Nonextensive Entropies of Black Hole and Cosmological Horizons

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Plan:

- 1. Motivation - horizon entropies as Holographic Dark Energy (HDE).
- 2. Composability, (non)additivity, and (non)extensivity in thermodynamics.
- 3. Comparable analysis of nonextensive entropies plethora.
- 4. Observational constraints on nonextensivity.
- 5. Conclusions.

References

- MPD, Look beyond additivity and extensivity of entropy for black hole and cosmological horizons, https://arxiv.org/abs/2409.00802, invited paper to "Entropy".
- I. Çimdiker, H. Gohar, MPD - EPJC 83, 169 (2023); CQG 40, 145001 (2023).
- T. Denkiewicz, V. Salzano, MPD - Barrow nearly-extensive Gibbs-like entropy favoured by the full dynamical and geometrical data set in cosmology, PRD D 108, 103533 (2023) (arXiv: 2303.11680).
- I. Çimdiker, MPD, V. Salzano (2024) - work in progress

1. Horizon entropies as Holographic Dark Energy (HDE).

The dark energy density in the standard ΛCDM model is related to the cosmological constant Λ (with unit m^{-2}) as follows (e.g. Amendola, Tsujikawa, 2010)

$$\rho_\Lambda = \frac{\Lambda c^2}{8\pi G}, \tag{1}$$

where c is the speed of light, G is the gravitational constant, and ρ_Λ is the mass density in units $kg \cdot m^{-3}$.

Alternative models come from thermodynamics of black hole horizons (area entropy $S(A) \propto A$, A - horizon area), but **applied to cosmological horizons**, which also in the context of string theory are called **holographic screens** or **holographic dark energy (HDE)** (Wang et al. 2016). Initially, only HDE based on Bekenstein entropy $S(L) \sim A \sim L^2$, L - radius of the cosmological horizon, was considered.

Unlike Boltzmann-Gibbs (BG), **Bekenstein entropy is nonextensive** (roughly speaking it scales with area and not volume) and **nonadditive** (cf. later).

Horizon entropies as HDE ctd.

The application of Bekenstein entropy to HDE inspired cosmologists to apply the plethora of nonextensive entropies which were introduced mainly in the context of statistical physics. Among them: Tsallis, Rényi, Tsallis-Cirto, Barrow, Landsberg, Tsallis-Jensen, Kaniadakis, Sharma-Mittal etc.. All of them are the specific functions of the cosmological horizon size i.e. $S = S(L)$. Bearing this in mind, we can write down a general expression for HDE as follows (Çimdiker, MPD, Salzano - 2024, in progress)

$$\rho_{HDE} = \frac{3c^2k^2L_0^2}{8\pi G}S(L)L^{-4} \tag{2}$$

where k is a dimensionless constant related to the holographic screen properties (Wang et al. 2016), and

$$L_0^2 = 4G\hbar/c^3 = A_0 \tag{3}$$

is the Planck area with L_0 being of the size of the Planck length $l_p = 2\sqrt{\hbar G/c^3}$. In (2) the appropriate quantities have been chosen in a way that ρ_{HDE} is given in units of mass density $kg \cdot m^{-3}$ as in (1).

Horizon entropies as HDE ctd.

Note: There is a selection of horizons (and so the distances L) in (2).

Future event horizon:

$$L \equiv a \int_t^{\infty} \frac{dt'}{a} = a \int_a^{\infty} \frac{da'}{H(a')a'^2} , \tag{4}$$

where a is the scale factor and $H(a)$ the Hubble parameter (Hsu 2004, Li 2004).

Hubble horizon:

$$L \equiv \frac{c}{H(a)} , \tag{5}$$

though it is not the ”true horizon” since it can be crossed (or has been crossed, in fact) (e.g. Pavon 2005).

Horizon with an infrared cut-off (Granda & Oliveros 2009):

$$L \equiv c \left[\alpha H^2(a) + \beta \dot{H}(a) \right]^{-1/2} , \tag{6}$$

with α, β - free dimensionless parameters.

2. Composability, (non)additivity, and (non)extensivity in thermodynamics.

- Boltzmann-Gibbs (BG) thermodynamics is based on **ignoring long-range forces** between thermodynamic subsystems i.e. on the assumption that the size of the system **exceeds** the range of interaction between its components.
- In particular, BG thermodynamics is based on the **additivity** and **extensivity** of entropy defined as

$$S_{BG} = -k_B \sum_{i}^{n} p_i \ln p_i, \tag{7}$$

where p_i is the probability distribution on configuration space Ω with the number of states n and k_B - Boltzmann constant.

- Taking equally probable states $p_i = 1/n$, one has $S_{BG}(n) = k_B \ln(n) \propto n$.

Composability, (non)additivity, and (non)extensivity ctd.

- **Composability:** Let us consider two independent systems A and B combined as a single Cartesian product $A \times B$ of the states of A and B with the requirement that (Tsallis 2024)

$$S(A \times B, \Upsilon) = k_B g \left(\frac{S(A)}{k_B}, \frac{S(B)}{k_B} \right), \tag{8}$$

  where g is a smooth function of $S(A)$ and $S(B)$ and Υ is a parameter. If the systems A and B fulfil the condition (8), then their combined system $A + B$ is called **composable**.

- **Additivity:** The entropy is additive when it fulfils the **composition rule**

$$S(A+B) \equiv S(P_A P_B) = S(P_A) + S(P_B) = S(A) + S(B), \tag{9}$$

  where A, B - independent, with configurations Ω_A and Ω_B, and probabilities P_A and P_B. BG entropy is additive and composable, i.e. (8) gives an additive composition rule (9 in the limit $\Upsilon \to 0$.

