Semiclassical Causal Geodesics

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Talk based on my collaboration with:
A. Cieślik, A. Góźdź, P. Mach, and A. Pędrak

Collaboration concerns the project:

Semiclassical Causal Geodesics:
I. Minkowski spacetime case, present talk
II. Schwarzschild spacetime case, in progress
III. Kerr spacetime case, in plan.

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OUTLINE

Introduction

Near gravitational singularities general relativity (GR) breaks down (curvature and matter field invariants diverge). It is believed that taking into account quantum effects may lead to regular theory called quantum gravity (QG).

Struggle for the construction of QG lasts more than 50 years; it turns out to be enormously difficult issue. There are a few candidates pretending to meet the problem: string theory, loop quantum gravity, causal dynamical triangulations, integral quantization (to be used in my talk), and others.

One of the main sources of difficulties:
lack of experimental data on extremal gravitational fields to be used in reducing the freedom of mathematical structures underlying constructed QG theories.

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Hope to get some useful information for the construction of QG from recent astrophysical observations

- detection of gravitational waves from binary black hole (BH) mergers: waves from inspiral, merger and ringdown phases of remnant BH
- detection of shadows of trapped photons performing motion around massive stars with BHs inside[1]

We expect that the comparison of our quantum description of shadows, done within the integral quantization (IQ) scheme, with observed shadows of BHs may

- impose some constraints on our method to reduce its ambiguity
- indicate some modifications of the method to be made to fit better the data

[1] Shadows of supermassive BHs: Messier 87* and Saggittarius A^{*} discovered by Event Horizon Telescope Collaboration in 2019 and 2022, respectively.

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Geodesics in Minkowski spacetime: preliminaries

Hamiltonian H describing the geodesic motion of test particle reads

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu . \qquad (1)$$

Causal geodesics satisfy the constraint: $H = -1/2\, m^2$, where
$m > 0$ corresponds to the timelike geodesics of a particle with rest mass m,
$m = 0$ concerns the null geodesics of a photon.
Hamilton's equations lead to standard geodesic equations.
$g^{\mu\nu} = g_{\mu\nu}$ is the Minkowski metric with the signature $(-,+,+,+)$, so that we have $g^{\mu\nu} p_\mu p_\nu = -p_0^2 + p_1^2 + p_2^2 + p_3^2$. The four momenta are defined as $-p_0 = p^0 = dx^0/d\tilde{s},\ p_n = p^n = dx^n/d\tilde{s},\ n = 1,2,3$, where x^μ are the metric coordinates, and $\tilde{s}$ is an affine parameter related with the proper time s of the test particle by $\tilde{s} = s/m$.
The canonical variables p_μ and x^ν define the phase space
$\mathcal{F} := \{(p_\mu, x^\mu) : \mu = 0,1,2,3\} \subseteq \mathbb{R}^4 \times \mathbb{R}^4$,
which is cotangent bundle $T^*\mathcal{M}$ of Minkowski spacetime $(\mathcal{M}, g)$.
Solution to Hamilton's equations reads:
$p_\mu = const\,, \quad x^\mu = p^\mu \tilde{s}, \quad \mu = 0,1,2,3; \quad x^\mu = \frac{1}{m} p^\mu\, s$ for timelike geodesics.
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Integral quantization

A general idea of this quantization requires the existence of one-to-one transformation of the space of elementary variables (extended configuration or phase space) of a physical system under consideration onto some group G.

The group G should have an unitary irreducible representation in a carrier Hilbert space $\mathcal{K}$, which enables to construct the space of coherent states in $\mathcal{K}$.

In what follows we use the Heisenberg-Weyl group, $HW(N)$, which is known to have the unitary irreducible representation in the Hilbert space $L^2(\mathbb{R}^N, d^N\xi)$, where $d^N\xi := d\xi^0\, d\xi^1 \dots d\xi^{N-1}$.

