κ-deformed complex scalar field: from theory to phenomenology [1][2]

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[1] M. Arzano, **A. B.**, J. Kowalski-Glikman, G. Rosati, and J. Unger, κ-deformed complex fields and discrete symmetries. Phys.Rev.D, 103:106015

[2] **A.B.**, J. Kowalski-Glikman, and W. Wislicki, κ-deformed complex scalar field: conserved charges, symmetries and their impact on physical observables. Phys.Rev.D, 105:105004.

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- One such theory is based on κ-Minkowski spacetime.
- We will see how to start from the basics in this model, and build up to physical predictions.

Introduction, action, and fields

Preamble: What is a Lie algebra? What is a group?

An algebra $\mathfrak{g}$ is a set of objects such that if $A, B \in \mathfrak{g}$ then $A \bullet B, (A + B) \in \mathfrak{g}$. If one chooses

$$A_i \bullet A_j := [A_i, A_j] = f_{ijk} A_k.$$

and Jacobi identity $\implies$ Lie algebra.
Example: Angular momentum generators J_i, $[J_i, J_j] = i\epsilon_{ijk} J_k$.
The type of commutator gives the name to the algebra.

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(Intuitively and sometimes false) if $A \in \mathfrak{g}$, then $e^{iA} \in G$ where G is a group. Example: the real number form a trivial Lie algebra (if $a, b \in \mathbb{R}$, then $a + b \in \mathbb{R}$, and also $[a, b] = 0 \in \mathbb{R}$) and their complex exponential e^{ia} form the group $U(1)$.

κ-Minkowski spacetime and momentum space picture

Non-commutative coordinates: $[\hat{x}^0, \hat{x}^i] = \frac{i}{\kappa}\hat{x}^i$ ($\mathfrak{an}(3)$ algebra)

Physical insight: $\left[\frac{1}{\kappa}\right] = L$. However, this κ-deformed theory is intended as an effective theory modelling quantum gravitational effects $\implies \frac{1}{\kappa} \approx l_p$. The *formal* limit '$\kappa \to \infty$' gives the 'classical' limit.

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$$\hat{x}^0 = -\frac{i}{\kappa}\begin{pmatrix} 0 & \mathbf{0} & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 1 & \mathbf{0} & 0 \end{pmatrix} \quad \hat{\mathbf{x}} = \frac{i}{\kappa}\begin{pmatrix} 0 & \epsilon^T & 0 \\ \epsilon & \mathbf{0} & \epsilon \\ 0 & -\epsilon^T & 0 \end{pmatrix}$$

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$$\hat{e}_k = e^{ik_i\hat{x}^i}e^{ik_0\hat{x}^0} \in AN(3) \quad \leftarrow \quad \text{Time to the right + dim.ful}$$

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$$e^{A\hat{x}} = \sum_{n=0}^{\infty} \frac{(A\hat{x})^n}{n!} \quad \leftarrow \quad \text{Definition of exp}$$

κ-Minkowski spacetime and momentum space picture

$$\hat{e}_k = \begin{pmatrix} \frac{\bar{p}_4}{\kappa} & \frac{\mathbf{k}}{\kappa} & \frac{p_0}{\kappa} \\ \frac{\mathbf{p}}{\kappa} & \mathbf{1} & \frac{\mathbf{p}}{\kappa} \\ \frac{\bar{p}_0}{\kappa} & -\frac{\mathbf{k}}{\kappa} & \frac{p_4}{\kappa} \end{pmatrix}$$

$$p_0 = \kappa \sinh \frac{k_0}{\kappa} + \frac{\mathbf{k}^2}{2\kappa} e^{k_0/\kappa}$$
$$p_i = k_i e^{k_0/\kappa}$$
$$p_4 = \kappa \cosh \frac{k_0}{\kappa} - \frac{\mathbf{k}^2}{2\kappa} e^{k_0/\kappa}$$