Composability, (non)additivity, and (non)extensivity ctd.

- **Extensivity:** Assume the set of thermodynamical variables $(X_0, X_1, X_2, \ldots, X_k)$ such that $X_0 = f(X_1, X_2, \ldots, X_k)$. Extensivity means that the function f is homogeneous degree one i.e. that

$$f(aX_1, aX_2, ..., aX_k) = af(X_1, X_2, ..., X_k) \tag{10}$$

for a positive real number a, for all $X_1, X_2, ...X_k$. For entropy S, energy U, volume V, mole number N (i.e. k=3), we have
$S(aU, aV, aN) = aS(U, V, N)$.
Most common, but not so precise definition states that if a system's total number of microstates, Ω, is proportional to its number of particles or degrees of freedom (size), the entropy is extensive.

- Attention! Extensive quantity can be nonadditive (e.g. if $f(x_1, x_2) = x_1^2/\sqrt{x_1^2 + x_2^2}$) (Landsberg, 1999); additive quantity can be nonextensive.

Composability, (non)additivity, and (non)extensivity ctd.

- **Gravitational systems** are long-range interacting and highly nonlinear and so they **cannot be adopted** as thermodynamical systems of Gibbs additive and extensive type. Dealing with them one needs to go beyond additivity and extensivity.
- Nonadditivity: it is violated when (9) does not hold.
- Superadditivity: $S(A+B) \geq S(A) + S(B)$ and the systems have the tendency to clump its pieces (subsystems).
- Subadditivity: $S(A+B) < S(A) + S(B)$ and the system tends to fragment its pieces rather than clump (a cosmological example of it is phantom matter).
- Nonextensivity: the entropy S is nonextensive if $S(aX) \neq aS(X)$, where X is a thermodynamical quantity and $a > 0$, i.e. when the relation (10) is violated.

3. Comparable analysis of nonextensive entropies plethora.

Bekenstein entropy: it is not motivated by anything like statistical mechanics, but it is well established notion in gravity theory. For a Schwarzschild black hole it reads

$$S_{Bek} = 4\pi k_B \left(\frac{M}{m_p}\right)^2 = \frac{4\pi k_B G M^2}{\hbar c}, \tag{11}$$

and it is usually presented with its accompanying Hawking temperature

$$T_H = \frac{\hbar c^3}{8\pi G k_B M}, \tag{12}$$

where M - mass of a black hole, $\hbar$ - reduced Planck constant, m_p - Planck mass. Bekenstein entropy is proportional to its mass/length scale $S \propto M^2 \propto L^2$ and fulfils the square root composition rule

$$S(A+B) = S(A) + S(B) + 2\sqrt{S(A)}\sqrt{S(B)}. \tag{13}$$

This is since $S(A) = M_A^2/4$, $S(B) = M_B^2/4$ and after a merge one has $S(A+B) = (M_A + M_B)^2/4$, which gives an extra term $M_A M_B/2$.

Tsallis q-entropy.

The Tsallis q-entropy (Tsallis, J. Stat. Phys. 52, 479 (1988); book of 2009) is one of the earliest proposals for generalization of BG entropy. It encompasses an issue of the long-range interaction between thermodynamical subsystems by introducing a **nonextensivity parameter** q ($q \in R$) into the BG entropy definition (7) keeping the standard BG condition that the sum of all the probabilities is equal to one ($\sum p_i = 1$), and reads in 3 alternative forms as follows

$$\mathcal{S}_q = k_B \sum_{i=1}^{n} p_i \ln_q \frac{1}{p_i} = -k_B \sum_{i=1}^{n} (p_i)^q \ln_q p_i = -k_B \sum_{i=1}^{n} \ln_{2-q} p_i, \qquad (14)$$

where a newly defined q-logarithmic function $\ln_q p$ is introduced

$$\ln_q p \equiv \frac{p^{1-q} - 1}{1-q} \qquad (15)$$

Tsallis q-entropy.

with the property that

$$\ln_q ab = \ln_q a + \ln_q b + (1-q)\ln_q a \ln_q b, \quad \text{q} \to 1 \;\; \ln ab = \ln a + \ln b, \tag{16}$$

which is in fact the origin of the **Abé composition rule**

$$S(A+B) = S(A) + S(B) + \frac{\Upsilon}{k_B} S(A)S(B), \tag{17}$$

where Υ takes numerical values according to a statistical definition of a specific entropy. For Tsallis q-entropy $\Upsilon = 1-q$.

Using the definition of q-logarithm (15), all 3 forms (14) can be brought into the same form

$$\mathcal{S}_q = k_B \frac{1 - \sum_{i=1}^{n} (p_i)^q}{q-1} \tag{18}$$

All the forms of Tsallis q-entropy in the limit $q \to 1$ give BG entropy.

Rényi entropy.

