Perfect fluid cosmology in conformal Killing gravity

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Harada's field equations

- Conformal Killing gravity is a relativistic theory of gravitation proposed by J. Harada in 2023.
- The theory was constructed under the following assumptions
  - The cosmological constant may appear in the solutions of the field equations only as an integration constant.
  - The divergencelessness of the energy-momentum tensor should be a consequence of the field equations.
  - Conformally flat metrics should not necessarily be solutions of the vacuum field equations.
- The name of the theory comes from the fact that it has an alternative formulation in which the Einstein field equations are modified by the conformal Killing tensor.

Harada's field equations

- The field equations of the conformal Killing gravity are expressed as follows

$$H_{lmn} = \varkappa T_{lmn},$$

where the coupling constant is $\varkappa = \frac{8\pi G}{c^4}$, where c is the speed of light, G is the gravitational constant and we define

$$H_{lmn} = \nabla_l h_{mn} + \nabla_n h_{lm} + \nabla_m h_{nl},$$
$$T_{lmn} = \nabla_l t_{mn} + \nabla_n t_{lm} + \nabla_m t_{nl},$$

where

$$h_{mn} = G_{mn} - \frac{1}{6} g_{mn} G^a{}_a, \qquad t_{mn} = T_{mn} - \frac{1}{6} g_{mn} T^a{}_a,$$

where G_{mn} is the Einstein tensor and T_{mn} is the energy-momentum tensor.

- We note that when $G_{mn} = \varkappa T_{mn}$ the field equations are identically satisfied.

- The tensor H_{lmn} has the following properties

$$H_{lmn} = H_{lnm}, \qquad H_{lmn} = H_{mln}, \qquad H_n{}^a{}_a = 0,$$

which means, it is a trace-free totally symmetric rank-3 tensor.

- The tracelessness of the tensor H_{lmn} taken together with the field equations yields the divergencelessness of the energy-momentum tensor.
- In D dimensions, the number of independent components of a tensor field with the symmetries of the tensor H_{lmn} equals

$$\binom{D+2}{3} - D,$$

which is the number of 3-element combinations with repetitions of the D-element set minus the number of dimensions. In 4 dimensions this number equals 16.

Decomposition of the tensor H_{lmn}

- We are going to consider some cosmological space-time with a metric tensor g_{mn} and some matter flow with a 4-velocity vector u_n normalized as $u^a u_a = -1$. We use the projection tensor P_{mn} defined as

$$P_{mn} = u_m u_n + g_{mn}.$$

- The tensor H_{lmn} can be decomposed into irreducible parts with respect to the 4-velocity u_n as

$$\begin{aligned} H_{lmn} = -u_l u_m u_n a + \left(u_l P_m{}^i u_n + P_l{}^i u_m u_n + u_l u_m P_n{}^i\right) b_i \\ - \left(P_l{}^i P_m{}^j u_n + u_l P_m{}^j P_n{}^i + P_l{}^j u_m P_n{}^i\right)\left(c_{ji} + \frac{1}{3} P_{ji} d\right) \\ + e_{lmn} + \frac{1}{5}\left(P_l{}^i P_{mn} + P_n{}^i P_{lm} + P_m{}^i P_{nl}\right) f_i. \end{aligned}$$

Decomposition of the tensor H_{lmn}

- The newly introduced quantities are defined as follows
  - two scalars
$$a = u^c u^b u^a H_{cba}, \qquad d = u^c P^{ba} H_{cba},$$
    where because of the tracelessness of the tensor H_{lmn}, we have $d = a$,
  - two spatial vectors
$$b_n = u^c u^b P_n{}^a H_{cba}, \qquad f_n = P_n{}^c P^{ba} H_{cba}, \qquad u^i b_i = u^i f_i = 0,$$
    where similarly because of the tracelessness of the tensor H_{lmn}, we have $f_n = b_n$,
  - trace-free symmetric rank-2 spatial tensor
$$c_{mn} = u^c \left(P_m{}^b P_n{}^a - \frac{1}{3} P_{mn} P^{ba} \right) H_{cba}, \qquad u^i c_{ni} = 0, \qquad c_{mn} = c_{nm}, \qquad c^i{}_i = 0,$$
  - trace-free totally symmetric rank-3 spatial tensor
$$e_{lmn} = \left(P_l{}^c P_m{}^b P_n{}^a - \frac{1}{5} \left(P_l{}^c P_{mn} + P_n{}^c P_{lm} + P_m{}^c P_{nl} \right) P^{ba} \right) H_{cba}$$
$$u^i e_{mni} = 0, \qquad e_{lmn} = e_{lnm}, \qquad e_{lmn} = e_{mln}, \qquad e_n{}^i{}_i = 0.$$
- The field b_n has 3 independent components, the field c_{mm} has 5 and the field e_{lmn} has 7.