Notice that $\hat{e}_k \Leftrightarrow (p_0, p_i, p_4)^T$ and if $\mathcal{O} = (0, \ldots, 0, \kappa)^T$ then $(p_0, p_i, p_4)^T = \hat{e}_k \mathcal{O}$, and

$$-p_0^2 + \mathbf{p}^2 + p_4^2 = \kappa^2$$

$$p_4 > 0, \qquad p_+ := p_0 + p_4 > 0$$

κ-Minkowski spacetime and momentum space picture

Notice: both the k_A and the p_A can be interpreted as coordinates in (intrinsically curved) momentum space, and their sum is now non-trivial.

$$\hat{e}_k \hat{e}_l := \hat{e}_{k \oplus l} \quad \leftarrow \quad \text{Group property}$$

$$(k \oplus l)_0 = k_0 + l_0$$
$$(k \oplus l)_i = k_i + e^{-k_0/\kappa} l_i$$

$$(p \oplus q)_0 = \frac{p_0}{\kappa} q_+ + \frac{\mathbf{pq}}{p_+} + \frac{\kappa}{p_+} q_0$$
$$(p \oplus q)_i = \frac{\mathbf{p}_i}{\kappa} q_+ + \mathbf{q}_i$$
$$(p \oplus q)_4 = \frac{p_4}{\kappa} q_+ - \frac{\mathbf{pq}}{p_+} - \frac{\kappa}{p_+} q_0$$

For similar reasons, $-(.) \mapsto S(.)$ with $p \oplus S(p) = S(p) \oplus p = 0$.

κ-Minkowski spacetime and momentum space picture

Why p and not k? Using p, we can now work in a commutative spacetime.

In particular, using an object called Weyl map, one can send a group element $\hat{e}_k$ into a canonical plane wave e_p

$$\mathcal{W}(\hat{e}_k) = e_p \qquad e_p = e^{ip_\mu x^\mu} = e^{i(\omega t - \mathbf{px})}$$

$$\mathcal{W}(\hat{e}_{k\oplus l}) = e_{p(k)\oplus q(l)} = e_p \star e_q$$

This $\star$ product is in general non-commutative.

Action, EOM, and fields

Because of the star product we have two possible orderings $\implies$ two possible actions.

$$S_1 = \int_{\mathbb{R}^4} d^4x \; (\partial^\mu \phi)^\dagger \star (\partial_\mu \phi) - m^2 \phi^\dagger \star \phi$$
$$S_2 = \int_{\mathbb{R}^4} d^4x \; (\partial_\mu \phi) \star (\partial^\mu \phi)^\dagger - m^2 \phi \star \phi^\dagger .$$

Therefore

$$S = \frac{1}{2} \int_{\mathbb{R}^4} d^4x \; [(\partial^\mu \phi)^\dagger \star (\partial_\mu \phi) + (\partial^\mu \phi) \star (\partial_\mu \phi)^\dagger - m^2 (\phi^\dagger \star \phi + \phi \star \phi^\dagger)]$$

Action, EOM, and fields

How to compute the EOM? Usually, one uses integration by parts. Here, however, the Leibniz rule for derivatives does not work!

$$i(p \oplus q)_\mu e_{p\oplus q} = \partial_\mu(e_p \star e_q) = (\partial_\mu e_p) \star e_q + e_p \star \partial_\mu e_q$$
$$= i(p+q)_\mu e_{p\oplus q}$$

Instead, more complicated rules need to be applied. Example:

$$\partial_0(\phi \star \psi) = \frac{1}{\kappa}(\partial_0 \phi) \star (\Delta_+ \psi) + \kappa(\Delta_+^{-1}\phi) \star (\partial_0 \psi) + i(\Delta_+^{-1}\partial_i \phi) \star (\partial_i \psi)$$

Action, EOM, and fields

After some computations, we obtain the following equations of motion

$$(\partial_\mu \partial^\mu - m^2)\phi = 0$$

The field satisfies the Klein-Gordon equations.
Any complex scalar field satisfying these eom can be written as

$$\phi(x) = \int \frac{d^3p}{\sqrt{2\omega_p}} \xi(p) a_{\mathbf{p}}\, e^{-i(\omega_p t - \mathbf{p}\mathbf{x})} + \int \frac{d^3p^*}{\sqrt{2|\omega_p^*|}} \xi(p) b^\dagger_{\mathbf{p}^*}\, e^{i(S(\omega_p^*)t - S(\mathbf{p}^*)\mathbf{x})}$$

Short summary up to now: We have **fields**, they satisfy the **KG** equations, and we know how to **integrate by parts**.