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Unitary representation of the $HW(N)$ group

The unitary irreducible representation of the group $HW(N)$ on the Hilbert space $\mathcal{K} := L^2(\mathbb{R}^N, d^N\xi)$ is defined as follows

$$\hat{\mathcal{U}}(p,x)\psi(\xi) = \exp\left(\frac{-ip_\mu x^\mu}{2\hbar}\right)\exp\left(\frac{ip_\mu \xi^\mu}{\hbar}\right)\psi(\xi - x)\,, \qquad (2)$$

where $\psi(\xi) := \langle\xi|\psi\rangle \in \mathcal{K}$.

Coherent states quantization

The coherent states, $|p, x\rangle \in \mathcal{K}$, are defined as follows

$$|p, x\rangle = \hat{\mathcal{U}}(p, x)|\Phi_0\rangle, \quad \langle\xi|p, x\rangle = \hat{\mathcal{U}}(p, x)\langle\xi|\Phi_0\rangle = \hat{\mathcal{U}}(p, x)\Phi_0(\xi)\,. \qquad (3)$$

where $\Phi_0(\xi) : \mathbb{R}^N \to \mathbb{C}$ is the so-called fiducial vector;
$|\Phi_0\rangle \in \mathcal{K}$ such that $\langle\Phi_0|\Phi_0\rangle = 1$.
The fiducial vector is a sort of parameter of the coherent states quantization (see next talk by Ola).

Since the representation is irreducible, the operators $|p, x\rangle\langle p, x| : \mathcal{K} \to \mathcal{K}$ satisfy

$$(2\pi\hbar)^{-N} \int_{\mathbb{R}^{2N}} d\rho(p, x)\, |p, x\rangle\langle p, x| = \hat{\mathbb{I}}\,, \qquad (4)$$

where $d\rho(p, x) := dp_0\, dp_1 \ldots dp_{N-1}\, dx^0\, dx^1 \ldots dx^{N-1}$,
so that we have the resolution of the unity operator in $\mathcal{K}$.

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Coherent states quantization (cont)

Eq. (4) can be used for mapping (quantization) of almost any classical observable $f : \mathbb{R}^{2N} \rightarrow \mathbb{R}$ onto an operator $\hat{f} : \mathcal{K} \rightarrow \mathcal{K}$ as follows

$$f \longrightarrow \hat{f} := (2\pi\hbar)^{-N} \int_{\mathbb{R}^{2N}} d\rho(p,x) |p,x\rangle f(p,x) \langle p,x| \,. \tag{5}$$

The mapping (5) leads to symmetric operator, and if the classical observable $f(p,x)$ is either bounded or integrable function, $L^1(\mathbb{R}^{2N}, d\rho(p,x))$, the mapping (5) defines self-adjoint operator.

If $\hat{f}$ is not self-adjoint, the problem can be solved, e.g., by making use of the so-called theory of positive operator valued measure, POVM.

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Spectrum of quantum Hamiltonian

Classical Hamiltonian of a test particle reads

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu = -\frac{1}{2} m^2 \,. \tag{6}$$

Quantum Hamiltonian $\hat{H}$, due to (5), has the form

$$\hat{H} = (2\pi\hbar)^{-4} \int_{\mathbb{R}^8} d\rho(p,x) |p,x\rangle H(p,x) \langle p,x| \,. \tag{7}$$

One can show that the functions defined as

$$\eta_p(\xi) = \langle \xi | \eta_p \rangle := \left(\frac{1}{\sqrt{2\pi\hbar}} \right)^4 \exp(i \frac{p\,\xi}{\hbar}) \,, \tag{8}$$

where $p\,\xi := p_\mu \xi^\mu$, with $\mu = 0, 1, 2, 3$, are generalized eigenstates of $\hat{H}$, defined by (7), if

- $|p, x\rangle$ are generated from a suitably chosen fiducial vector $|\Phi_0\rangle$
- p_μ satisfy specific constraint

Spectrum of quantum Hamiltonian (cont)

The key element is making use of the orthogonal decomposition of the unity in the carrier space $\mathcal{K}$ in terms of the generalized states (8), which reads

$$\int_{\mathbb{R}^4} d^4p \, |\eta_p\rangle\langle\eta_p| = \hat{\mathbb{I}} \, . \tag{9}$$

The validity of (9) results from the theory of Fourier transforms in the context of distributions.