Basic identities

- In the temporal-spatial splitting method applied to cosmology, there are generally used some basic identities, which we now list.
- The divergencelessness of the energy-momentum tensor

$$\nabla^a T_{na} = 0.$$

- The Ricci identities for the flow 4-velocity

$$\nabla_l \nabla_m u_n - \nabla_m \nabla_l u_n - R_{lmn}{}^a u_a = 0,$$

  where R_{klmn} is the Riemann tensor.
- The divergencelessness of the Einstein tensor

$$\nabla^a G_{na} = 0.$$

  Since we do not assume the Einstein field equations, the above identity is independent of the divergencelessness of the energy-momentum tensor.
- The Bianchi identities

$$\nabla^a C_{lmna} - C_{nml} = 0,$$

  where C_{klmn} is the Weyl tensor and C_{lmn} is the Cotton tensor.

- We define the cosmological model under consideration as one filled with a purely expanding perfect fluid. This means that for the 4-velocity it holds

$$\nabla_m u_n = \frac{1}{3} P_{mn} \theta,$$

where θ is the expansion rate and for the energy-momentum tensor

$$T_{mn} = u_m u_n \rho + P_{mn} p,$$

where ρ is the energy density and p is the pressure.

- Moreover, we decide to assume that the Einstein tensor is built entirely from the free scalar fields appearing in the problem

$$G_{mn} = u_m u_n \Big((1+3\alpha)R + 3\beta\theta^2 + 3\gamma\rho + 3\delta p + 3\epsilon \Big) + P_{mn} \Big(\alpha R + \beta\theta^2 + \gamma\rho + \delta p + \epsilon \Big),$$

where R is the curvature scalar and α, β, γ, δ, and ϵ are constants. The splitting is chosen so that it occurs $G^a{}_a = -R$. There is no room for this type of assumption in the general theory of relativity because of the form of Einstein's field equations.

- We continue by trying to fully exploit the mentioned identities and Harada's field equations. After temporal-spatial decomposition, nontrivial constraints give the following projected equations
  - from the divergencelessness of the energy-momentum tensor

$$u^b\nabla^a T_{ba} = 0, \qquad P_n{}^b\nabla^a T_{ba} = 0,$$

  - from the Ricci identities

$$u^c P^{db}\left(\nabla_d\nabla_c u_b - \nabla_c\nabla_d u_b - R_{dcb}{}^a u_a\right) = 0,$$

$$P_n{}^c P^{db}\left(\nabla_d\nabla_c u_b - \nabla_c\nabla_d u_b - R_{dcb}{}^a u_a\right) = 0,$$

  - from the divergencelessness of the Einstein tensor

$$u^b\nabla^a G_{ba} = 0, \qquad P_n{}^b\nabla^a G_{ba} = 0,$$

  - from the Bianchi identities

$$u^d P_n{}^c u^b\left(\nabla^a C_{dcba} - C_{bcd}\right) = 0,$$

  - from the Harada field equations

$$u^c u^b u^a\left(H_{cba} - \varkappa T_{cba}\right) = 0.$$

- From these eight equations it follows that the spatial derivatives of the free scalar fields vanish

$$P_n{}^a \nabla_a R = 0, \qquad P_n{}^a \nabla_a \theta = 0, \qquad P_n{}^a \nabla_a \rho = 0, \qquad P_n{}^a \nabla_a p = 0,$$

and the temporal derivatives are given by

$$\begin{aligned}
u^a \nabla_a R &= -\frac{1}{1+3\alpha}\theta\Big((1+4\alpha)R + 4\beta\theta^2 + 4\gamma\rho + 4\delta p + 4\epsilon\Big) \\
&\quad - \frac{3\beta}{1+3\alpha}u^a\nabla_a\theta^2 - \frac{3\gamma}{1+3\alpha}u^a\nabla_a\rho - \frac{3\delta}{1+3\alpha}u^a\nabla_a p, \\
u^a \nabla_a \theta &= -\left(\frac{1}{2} + 3\alpha\right)R - \left(\frac{1}{3} + 3\beta\right)\theta^2 - 3\gamma\rho - 3\delta p - 3\epsilon, \\
u^a \nabla_a \rho &= -\theta(\rho + p), \\
u^a \nabla_a p &= -\frac{5+18\alpha}{3\varkappa(1+3\alpha) - 3\delta}\theta\Big((1+4\alpha)R + 4\beta\theta^2 + 4\gamma\rho + 4\delta p + 4\epsilon\Big) \\
&\quad + \frac{3\beta}{3\varkappa(1+3\alpha) - 3\delta}u^a\nabla_a\theta^2 - \frac{5\varkappa(1+3\alpha) - 3\gamma}{3\varkappa(1+3\alpha) - 3\delta}u^a\nabla_a\rho.
\end{aligned}$$

Model properties

- In the absence of dependence on spatial coordinates, we can replace derivatives in the direction of the 4-velocity $u^a \nabla_a$ with derivatives with respect to the time coordinate ∂_t.
- The equations for temporal derivatives of the scalar fields R, θ, ρ, and p form a dynamical system with 5 constant parameters (α, β, γ, δ, and ϵ).
- Two important identities can be found for this dynamical system.