Rényi entropy (Rényi 1959) (a measure of entanglement in quantum information theory) is defined as

$$S_R = k_B \frac{\ln \sum_{i=1}^{n} (p_i)^q}{1-q}. \tag{19}$$

It is additive since it can be written in terms of Tsallis q-entropy

$$S_R = \frac{k_B}{1-q} \ln[1 + \frac{1-q}{k_B} S_T] \tag{20}$$

and brought into an additive form by the application of a more general Abé composition rule given by

$$H(S_{A+B}) = H(S_A) + H(S_B) + \frac{\Upsilon}{k_B} H(S_A) H(S_B), \tag{21}$$

together with redefinition using the logarithm in the form

Rényi entropy

$$L(S) = \frac{k_B}{\Upsilon} \ln \left(1 + \frac{\Upsilon}{k_B} H(S) \right) \qquad (22)$$

and applied to (20) giving an additive formula

$$L(S_{A+B}) = L(S_A) + L(S_B), \qquad (23)$$

where $L(S)$ corresponds to Rényi entropy and $H(S)$ corresponds to Tsallis q-entropy. In such a formulation, one can write that Rényi entropy fulfils the Abé rule (17) with the parameter $\Upsilon = 0$.

Often, one assumes that Tsallis q-entropy is the Bekenstein entropy, and so **one defines Rényi entropy on the horizon** of a black hole.

Tsallis-Cirto δ-entropy

The Tsallis-Cirto δ-entropy (Tsallis & Cirto 2012, Tsallis 2019) (sometimes also improperly called in the literature as just the Tsallis entropy) is yet another generalization of BG entropy (7) made by the introduction of another nonextensivity parameter δ as follows

$$\mathcal{S}_\delta = k_B \sum_{i=1}^{n} p_i \left(\ln p_i\right)^\delta \quad (\delta > 0, \delta \in R), \tag{24}$$

and this difference is easily recognized when one compares it with the Tsallis q-entropy (14) and with BG entropy (7).
The composition rule for the Tsallis-Cirto entropy δ-entropy

$$\left(\frac{S_{\delta,A+B}}{k_B}\right)^{1/\delta} = \left(\frac{S_{\delta,A}}{k_B}\right)^{1/\delta} + \left(\frac{S_{\delta,B}}{k_B}\right)^{1/\delta}, \tag{25}$$

is another example of a composition rule, which is *different* from the Abé composition rule (17). We call it *δ-addition rule*.

Tsallis-Cirto δ-entropy.

In fact, Tsallis and Cirto suggest that

$$S_\delta = k_B \left(\frac{S_{Bek}}{k_B} \right)^\delta , \tag{26}$$

where S_{Bek} is the Bekenstein entropy (11).
Some remarks are as follows:

- According (25), one realizes that the Bekenstein entropy as given by $S_{Bek} \propto (S_\delta)^{1/\delta}$ can be additive, while the Tsallis-Cirto entropy S_δ itself is nonadditive.
- Bearing in mind the definition of Bekenstein entropy for a Schwarzschild black hole (11), one can easily notice that for $\delta = 3/2$ the Tsallis-Cirto entropy (26) is proportional to the volume $S_\delta \propto M^3$ and so it is an extensive quantity.

Tsallis q, δ-entropy

Tsallis q, δ-entropy generalizes both the Tsallis q-entropy (14) and the Tsallis-Cirto δ-entropy (24) combining them as follows (Tsallis 2019)

$$\mathcal{S}_{q,\delta} = k_B \sum_{i=1}^{n} p_i \left(\ln_q p_i\right)^{\delta} \quad (\delta > 0, q \in R, \delta \in R). \tag{27}$$

Now both q and δ play the role of **two independent** nonextensivity parameters. By assuming that all the states are equally probable, one gets from (27) that

$$\mathcal{S}_{q,\delta} = k_B \left(\ln_q n\right)^{\delta} \equiv k_B \ln_q^{\delta} n. \tag{28}$$

The Tsallis q, δ-entropy **fulfils neither** Abé addition rule **nor** δ-addition rule though it does the former in the limit $\delta \to 0$ and the latter in the limit $q \to 0$.

Tsallis-Jensen q, γ entropy.

Recently, Tsallis and Jensen (arXiv: 2408.08820) proposed another generalization of BG entropy which reads

$$S_{q,\gamma} = k_B \left[\frac{\ln \sum_{i=1}^{n} p_i^q}{1-q} \right]^{\frac{1}{\gamma}} = k_B \left(\frac{S_R}{k_B} \right)^{\frac{1}{\gamma}} , \qquad (29)$$

where S_R is the Rényi entropy and γ is a new parameter somewhat analogous to the parameter δ in Tsallis-Cirto entropy (24).
Since the Rényi entropy has the BG limit for $q \to 1$, then we can write

$$S_{1,\gamma} = k_B \left(\frac{S_{BG}}{k_B} \right)^{\frac{1}{\gamma}} , \qquad (30)$$

and analogously if we take Bekenstein entropy (11) instead of BG in (30) in the same limit

$$S_{1,\gamma}^{Bek} = k_B \left(\frac{S_{Bek}}{k_B} \right)^{\frac{1}{\gamma}} . \qquad (31)$$

Tsallis-Jensen entropy.

Bearing in mind the additive composition formula for the BG entropy (9) and using (30), one can write the additivity for $S_{1,\gamma}$ in such a case as

$$[S_{1,\gamma}(A+B)]^{\gamma} = [S_{1,\gamma}(A)]^{\gamma} + [S_{1,\gamma}(B)]^{\gamma} \,. \tag{32}$$

Similarly, taking into account the square root additivity rule (13) for (31) as

$$\left[S_{1,\gamma}^{Bek}(A+B)\right]^{\gamma} = \left[S_{1,\gamma}^{Bek}(A)\right]^{\gamma} + \left[S_{1,\gamma}^{Bek}(B)\right]^{\gamma} + 2\left[S_{1,\gamma}^{Bek}(A) S_{1,\gamma}^{Bek}(B)\right]^{\frac{\gamma}{2}} \,. \tag{33}$$

Finally, since the Rényi entropy S_R in (29) is in general additive according to the composition rule (23), then we can write quite generally the composition rule for the Tsallis-Jensen entropy (29) as

$$[S_{q,\gamma}(A+B)]^{\gamma} = [S_{q,\gamma}(A)]^{\gamma} + [S_{q,\gamma}(B)]^{\gamma} \,, \tag{34}$$

and this is exactly the δ composition rule (25) with $\gamma = 1/\delta$.