What do we need now: How do the fields (and the **creation/annihilation** operators) behave under C, P, T? What is the commutator between **creation/annihilation** operators? In other words, what are the properties of $a, a^\dagger, b, b^\dagger$?

Properties of the fields under C, P, T

Properties of the fields under C, P, T

How do κ-deformed fields transform? P and T can consistently be defined as acting like in the undeformed case (they leave $[\hat{x}^0, \hat{x}^i] = \frac{i}{\kappa}\hat{x}^i$ invariant)

$$\mathcal{T}\phi(t,\mathbf{x})\mathcal{T}^{-1} = \phi(-t,\mathbf{x}) \quad \Longrightarrow \quad \mathcal{T}a_{\mathbf{p}}\mathcal{T}^{-1} = a_{-\mathbf{p}}$$
$$\mathcal{P}\phi(t,\mathbf{x})\mathcal{P}^{-1} = \phi(t,-\mathbf{x}) \quad \Longrightarrow \quad \mathcal{P}a_{\mathbf{p}}\mathcal{P}^{-1} = a_{-\mathbf{p}}$$

Because of the presence of the antipode $S(.)$ in the fields and because of the form of the action (sum of the two orderings), also C can be shown to behave like in the undeformed case (in its action on $a, a^\dagger, b, b^\dagger$) when acting on fields.

$$\mathcal{C}\phi(t,\mathbf{x})\mathcal{C}^{-1} = \phi(t,\mathbf{x})^\dagger \quad \Longrightarrow \quad \boxed{\mathcal{C}a_{\mathbf{p}}\mathcal{C}^{-1} = b_{\mathbf{p}^*}}$$

The action is manifestly invariant under C, P, T *and* under (deformed) Lorentz transformations.

Symplectic form and $[a, a^\dagger]$, $[b, b^\dagger]$

Symplectic form and $[a, a^\dagger]$, $[b, b^\dagger]$

To proceed we need the symplectic form. What is it? It is a 2-*form* Ω in phase space which is *closed* (i.e. $d\Omega = 0$).

Why do we need it? Because, given a (non-degenerate) symplectic form $\Omega = \omega_{ij} dq^i \wedge dp^j$ then one can define

$$\{A, B\} = \omega^{ij} \frac{\partial A}{\partial Q^i} \frac{\partial B}{\partial Q^j}.$$

$(Q = (p_1, \dots, q_1, \dots))$. In general, the inverse of ω_{ij} can be used to define Poisson brackets $\rightarrow$ commutator.

How to get it? From the surface term coming from the variation of the action (no details here)!

How can one **concretely** obtain Poisson brackets from the symplectic form? Example: two spatial dimensions. We have $Q = (p^1, p^2, x^1, x^2)$

$$\Omega = \frac{1}{2}\omega_{ab} dQ^a \wedge dQ^b = f\, dp^1 \wedge dx^1 + g\, dp^2 \wedge dx^2$$

where f, g are chosen such that $d\Omega = 0$. Therefore we have

$$\omega = \begin{pmatrix} 0 & 0 & f & 0 \\ 0 & 0 & 0 & g \\ -f & 0 & 0 & 0 \\ 0 & -g & 0 & 0 \end{pmatrix} \Longrightarrow \omega^{-1} = \begin{pmatrix} 0 & 0 & -1/f & 0 \\ 0 & 0 & 0 & -1/g \\ 1/f & 0 & 0 & 0 \\ 0 & 1/g & 0 & 0 \end{pmatrix}.$$

Symplectic form and $[a, a^\dagger]$, $[b, b^\dagger]$

Recall that we have

$$\{A, B\} = (\omega^{-1})^{ab} \frac{\partial A}{\partial Q^a} \frac{\partial B}{\partial Q^b}.$$