- The first one follows from the general result that when the vorticity rate, the acceleration and the temporal-spatial part of the Einstein tensor vanish

$$\omega_{mn} = 0, \qquad \eta_n = 0, \qquad u^b P_n{}^a G_{ba} = 0,$$

then the tensor field Υ_{mn} defined as

$$\Upsilon_{mn} = \mathcal{R}_{mn} - \frac{1}{2} g_{mn} \mathcal{R},$$

where $\mathcal{R}_{mn}$ is the spatial Ricci tensor and $\mathcal{R}$ is its trace, is divergenceless

$$\nabla^a \Upsilon_{na} = 0.$$

Applying this identity to the considered model, we obtain

$$u^a \nabla_a \mathcal{R} + \frac{2}{3} \theta \mathcal{R} = 0,$$

where here

$$\mathcal{R} = (2 + 6\alpha) R - \left(\frac{2}{3} - 6\beta \right) \theta^2 + 6\gamma\rho + 6\delta p + 6\epsilon.$$

- The second one is specific for the conformal Killing gravity and reads

$$u^a \nabla_a \mathcal{I} - \frac{2}{3}\theta \mathcal{I} = 0,$$

where

$$\mathcal{I} = \mathcal{R} - \frac{1}{2}R + \frac{2}{3}\theta^2 - \frac{3}{2}\varkappa\rho - \frac{3}{2}\varkappa p.$$

In Einstein's general theory of relativity, the scalar $\mathcal{I}$ vanishes in the case of a purely expanding perfect fluid.

Equation of state

- Let us further assume that the matter in the model under consideration obeys the following linear barotropic equation of state

$$p = w\rho,$$

where w is a constant. This assumption implies that

$$\alpha = -\frac{5}{18}, \qquad \beta = 0, \qquad \gamma = \frac{5+3w}{18}\varkappa - \delta w,$$

and the dynamical system takes the form

$$\begin{aligned}
\partial_t R &= \theta\left(\frac{2}{3}R - \frac{(1-3w)(5+3w)}{3}\varkappa\rho - 24\epsilon\right),\\
\partial_t \theta &= \frac{1}{3}R - \frac{1}{3}\theta^2 - \frac{5+3w}{6}\varkappa\rho - 3\epsilon,\\
\partial_t \rho &= -(1+w)\theta\rho.
\end{aligned}$$

This is a 3-dimensional dynamical system for scalars R, θ, and ρ with two constant parameters (ϵ and w).

Equation of state

- Given the assumed equation of state, the scalars $\mathcal{R}$ and $\mathcal{I}$ take the following form

$$\mathcal{R} = \frac{1}{3}R - \frac{2}{3}\theta^2 + \frac{5+3w}{3}\varkappa\rho + 6\epsilon, \qquad \mathcal{I} = -\frac{1}{6}R + \frac{1-3w}{6}\varkappa\rho + 6\epsilon.$$

Using the differential equations for these scalars, we give two first integrals for the above dynamical system

$$\mathcal{R}(\varkappa\rho)^{-\frac{2}{3(1+w)}} = c_1, \qquad \mathcal{I}(\varkappa\rho)^{\frac{2}{3(1+w)}} = c_2,$$

where c_1 and c_2 are constants of motion.

- The dynamical system under consideration is completely integrable. Its solution can be given in the form of the inverse of the integral

$$\theta^2 = \frac{1}{2}R + 9\epsilon + \frac{5+3w}{2}\varkappa\rho - \frac{3c_1}{2}(\varkappa\rho)^{\frac{2}{3(1+w)}},$$

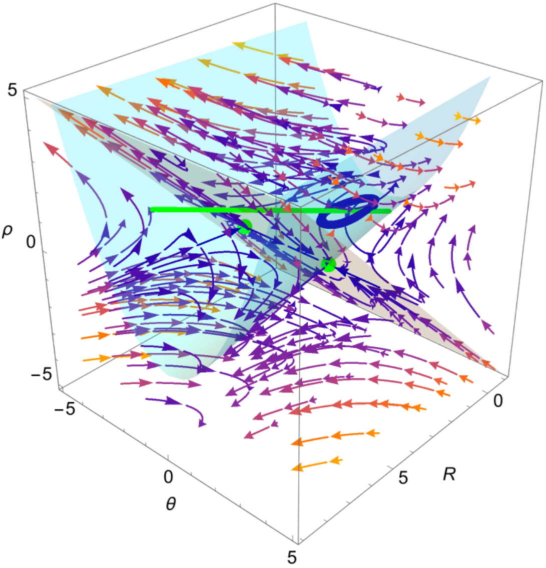
$$R = 36\epsilon + (1-3w)\varkappa\rho - 6c_2(\varkappa\rho)^{-\frac{2}{3(1+w)}},$$