Tsallis invented entropies with characteristics

Entropy Type	Extensivity	Additivity	Abé addition rule	$\delta-$addition rule
Boltzmann-Gibbs S_{BG}	yes	yes	yes, $\Upsilon = 0$	yes, $\delta = 1$
Tsallis $S_{q,1} = S_q$	no	no	yes, $\Upsilon = 1 - q$	no
Tsallis-Cirto $S_{1,\delta} = S_\delta$	no	no	no	yes
General Tsallis $S_{q,\delta}$	no	no	no	no
Tsallis-Jensen $S_{q,\gamma}$	no	no	no	yes, $\delta = 1/\gamma$

Barrow fractal Δ-entropy

Barrow entropy (2020) has no statistical roots at all. It is closely tied to black hole horizon geometry influenced by quantum fluctuations which make initially smooth black hole horizon a fractal composed of spheres forming the so-called **”sphereflake”**: https://www.youtube.com/watch?v=pPb5NKEYCD8;

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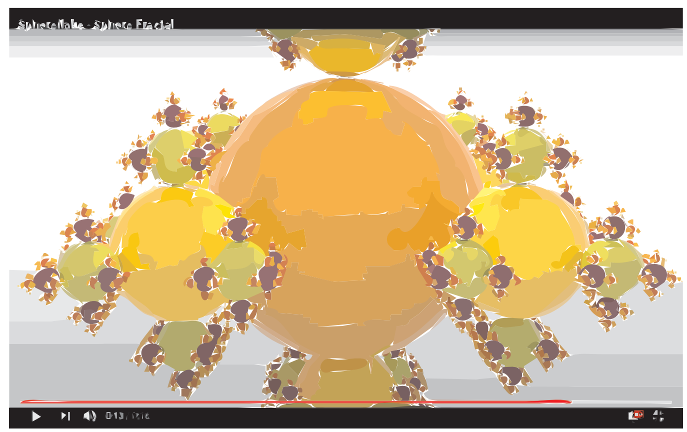

Barrow entropy vs Bekenstein entropy

- Barrow entropy reads

$$S_{Bar} = k_B \left(\frac{A}{A_0} \right)^{1+\frac{\Delta}{2}} = k_B \left(\frac{S_{Bek}}{k_B} \right)^{1+\frac{\Delta}{2}}, \tag{35}$$

  where S_{Bek} is Bekenstein entropy, A - the horizon area, A_0 - the Planck area, $A_0 \sim L_0^2 \sim l_p^2$ with l_p - Planck length, and Δ is the parameter related to the fractal dimension d_f by the relation $\Delta = d_f - 2$.

- This structure is characterised by the fractal dimension d_f which in the extreme cases is the surface or the volume i.e. $2 \leq d_f \leq 3$, and results in an **effective horizon area** of r^{d_f}, where r is the black hole horizon radius.

- In fact, $0 \leq \Delta \leq 1$ with $\Delta \to 1$ limit yielding maximally fractal structure, where the horizon area behaves effectively like a $3-$dimensional volume, and with $\Delta \to 0$ limit yielding the Bekenstein area law, where no fractalization occurs.

Barrow entropy vs Tsallis-Cirto δ-entropy

- Although Barrow entropy has geometrical roots, and is not motivated by thermodynamics, it has the same form as Tsallis-Cirto δ entropy (26) being also related to Bekenstein entropy S_{Bek} as in (11), provided that

$$\delta = 1 + \frac{\Delta}{2}. \tag{36}$$

- However, the ranges of parameters δ and Δ are different - δ has only the bound $\delta > 0$ while $0 \leq \Delta \leq 1$ is equivalent to $1 \leq \delta \leq 3/2$. Thus, qualitatively, both entropic forms yield the same temperatures as a function of a black hole mass.
- Both Tsallis-Cirto entropy limit $\delta \to 3/2$ and Barrow limit $\Delta \to 1$ yield an extensive, but still nonadditive entropy for black holes.

Landsberg $\mathcal{U}$-entropy.

The Landsberg U-entropy is defined in relation to Tsallis $q-$entropy (18) as (Landsberg 1999)

$$S_U = \frac{k_B}{1-q}\left(1 - \frac{1}{\sum_{i=1}^{n}(p_i)^q}\right) = k_B \frac{1-\sum_{i=1}^{n}(p_i)^q}{q-1}\frac{1}{(p_i)^q} = \frac{\mathcal{S}_q}{\sum_{i=1}^{n}(p_i)^q}, \tag{37}$$

and it fulfils the Abé rule (17) for $\Upsilon = q-1$. By assuming that all the states are equally probable, it simplifies (37) to the form

$$S_U = n^{q-1} S_q, \tag{38}$$

so it simply relates to Tsallis q-entropy.