One can immediately see from the above relations that

$$\{x^1, p^1\} = \frac{1}{f} \qquad \{x^2, p^2\} = \frac{1}{g}$$

Symplectic form and $[a, a^\dagger]$, $[b, b^\dagger]$

In our case, we can additionally use the explicit definition of the fields which appeared previously. The result is very similar to the undeformed case:

$$\Omega = i \int d^3p \, \alpha \, (\delta a_{\mathbf{p}} \wedge \delta a^\dagger_{\mathbf{p}} - \delta b^\dagger_{\mathbf{p}^*} \wedge \delta b_{\mathbf{p}^*}).$$

and therefore

$$\left[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}\right] = \frac{1}{\alpha}\delta(\mathbf{p} - \mathbf{q})$$
$$\left[b_{\mathbf{p}^*}, b^\dagger_{\mathbf{q}^*}\right] = \frac{1}{\alpha}\delta(\mathbf{p} - \mathbf{q})$$

Short summary up to now: We have **fields**, they satisfy the **KG** equations, and we know how to **integrate by parts**. We now also know that the **creation/annihilation** operators work reasonably well.

What do we need now: What are the **charges** in our model?

Charges and how to compute them

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We need the charges. How to get them?

- Using the Noether theorem. However, difficult computations (recall integration by parts), so only the translational charges are easily obtainable in this way;

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- Using the Noether theorem. However, difficult computations (recall integration by parts), so only the translational charges are easily obtainable in this way;

- More pragmatic approach: use the canonical formalism (Noether theorem) to compute translational charges, then switch to covariant phase space formalism for the others. Keep in mind, a kind of "matching" is necessary!

Translation charges: direct approach

After some (very long an tedious) computations one finally gets to the following translation charges.

$$\mathcal{P}_0 = \int d^3p\alpha(p) \left\{ -a^\dagger_{\mathbf{p}}\, a_{\mathbf{p}}\, S(\omega_p) + b^\dagger_{\mathbf{p}^*}\, b_{\mathbf{p}^*}\, \omega_p \right\}$$

$$\xrightarrow{\kappa\to\infty} \int d^3p\omega_p \left\{ a^\dagger_{\mathbf{p}}\, a_{\mathbf{p}}\ + b^\dagger_{\mathbf{p}^*}\, b_{\mathbf{p}^*}\ \right\}$$

$$\mathcal{P}_i = \int d^3p\alpha(p) \left\{ -a^\dagger_{\mathbf{p}}\, a_{\mathbf{p}}\, S(\mathbf{p})_i + b^\dagger_{\mathbf{p}^*}\, b_{\mathbf{p}^*}\, \mathbf{p}_i \right\}$$

$$\mathcal{P}_4 = \int d^3p\, (p_4 - \kappa)\, \alpha(p) \left\{ -a^\dagger_{\mathbf{p}}\, a_{\mathbf{p}}\ + b^\dagger_{\mathbf{p}^*}\, b_{\mathbf{p}^*}\ \right\}.$$

Notice that in the limit $\kappa \to \infty$ one gets the canonical $\mathcal{P}_0, \mathcal{P}_i$. First hint: particles and antiparticles behave differently (but same mass!).

Covariant phase space formalism for charges

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Assuming that the charges come from a symmetry described by some continuous vector field ξ in spacetime, then

$$-\delta_\xi \lrcorner\, \Omega \stackrel{!}{=} \delta Q_\xi$$

where δ is the exterior derivative in phase space, Q_ξ is the charge associated to the vector ξ. $\delta_\xi A$ measures the infinitesimal variation of the object A in phase space due to the symmetry of the action along ξ in spacetime.

Covariant phase space formalism for charges

$$-\delta_\xi \lrcorner \Omega \stackrel{!}{=} \delta Q_\xi$$

Example: Gauge symmetry in electrodynamics:

$$A_\mu \mapsto A_\mu + \epsilon \partial_\mu \Lambda \qquad \leftrightarrow \qquad A_\mu \mapsto A_\mu + \delta_\xi A_\mu$$

$$F_{\mu\nu} \mapsto F_{\mu\nu}$$

A symmetry of the physical system (i.e. of $F_{\mu\nu}$) is not necessarily a symmetry of the field (i.e. A_μ). In this case, ξ is the direction along the gauge orbits in phase space.