One can show that the eigenvalue problem for the Hamiltonian (7) reads

$$\left(\int_{\mathbb{R}^4} d^4p \, |\tilde{\Phi}_0(p)|^2\right) g^{\alpha\beta} k_\alpha k_\beta + \left(\int_{\mathbb{R}^4} d^4p \, p_\beta |\tilde{\Phi}_0(p)|^2\right) 2g^{\alpha\beta} k_\alpha + \int_{\mathbb{R}^4} d^4p \, g^{\alpha\beta} p_\alpha p_\beta |\tilde{\Phi}_0(p)|^2 = -m^2 \, , \tag{10}$$

where $\tilde{\Phi}_0(p)$ is the Fourier transform of some fiducial vector $\Phi_0(p)$.

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Spectrum of quantum Hamiltonian (cont)

Choosing the fiducial vector in the form of the 4D harmonic oscillator ground state wave function

$$\Phi_0(\xi) = \prod_{\mu=0}^{3} \left(\frac{\lambda_\mu}{\pi\hbar}\right)^{\frac{1}{4}} \exp\left(-\frac{\lambda_\mu(\xi^\mu)^2}{2\hbar}\right) \tag{11}$$

with $\lambda_0 = 3\lambda_1$ and $\lambda_1 = \lambda_2 = \lambda_3 > 0$, enormously reduces (10).
It turns out that the functions (8) are the eigenstates of the Hamiltonian (7) if

$$g^{\alpha\beta} p_\alpha p_\beta = -m^2 \,. \tag{12}$$

which is quite similar to the relationship satisfied by the classical momenta.

The quantum Hamiltonian $\hat{H}$ has two infinitely many degenerate eigenvalues: $m > 0$ corresponding to a particle, and $m = 0$ describing a photon.

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Transition amplitudes

We define the mass layer, $\mathcal{J}_{m,\epsilon}$, of thickness ϵ for a test particle $m \geq 0$ as follows

$$\mathcal{J}_{m,\epsilon} := \left\{ p \ : \ -\sqrt{m^2 + \vec{p}^2 + \epsilon} \leq p_0 \leq -\sqrt{m^2 + \vec{p}^2}, \ \ \vec{p} \in \mathbb{R}^3 \right\} , \tag{13}$$

to be used as a subsidiary set that will enable introducing well defined transition amplitudes. Final results will be obtained by taking the limit $\epsilon \to 0$, which would lead to the commonly used notion of mass shell. Our construction of the layer is based on one of the solutions to the equation $g^{\alpha\beta} p_\alpha p_\beta = -m^2$, which is compatible with the choice of the metric signature $(-, +, +, +)$ and the orthochronous part of the Lorentz group.

The operator projecting onto the mass layer (13) are constructed from the generalized eigenstates of the test particle Hamiltonian as follows

$$P_{\mathcal{J}_{m,\epsilon}} := \int_{\mathbb{R}^4} d^4 p \, |\eta_p\rangle \chi(p \in \mathcal{J}_{m,\epsilon}) \langle \eta_p| \, , \tag{14}$$

where $\chi(p \in Q) = 1$ iff the relationship Q is satisfied or equals 0 otherwise.

The transition amplitude of the particle of mass $m \geq 0$ from the state $|p' x'\rangle$ to the state $|p'' x''\rangle$ is given by the following matrix element of the projection operator

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Transition amplitudes

We define the mass layer, $\mathcal{J}_{m,\epsilon}$, of thickness ϵ for a test particle $m \geq 0$ as follows

$$\mathcal{J}_{m,\epsilon} := \left\{ p \;:\; -\sqrt{m^2 + \vec{p}^2 + \epsilon} \leq p_0 \leq -\sqrt{m^2 + \vec{p}^2}, \;\; \vec{p} \in \mathbb{R}^3 \right\}, \tag{13}$$

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Transition amplitude (cont)