Sharma-Mittal entropy

The Sharma-Mittal (SM) entropy (Sharma & Mittal, J. Comb. Inf. Syst. Sci. 2, 122 (1977)) combines the Rényi entropy with the Tsallis $q-$entropy, and is defined as

$$S_{SM} = \frac{k_B}{R}\left[\left(\sum_{i=1}^{n}(p_i)^q\right)^{\frac{R}{1-q}} - 1\right], \tag{39}$$

where R is another dimensionless parameter apart from q. For equally probable states, one gets from (39) that

$$S_{SM} = \frac{k_B}{R}\left\{\left[1 + \frac{1-q}{k_B}S_q\right]^{\frac{R}{1-q}} - 1\right\}, \tag{40}$$

where $R \to 1-q$ limit yields the Tsallis entropy, and $R \to 0$ limit yields Rényi entropy. It is interesting to note that the SM entropy obeys the composition rule of Abé (17) for $\Upsilon = 1$.

Kaniadakis entropy.

- Kaniadakis entropy (Kaniadakis PRE 2002, PRE 2005) results from taking into account Lorentz transformations of special relativity. It is **a single K-parameter $(-1 < K < 1)$ deformation** of BG entropy (7) with K parameter related to the dimensionless rest energy of the various parts of a multibody relativistic system.
- The basic definition of Kaniadakis entropy which directly generalizes BG entropy reads

$$S_K = -k_B \sum_{i=1}^{n} p_i \ln_K p_i \tag{41}$$

- The formula (41) introduces the K-logarithm

$$\ln_K x \equiv \frac{x^K - x^{-K}}{2K} = \frac{1}{K} \sinh\left(K \ln x\right) \tag{42}$$

with some basic properties like $\ln_K x^{-1} = -\ln_K x$ and $\ln_{-K} x = \ln_K x$ and it gives the standard logarithm $\ln x$ in the limit $K \to 0$.

Kaniadakis entropy.

- An equivalent definition of Kaniadakis entropy which can be obtained after the application of $K-$logarithm (42) reads

$$S_K = -k_B \sum_{i=1}^{n} \frac{(p_i)^{1+K} - (p_i)^{1-K}}{2K}. \tag{43}$$

- The $K-$logarithm fulfils a generalized composition rule which reads

$$\ln_K(xy) = \ln_K x\sqrt{1+K^2(\ln_K y)^2} + \ln_K y\sqrt{1+K^2(\ln_K x)^2}, \tag{44}$$

and it admits the standard logarithm rule $\ln(xy) = \ln x + \ln y$ in the limit $K \to 0$.

- The rule (44) comes from the definition of $K-$sum

$$(x \oplus y)_K = x\sqrt{1+K^2y^2} + y\sqrt{1+K^2x^2}, \tag{45}$$

where one replaced $x \to \ln x$ and $y \to \ln y$ and gives standard additivity rule $(x \oplus y)_K = x + y$ in the limit $K \to 0$.

Kaniadakis entropy.

- Using the definition of Kaniadakis entropy (41) and $K-$logarithm composition rule, we can write down the Kaniadakis entropy additivity rule as follows

$$S_K(A+B) = S_K(A)\sqrt{1+\frac{K^2}{k_B^2}S_K(B)} + S_K(B)\sqrt{1+\frac{K^2}{k_B^2}S_K(A)} \quad (46)$$

which we call *$K-$addition rule*.

- It is interesting to note that by the application of the $K-$sum (Kaniadakis 2002) defined as

$$(x \otimes y)_K = \frac{1}{K}\sinh\left[\frac{1}{K}\text{arcsinh}(Kx)\text{arcsinh}(Ky)\right], \quad (47)$$

Kaniadakis entropy.

one has for the $K-$logarithm

$$\ln_K[(x \otimes y)_K] = \ln_K x + \ln_K y, \tag{48}$$

so that applying it to (41), the Kaniadakis entropy can take a completely *additive form* as below

$$S_K(A+B)_K = S_K(p_A p_B) = S_K(p_A) + S_K(p_B) = S_K(A) + S_K(B). \tag{49}$$

Finally, in analogy to the previous considerations, and under the assumption that all the states are equally probable, one gets from (41) that

$$\ln_K p_i = -\frac{1}{K}\frac{e^{K\ln n} - e^{-K\ln n}}{2}, \tag{50}$$

which can further be transformed into ($S = k_B \ln n$ is BG entropy):

$$S_K = \frac{k_B}{K}\sinh\left(\frac{K}{k_B}S\right). \tag{51}$$

The plethora of entropies

Entropy Type	Extensivity	Additivity	Abé rule	$\delta-$rule	$K-$rule
Boltzmann-Gibbs S_{BG}	yes	yes	yes, $\Upsilon = 0$	yes, $\delta = 1$	yes, $K = 0$
Bekenstein S_{Bek}	no	no*	no	no	no
Tsallis q-entropy S_q	no	no	yes, $\Upsilon = 1 - q$	no	no
Tsallis-Cirto S_δ $(\delta \neq \frac{3}{2})$	no	no	no	yes	no
Tsallis-Cirto S_δ $(\delta = \frac{3}{2})$	yes	no	no	yes, $\delta = \frac{3}{2}$	no
Barrow $S_{Bar} = S_{Bek}$ $(\Delta = 0)$	no	no*	no	no	no
Barrow S_{Bar} $(0 < \Delta < 1)$	no	no	no	yes	no
Barrow S_{Bar} $(\Delta = 1)$	yes	no	no	yes, $\delta = \frac{3}{2}$	no
Rényi S_R	no	yes	yes, $\Upsilon = 0$	no	no
Landsberg U-entropy S_U	no	no	yes, $\Upsilon = q - 1$	no	no
Kaniadakis S_K	no	no	no	no	yes
Sharma-Mittal $S_{SM}(q, R)$	no	no	yes, $\Upsilon = R$	no	no
Tsallis q, δ-entropy $S_{q,\delta}$	no	no	no	no	no
Tsallis-Jensen $S_{q,\gamma}$	no	no	no	yes, $\delta = 1/\gamma$	no
Tsallis-Jensen $S_{1,\gamma}$	no	no	no	yes, $\delta = 1/\gamma$	no
Tsallis-Jensen $S_{q,1} = S_R$	no	yes	yes, $\Upsilon = 0$	no	no

