TT} + \dot{h}^{TT}\left(\frac{\delta H^{int}}{\delta h^{TT}}\right)^{TT} - \frac{1}{2}\dot{h}^{TT}\left[\left(\frac{\delta I}{\delta \pi_{TT}}\right)^{\cdot}\right]^{TT} - \frac{1}{2}\dot{h}^{TT}\left[\left(\frac{\delta H^{int}}{\delta \pi_{TT}}\right)^{\cdot}\right]^{TT}$$

$$= \dot{h}^{TT}\left(\frac{\delta I}{\delta h^{TT}}\right)^{TT} + \dot{h}^{TT}\left(\frac{\delta H^{int}}{\delta h^{TT}}\right)^{TT} + \frac{1}{2}\ddot{h}^{TT}\left(\frac{\delta I}{\delta \pi_{TT}}\right)^{TT} + \frac{1}{2}\ddot{h}^{TT}\left(\frac{\delta H^{int}}{\delta \pi_{TT}}\right)^{TT} + (\text{total time derivative})_1$$

$$= \dot{h}^{TT}\left(\frac{\delta I}{\delta h^{TT}}\right)^{TT} + \dot{h}^{TT}\left(\frac{\delta H^{int}}{\delta h^{TT}}\right)^{TT} + \dot{\pi}_{TT}\left(\frac{\delta I}{\delta \pi_{TT}}\right)^{TT} + \dot{\pi}_{TT}\left(\frac{\delta H^{int}}{\delta \pi_{TT}}\right)^{TT}$$

$$+\frac{1}{2}\left[\left(\frac{\delta I}{\delta \pi_{TT}}\right)^{\cdot}\right]^{TT}\left(\frac{\delta I}{\delta \pi_{TT}}\right)^{TT} + \frac{1}{2}\left[\left(\frac{\delta I}{\delta \pi_{TT}}\right)^{\cdot}\right]^{TT}\left(\frac{\delta H^{int}}{\delta \pi_{TT}}\right)^{TT}$$

$$+\frac{1}{2}\left[\left(\frac{\delta H^{int}}{\delta \pi_{TT}}\right)^{\cdot}\right]^{TT}\left(\frac{\delta I}{\delta \pi_{TT}}\right)^{TT} + \frac{1}{2}\left[\left(\frac{\delta H^{int}}{\delta \pi_{TT}}\right)^{\cdot}\right]^{TT}\left(\frac{\delta H^{int}}{\delta \pi_{TT}}\right)^{TT} + (\text{total time derivative})_1$$

$$= \dot{h}^{TT}\left(\frac{\delta H^{int}}{\delta h^{TT}}\right)^{TT} + \dot{\pi}_{TT}\left(\frac{\delta H^{int}}{\delta \pi_{TT}}\right)^{TT} + (\text{total time derivative})_2;$$

6. using the property $\int A^{TT}_{ij}B_{ij} = \int A_{ij}B^{TT}_{ij} = \int A^{TT}_{ij}B^{TT}_{ij}$, one shows that

$$\frac{1}{2}\left\langle \int \dot{h}^{TT}S^{TT}\right\rangle = \left\langle \int \left[\dot{h}^{TT}\left(\frac{\delta H^{int}}{\delta h^{TT}}\right)^{TT} + \dot{\pi}_{TT}\left(\frac{\delta H^{int}}{\delta \pi_{TT}}\right)^{TT}\right]\right\rangle$$

$$= \left\langle \int \left(\dot{h}^{TT}\frac{\delta H^{int}}{\delta h^{TT}} + \dot{\pi}_{TT}\frac{\delta H^{int}}{\delta \pi_{TT}}\right)\right\rangle = \left\langle \frac{\partial}{\partial t}H^{int}\right\rangle.$$

D-Dimensional Retarded Green's Function

- For the wave equation in D-dimensional Minkowski spacetime:
$$\Box\phi(t,\mathbf{x}) = S(t,\mathbf{x}),$$
the retarded Green's function G_{ret} fulfills equation
$$\Box G_{\text{ret}}(t,\mathbf{x}) = \delta(t)\delta^d(\mathbf{x}).$$
- Using the momentum representation and spherical coordinates in the d-dimensional $\mathbf{k}$ space, G_{ret} can be written as
$$G_{\text{ret}}(t,\mathbf{x}) = -\frac{\Theta(t)}{(2\pi)^{(D-1)/2}}\frac{1}{r^{(D-3)/2}}\int_0^{+\infty} k^{(D-3)/2}J_{(D-3)/2}(kr)\sin(kt)\mathrm{d}k,$$
$r := |\mathbf{x}|$, $J_{(D-3)/2}$ is a Bessel function, Θ is the Heaviside step function.
- The structure of G_{ret} depends on the parity of D. For even D,
$$G_{\text{ret}}(t,\mathbf{x}) = \frac{1}{4\pi}\left(-\frac{1}{2\pi r}\frac{\partial}{\partial r}\right)^{(D-4)/2}\left(\frac{\delta(t-r)}{r}\right).$$

- The formal retarded solution of the wave equation

$$\Box h_{ij}^{\mathrm{TT}} = S_{ij}^{\mathrm{TT}},$$

one expresses in terms of the retarded Green's function,

$$h_{ij}^{\mathrm{TT}}(t, \mathbf{x}) = \left(\int \mathrm{d}t' \int \mathrm{d}^d x' \, G_{\mathrm{ret}}(t - t', \mathbf{x} - \mathbf{x}') S_{ij}(t', \mathbf{x}') \right)^{\mathrm{TT}}.$$

- This, after expanding $\delta(t - r)$ into the PN series (here $\overset{(n)}{\delta}$ denotes the nth derivative of the δ),

$$\delta(t - r) = \sum_{n=0}^{\infty} \frac{r^n}{n!} \overset{(n)}{\delta}(t),$$

is substituted into the expression $-\frac{1}{2} \left\langle \int \mathrm{d}^d x \, \dot{h}_{ij}^{\mathrm{TT}} S_{ij}^{\mathrm{TT}} \right\rangle$.

After some manipulations one gets the final formula for "instantaneous" GW luminosity (here $\overset{(k+1)}{S_{ij}}$ denotes the $(k+1)$th time derivative of the source S_{ij}):

$$\mathcal{L} = \frac{1}{8\pi} \left(-\frac{1}{\pi} \right)^{(d-3)/2} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(k+1)}{\Gamma(2k+2)\Gamma(k + \frac{5-d}{2})} \times \int \mathrm{d}^d x \int \mathrm{d}^d x' \left(|\mathbf{x} - \mathbf{x}'|^{2k-(d-3)} \right)^{\mathrm{TT}} \left\langle \overset{(k+1)}{S_{ij}}(t, \mathbf{x}) \overset{(k+1)}{S_{ij}}(t, \mathbf{x}') \right\rangle.$$

The Results

- Leading-order (Newtonian) and 1PN-order GW luminosities recomputed.
- 2PN-order GW luminosity is being calculated.
- Work on including tail-related effects into GW luminosity is in progress.

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