One can find that the transition amplitude is given by

$$\mathcal{A}_{m,\epsilon} = \exp(i\frac{p'\,x' - p''\,x''}{2\hbar})\,\mathcal{B}\int_{\mathbb{R}^3} dp_1\,dp_2\,dp_3 \int_{\mathbb{R}} dp_0\ \chi(p \in \mathcal{J}_{m,\epsilon}) \tag{16}$$

$$\cdot \exp\left(\frac{i}{\hbar}p_0(x''^0 - x'^0) - \frac{1}{\hbar\lambda_0}(p_0 - \bar{p}_0)^2\right) \exp\left(\frac{i}{\hbar}\vec{p}(\vec{x}'' - \vec{x}') - \frac{1}{\hbar\lambda_3}(\vec{p} - \vec{\bar{p}})^2\right) .$$

where

$$\mathcal{B} := \prod_{\mu=0}^{3} (\pi\hbar\lambda_\mu)^{-\frac{1}{2}} \exp\left(-\frac{(p''_\mu - p'_\mu)^2}{4\hbar\lambda_\mu}\right) ,$$

with $\lambda_1 = \lambda_2 = \lambda_3$ and $2\bar{p}_\mu := p''_\mu + p'_\mu$.

Application: quantum evolution of test particle

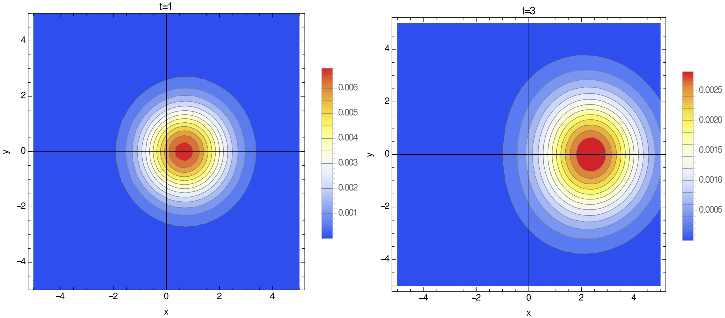

Figure: The density plot of $|\mathcal{A}_{m,\epsilon}(\vec{x}\,'')/\epsilon|^2$ for $t = 1$ and $t = 3$, where $t = x\,''^0$, $x' = (0,0,0,0)$, $p' = p\,''$, $m = 1$, $\vec{p}\,' = (1,0,0)$; geodesic goes along x-axis.

The plots present the probability distribution of particle transition from the space point $(0,0,0)$ to $\vec{x}\,''$ in time t. The distribution follows geodesic. Its maximum becomes smaller and wider for increasing time. It looks the same in the xz-plane. For all t the distribution is axially symmetric, where the symmetry axis is spatial momentum direction.

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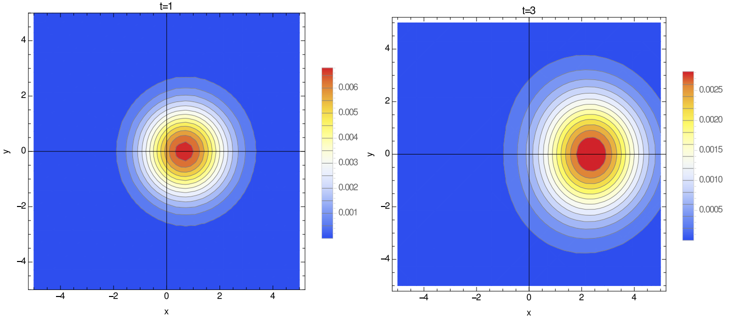

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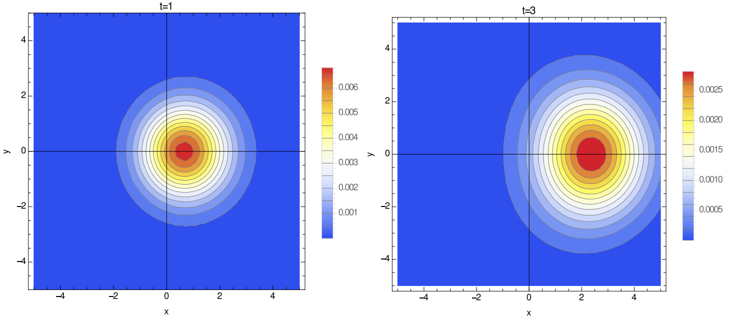