* obeys square root composition rule (13)

Some remarks:

- There is the whole group of **Tsallis invented** thermodynamical entropies which generalize BG entropy in some different ways (cf. Table). Obey either the Abé composition rule or δ-addition rule.
- Tsallis q-entropy **relates** to both the Rényi and the Landsberg $\mathcal{U}$ entropies, while it **is generalized** by the Sharma-Mittal entropy.
- Tsallis-Cirto δ-entropy is **related** to Barrow entropy and Tsallis-Jensen q, γ entropy.
- Kaniadakis entropy form **a separate branch** of nonextensive entropies because of its hyperbolic formulation as a consequence of relativity theory being taken into account, but it still has a BG limit.
- All the entropies in our study have **BG limit except Bekenstein** entropy, but it is composable though its composition rule is unique among any other rules (square root rule).

General 4-parameter and 5-parameter entropies

There exists a four-parameter entropic formula (Nojiri, Odinstov 2022) which reads (here $k_B = 1$)

$$S_g(\alpha_\pm, \beta, \sigma) = \frac{1}{\sigma}\left[\left(1 + \frac{\alpha_+}{\beta} S\right)^{\beta} - \left(1 + \frac{\alpha_-}{\beta} S\right)^{-\beta}\right], \tag{52}$$

as well as the five-parameter formula (Odintsov, Paul 2023) of the form

$$S_g(\alpha_\pm, \beta, \sigma, \epsilon) = \frac{1}{\sigma}\left\{\left[1 + \frac{1}{\epsilon}\tanh\left(\frac{\epsilon\alpha_+}{\beta} S\right)\right]^{\beta} - \left[1 + \frac{1}{\epsilon}\tanh\left(\frac{\epsilon\alpha_-}{\beta} S\right)\right]^{-\beta}\right\}, \tag{53}$$

General 4-parameter and 5-parameter entropies

Both these formulas **generalize some of the entropies** which are contained in the Table 31 and have the following limits:

1. if $\epsilon \to 0$, then one recovers Tsallis-Cirto (24) and Barrow (35) entropies;
2. if $\epsilon \to 0$, $\alpha_- \to 0$, $\beta \to 0$, and α_+/β finite, then one recovers Rényi entropy (20);
3. if $\epsilon \to 0$, $\alpha_- \to 0$, $\sigma = \alpha_+ = R$, and $\beta = R/\delta$, then one recovers Sharma-Mittal entropy formula (40), though only when one replaces Tsallis q-entropy S_q with Tsallis-Cirto S_δ entropy;
4. if $\epsilon \to 0$, $\beta \to \infty$, $\alpha_+ = \alpha_- = \sigma/2 = K$, then one recovers Kaniadakis entropy (51).

4. Observational constraints on nonextensivity.

- Interestingly, we applied Barrow entropy as HDE in (2) gives (e.g. Saridakis 2020, MPD & Salzano 2020, Denkiewicz et al. 2023, Çimdiker at al. 2024):

$$\rho_{BH} = \frac{3\,C^2}{8\pi G} L^{(\Delta-2)}\,, \tag{54}$$

where C is the *holographic parameter* with dimensions of $s^{-1}m^{(1-\Delta/2)}$ and

$$C = ckL_0^{-\Delta/2} = 3 \cdot 10^{2(4+9\Delta)} 2^{-2\Delta}. \tag{55}$$

- **Note:** ΛCDM ($\Delta = 2$, $\rho_{BH} =$ const.) is excluded in Barrow holography, but possible for Tsallis-Cirto $\Delta = \delta = 2$.
- **Note:** all the cosmological calculations are also valid for Tsallis entropy at least in the range of its parameter $1 \leq \delta \leq 3/2$.

Observational constraints on nonextensivity.

The Friedmann equation for Barrow (Tsallis) HDE is

$$H^2(a) = \frac{8\pi G}{3}\left(\rho_m(a) + \rho_r(a) + \rho_{BH}(a)\right) , \qquad (56)$$

where the suffices m and r refer respectively to matter and radiation. Standard continuity equation for matter and radiation is still valid, i.e.

$$\dot{\rho}_{m,r}(a) + 3H\left(\rho_{m,r}(a) + \frac{p_{m,r}(a)}{c^2}\right) = 0\,, \qquad (57)$$

where the pressure $p_i = w_i \rho_i$. We can rewrite (6) as

$$1 = \Omega_m(a) + \Omega_r(a) + \Omega_H(a)\,, \qquad (58)$$

introducing the dimensionless density parameters $\Omega_i(a)$, defined as

$$\Omega_{m,r}(a) = \frac{H_0^2}{H^2(a)}\Omega_{m,r}a^{-3(1+w_{m,r})}\,, \quad \Omega_{BH}(a) = \frac{C^2}{H^2(a)}L^{(\Delta-2)}\,. \qquad (59)$$

Observational constraints on nonextensivity.