$$t + c_0 = \int_0^\rho -\frac{1}{(1+w)o}\left(27\epsilon + 3\varkappa o - \frac{3c_1}{2}(\varkappa o)^{\frac{2}{3(1+w)}} - 3c_2(\varkappa o)^{-\frac{2}{3(1+w)}}\right)^{-\frac{1}{2}} \mathrm{d}o,$$

where c_0 is a constant.

Phase space

- Phase space diagram (R, θ, ρ) of the considered dynamical system with parameter values: $\varkappa = 1$, $\epsilon = \frac{1}{9}$, $w = 0$.

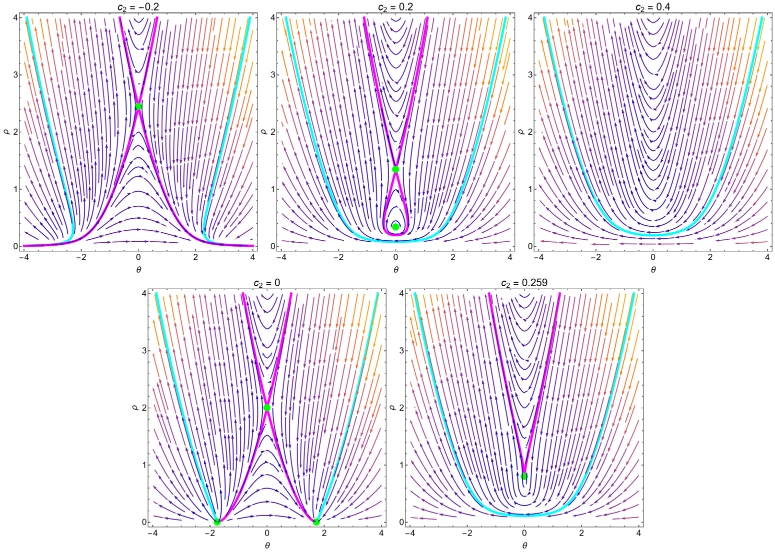


Phase space

- Fixed points in the phase space (R, θ, ρ):
  - point $(R, \theta, \rho) = (36\epsilon, -3\sqrt{3\epsilon}, 0)$, which is unstable,
  - point $(R, \theta, \rho) = (36\epsilon, 3\sqrt{3\epsilon}, 0)$, which is unstable,
  - line $R = 9\epsilon + \frac{5+3w}{2}\varkappa\rho$, $\theta = 0$, of non-isolated points that locally change type.
- Fixed points in the phase plane (θ, ρ) at a fixed value of the constant c_2:
  - for $c_2 < 0$ one point: saddle,
  - for $c_2 = 0$ three points: unstable node, stable node and saddle, (exactly as in the general theory of relativity),
  - for $0 < c_2 < c_2^*$ two points: saddle and center,
  - for $c_2 = c_2^*$ one point: degenerate saddle,
  - for $c_2^* < c_2$ no fixed points.
- The value of the constant c_2^* and the corresponding position of the degenerate saddle fixed point:

w	c_2^*	$(\theta, \varkappa\rho)$
1	$\frac{27\times 3^{\frac{2}{3}}}{16}\epsilon^{\frac{4}{3}}$	$(0, \frac{9\epsilon}{8})$
$\frac{2}{3}$	$\frac{45\times 3^{\frac{2}{5}}}{7\times 2^{\frac{1}{5}}7^{\frac{2}{5}}}\epsilon^{\frac{7}{5}}$	$(0, \frac{12\epsilon}{7})$
$\frac{1}{3}$	$3\sqrt{3}\epsilon^{\frac{3}{2}}$	$(0, 3\epsilon)$
0	$\frac{81\times 6^{\frac{1}{3}}}{5\times 5^{\frac{2}{3}}}\epsilon^{\frac{5}{3}}$	$(0, \frac{36\epsilon}{5})$

- Phase plane diagrams (θ, ρ) of the considered dynamical system at a fixed value of the constant c_2 with parameter values: $\varkappa = 1$, $\epsilon = \frac{1}{9}$, $w = 0$.



Summary

- We have considered a purely expanding perfect fluid in the context of the conformal Killing gravity.
- By making an appropriate assumption about the Einstein tensor, we have integrated the field equations in the case of a linear barotropic equation of state.
- Phase space diagrams show the variety of possible evolutionary scenarios for the class of cosmological models under consideration.