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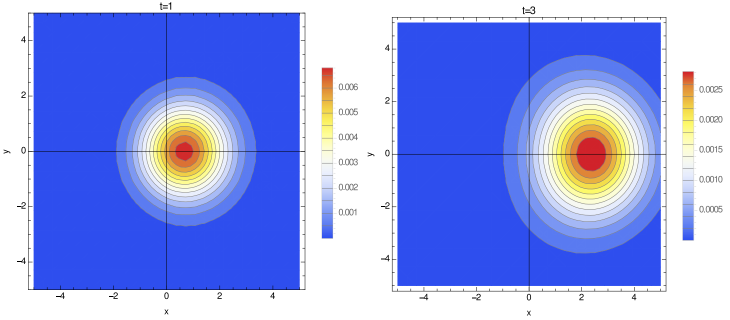

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Application: quantum random walk of test particle

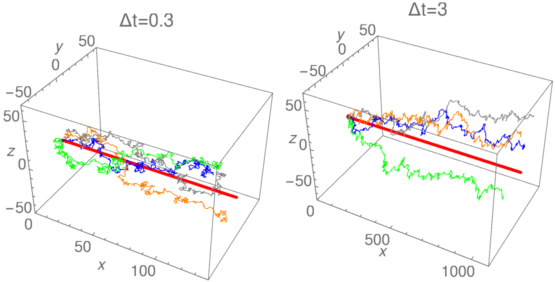
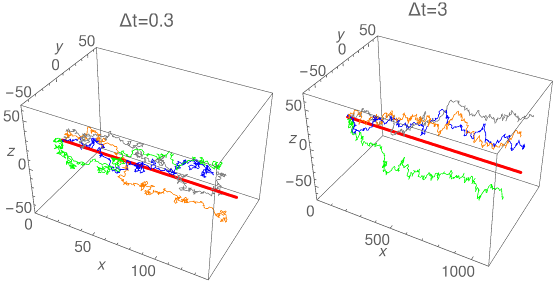


Figure: The stochastic random walk of particle, where $\Delta t = 0.3$ and 3 are time periods between "measurements" of particle positions, and where $m = 1$, $\vec{p}\,' = (1, 0, 0)$, $\vec{x}\,' = (0, 0, 0)$, $p'_0 = -\sqrt{m^2 + \vec{p}\,'^2}$, $\lambda_0 = 3\lambda_3 = 3$. Each stochastic trajectory consists of 500 points.

The trajectories are constructed via computer's generator of random numbers combined by Mathematica with the probability distribution resulting from (16). One can see that particle's position is close to the geodesic (red line). For larger Δt quantum particle has tendency of having larger deviation from the geodesic.

Application: quantum random walk of test particle



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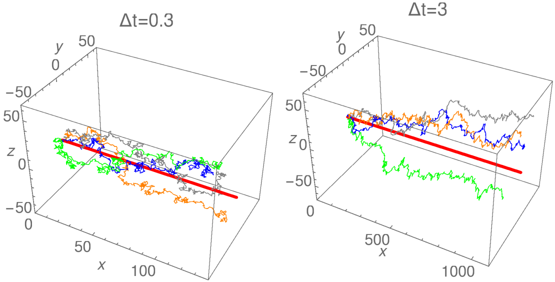

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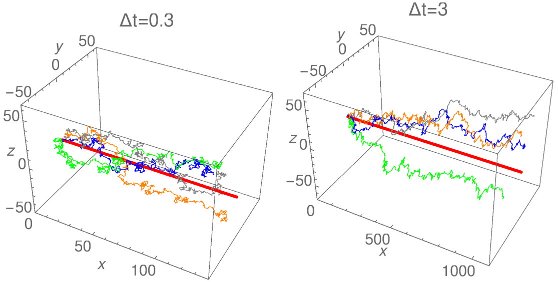

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Application of the formalism to Schwarzschild[2] and Kerr[3] spacetimes in order to

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- find possibly general and unique integral quantization scheme

Successful consolidation of these two issues may help in the construction of quantum gravity.

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Thank you?