Data applied (Denkiewicz et al. 2023):

- Type Ia Supernovae (SNeIa) from the Pantheon sample;
- Cosmic Chronometers (CC);
- the “Mayflower” sample of Gamma Ray Bursts (GRBs);
- latest *Planck* 2018 release for Cosmic Microwave Background radiation (CMB) (shift parameter);
- Baryon Acoustic Oscillations (BAO) from several surveys.

Considered 2 cases:

- “full data”, where we join both early- (CMB + BAO data from SDSS) and late-time observations (SNeIa, CC, GRBs + BAO data from WiggleZ);
- “late-time” data set - includes only late-time data **(after recombination)**.

Results of statistical analysis ("geo" - geometrical, "late" - late time data, "dyn" - dynamical, BH1 - event horizon (not Hubble))

	Ω_m	Ω_b	h	$\mathcal{M}$	$\sigma_{8,0}$	Δ	C	$S_{8,0}$	$\log \mathcal{B}^i_j$
"Revision" of MPD & Salzano 2020									
LCDM (geo-late)	$0.293^{+0.016}_{-0.016}$	–	$0.713^{+0.013}_{-0.013}$	–	–	–	–	–	*0*
LCDM (geo-full)	$0.319^{+0.005}_{-0.005}$	$0.0494^{+0.0004}_{-0.0004}$	$0.673^{+0.003}_{-0.003}$	–	–	–	–	–	*0*
BH1 (geo-late)	$0.290^{+0.020}_{-0.019}$	–	$0.715^{+0.014}_{-0.013}$	–	–	**> 0.63**	$3.93^{+1.77}_{-1.88}$	–	$-0.71^{+0.03}_{-0.02}$
BH1 (geo-full)	$0.314^{+0.006}_{-0.006}$	$0.049^{+0.001}_{-0.001}$	$0.676^{+0.007}_{-0.007}$	–	–	**> 0.84**	$4.66^{+0.87}_{-1.07}$	–	$-0.05^{+0.03}_{-0.03}$
Updated and newest constraints from Denkiewicz et al. 2023									
LCDM (geo-late)	$0.321^{+0.015}_{-0.015}$	–	$0.730^{+0.010}_{-0.009}$	$-19.263^{+0.028}_{-0.028}$	–	–	–	–	*0*
LCDM (geo-full)	$0.318^{+0.007}_{-0.006}$	$0.0493^{+0.0006}_{-0.0006}$	$0.674^{+0.004}_{-0.004}$	$-19.437^{+0.012}_{-0.012}$	–	–	–	–	*0*
LCDM (geo-late+dyn)	$0.315^{+0.014}_{-0.014}$	–	$0.731^{+0.010}_{-0.010}$	$-19.263^{+0.028}_{-0.028}$	$0.770^{+0.018}_{-0.017}$	–	–	$0.790^{+0.023}_{-0.022}$	*0*
LCDM (geo-full+dyn)	$0.314^{+0.006}_{-0.005}$	$0.0490^{+0.0006}_{-0.0006}$	$0.677^{+0.004}_{-0.004}$	$-19.429^{+0.011}_{-0.011}$	$0.779^{+0.017}_{-0.017}$	–	–	$0.796^{+0.019}_{-0.019}$	*0*
BH1 (geo-late)	$0.300^{+0.020}_{-0.019}$	–	$0.729^{+0.010}_{-0.010}$	$-19.263^{+0.028}_{-0.029}$	–	**> 0.63**	$4.50^{+2.20}_{-2.13}$	–	$-0.39^{+0.02}_{-0.04}$
BH1 (geo-full)	$0.311^{+0.006}_{-0.006}$	$0.0486^{+0.0008}_{-0.0008}$	$0.679^{+0.006}_{-0.006}$	$-19.438^{+0.013}_{-0.013}$	–	**> 0.82**	$4.58^{+0.90}_{-1.16}$	–	$-2.99^{+0.04}_{-0.04}$
BH1 (geo-late+dyn)	$0.290^{+0.018}_{-0.017}$	–	$0.729^{+0.010}_{-0.010}$	$-19.261^{+0.028}_{-0.028}$	$0.791^{+0.022}_{-0.022}$	**> 0.69**	$5.31^{+1.97}_{-2.27}$	$0.778^{+0.021}_{-0.022}$	$-0.35^{+0.03}_{-0.03}$
BH1 (geo-full+dyn)	$0.307^{+0.006}_{-0.006}$	$0.0484^{+0.0009}_{-0.0008}$	$0.681^{+0.006}_{-0.005}$	$-19.431^{+0.013}_{-0.013}$	$0.777^{+0.017}_{-0.017}$	**> 0.86**	$4.89^{+0.76}_{-1.03}$	$0.786^{+0.020}_{-0.020}$	$-4.36^{+0.04}_{-0.04}$

Calculated quantities - explanation of the table:

In the following table for each parameter we provide the median and the 1σ constraints. The columns show:

1. considered theoretical scenario;
2. dimensionless matter parameter, Ω_m;
3. dimensionless baryonic parameter, Ω_b;
4. dimensionless Hubble constant, h;
5. fiducial absolute magnitude, $\mathcal{M}$;
6. amplitude of the linear power spectrum at present time, $\sigma_{8,0}$;
7. Barrow entropic parameter, Δ;
8. holographic parameter, C;
9. amplitude of the weak lensing measurement (secondary derived parameter), $S_{8,0} = \sigma_{8,0}\sqrt{\Omega_m/0.3}$;
10. logarithm of the Bayes Factor, $\log \mathcal{B}^i_j$.

Data pointing towards the extensive HDE!

- Our bound on Barrow parameter $\Delta > 0.86$ (Tsallis-Cirto $\delta > 1.43$) strongly points towards its maximum value $\Delta = 1$ ($\delta = 3/2$) in which case Barrow/Tsallis-Cirto entropy **recovers extensivity** (though still remains nonadditive fulfilling the rule (25)) - this is why we call it **”nearly extensive Gibbs-like entropy”**.
- This has been **recently pointed out also by Tsallis and Jensen** (arXiv: 2408.08820) who claim that their

$$S_{1,\gamma} = S_\delta \propto (S_{Bek})^\delta \quad \text{with} \quad \delta = \frac{1}{\gamma} = 1 + \frac{\Delta}{2}. \tag{60}$$

Other data pointing towards extensive HDE.

The following bounds confirm that claim:

- High-energy neutrino data (IceCube Neutrino Observatory on South Pole) give $\delta = 1.565$ (Jizba & Lambiase EPJC 82, 1123 (2022));
- 2 different models of neutrino data from Planck Observatory (ESA) give $\delta = 1.87$ and $\delta = 1.26$ (Salehi et al. GRG 55, 57 (2023));
- Big-Bang nucelosynthesis bound based on analysis of abundance of CDM particles gives $\delta = 1.499$ (Jizba & Lambiase, Entropy 25, 1495 (2023)).
- HDE approach + different horizon + interaction between DM and DE, tested against combined Pantheon, SNIa, BAO, CMB, and GRB, got $\delta = 1.360$ (Mamon arXiv: 2007.01591)

Tension with other bounds on Barrow (Tsallis-Cirto) parameter Δ (δ).

- However, this result is **in tension with all the other bounds** in the literature. Possible explanations lie in the fact that **we use directly holographic principle (HP) or HDE approach** while most of the **other papers use gravity-thermodynamics conjecture (GT)** of Jacobson (1995).
- Jacobson method of obtaining gravity from thermodynamics relies on the fact that the Barrow entropy gives a general relativity-like gravity with a rescaled cosmological constant $\tilde{\Lambda} = \Lambda[(1 + \Delta/2)A^{\Delta/2}]^{-1}$, and having the limit $\Delta \to 0$ as standard Λ.
- In view of that, it is **already Λ-term dominated model** which solves dark energy problem and is statistically preferable with some ”small correction” coming from the nonextensive entropy.
- This is what happens in most of these types of bounds which obtain $\Delta \sim 0$.

Tension with other bounds on Barrow/Tsallis-Cirto parameter Δ/δ.

- From obs. bounds on baryon asymmetry: Barrow entropy $\Delta \sim 0.005 - 0.008$ (Luciano & Giné arXiv: 2210.09755); Tsallis entropy $0.002 \lesssim |\ \delta - 1\ | = |\ \Delta/2\ | \lesssim 0.004$ (arXiv: 2204.02723);
- From Big-Bang nucleosynthesis: Tsallis entropy $1 - \delta < 10^{-5}$ (Ghoshal, Lambiase arXiv:2104.11296); Barrow entropy $\Delta \lesssim 1.4 \cdot 10^{-4}$ (Barrow, Basilakos, Saridakis PLB 815, 136134 (2021);
- GT approach applied to Pantheon + BAO ("late-time") data: $\Delta \sim 10^{-4}$ (Leon et al. JCAP 12, 032 (2021) - radiation is neglected;
- GT approach applied also to cosmol. perturbations with an extra scalar field acting as dark energy which is effectively Λ (Aghari, Sheykhi arXiv:2106.15551) - Tsallis $\delta \approx 0.9997$ ($\Delta = -0.0006 < 0$);
- GT application of Planck data to Barrow restricts to $\Delta \lesssim 10^{-4}$ (though assuming only 30 e-folds) (Luciano arXiv: 2301.12509)

Tension with other bounds on Barrow/Tsallis-Cirto parameter Δ/δ.

- HP approach; using SNeIa, CC, GRBs they obtain $\Delta \approx -1.68$ which is behind the domain of Barrow parameter though consistent with Tsallis $\delta \approx 0.16$ (Mangoudehi arXiv: 2211.17212);
- HP approach, usage of early-times data (BAO, CMB), no radiation, Tsallis $\delta \approx 1.07$ corresponding Barrow $\Delta \approx 0.14$ (Sadri arXiv: 1905.11210);
- HP approach, late-time data only, Barrow $\Delta \sim 0.09$ (Saridakis et al. JCAP 12, 012 (2018), Anagnostopoulos et al. EPJS 80, 826 (2020);
- HP approach, SNeIa + CC only, Barrow $\Delta \sim 0.06 \div 0.2$ (Adhikary et al. PRD 104, 123519 (2021));

6. Conclusions

- There is a **plethora of nonextensive entropies** which can be applied to black hole horizon models and to Holographic Dark Energy models.
- Interestingly, these entropies have **BG limit except Bekenstein**.
- They are mostly **composable** though the composition rules take many forms. Bekenstein entropy fulfils a specific square root composition rule.
- There are 2 different approaches to holographic models which are observationally **in strong tension**. One kind of them (Jacobson approach modifying Λ-term) point out towards nonextensive entropies and another (pure HDE, no Λ) point out towards extensivity.