Undeformed case: time translations

Example: Translation charge in undeformed context.

$$\Omega^U = i \int d^3p \, (\delta a_{\mathbf{p}} \wedge \delta a^\dagger_{\mathbf{p}} - \delta b^\dagger_{\mathbf{p}^*} \wedge \delta b_{\mathbf{p}^*})$$

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We know that after a time translation ($\xi = \partial_0$) we have

$$a_{\mathbf{p}} \mapsto e^{i\epsilon\omega_p} a_{\mathbf{p}} = a_{\mathbf{p}} + i\epsilon\omega_p a_{\mathbf{p}}$$

and therefore ($\delta_\xi = \delta_{\partial_0}$)

$$\delta_{\partial_0} a_{\mathbf{p}} = i\epsilon\omega_p a_{\mathbf{p}} \quad \Leftrightarrow \quad \delta_{\partial_0} a^\dagger_{\mathbf{p}} = -i\epsilon\omega_p a^\dagger_{\mathbf{p}}$$

Undeformed case: time translations

$$-\delta_{\partial_0} \lrcorner \Omega^U = -i \int d^3p \, (\delta_{\partial_0} a_{\mathbf{p}} \delta a^\dagger_{\mathbf{p}} - \delta a_{\mathbf{p}} \delta_{\partial_0} a^\dagger_{\mathbf{p}}$$

$$- \delta_{\partial_0} b^\dagger_{\mathbf{p}^*} \delta b_{\mathbf{p}^*} + \delta b^\dagger_{\mathbf{p}^*} \delta_{\partial_0} b_{\mathbf{p}^*})$$

$$= -i \int d^3p \, [i\epsilon\omega_p a_{\mathbf{p}} \delta a^\dagger_{\mathbf{p}} - \delta a_{\mathbf{p}} (-i\epsilon\omega_p a^\dagger_{\mathbf{p}})$$

$$- (-i\epsilon\omega_p b^\dagger_{\mathbf{p}^*}) \delta b_{\mathbf{p}^*} + \delta b^\dagger_{\mathbf{p}^*} i\epsilon\omega_p b_{\mathbf{p}^*}]$$

$$= \epsilon\delta \int d^3p \, \omega_p (a_{\mathbf{p}} a^\dagger_{\mathbf{p}} + b^\dagger_{\mathbf{p}^*} b_{\mathbf{p}^*})$$

$$= \epsilon \mathcal{P}_0$$

Undeformed case: boosts

Example: Boost charges in undeformed context.

$$\Omega^U = i\int d^3p\,(\delta a_{\mathbf{p}} \wedge \delta a^\dagger_{\mathbf{p}} - \delta b^\dagger_{\mathbf{p}^*} \wedge \delta b_{\mathbf{p}^*})$$

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$$\delta^B a_{\mathbf{p}} = i\omega_{\mathbf{p}} \, \lambda^j \frac{\partial a_{\mathbf{p}}}{\partial \mathbf{p}^j} + i \, a_{\mathbf{p}} \lambda^j \frac{\mathbf{p}_j}{2\omega_{\mathbf{p}}},$$

$$\delta^B a^\dagger_{\mathbf{p}} = i\omega_{\mathbf{p}} \, \lambda^j \frac{\partial a^\dagger_{\mathbf{p}}}{\partial \mathbf{p}^j} + i \, a^\dagger_{\mathbf{p}} \lambda^j \frac{\mathbf{p}_j}{2\omega_{\mathbf{p}}}$$

Undeformed case: boosts

$$
\begin{aligned}
-\delta^B \lrcorner\, \Omega^{\kappa\to\infty} &= i\int d^3p\, (\delta^B a^\dagger_{\mathbf{p}} \delta a_{\mathbf{p}} - \delta a^\dagger_{\mathbf{p}} \delta^B a_{\mathbf{p}} - \delta^B b_{\mathbf{p}} \delta b^\dagger_{\mathbf{p}} + \delta b_{\mathbf{p}} \delta^B b^\dagger_{\mathbf{p}}) \\
&= \lambda^i \int d^3p\, \omega_{\mathbf{p}} \left(\frac{\partial a_{\mathbf{p}}}{\partial \mathbf{p}^i} \delta a^\dagger_{\mathbf{p}} - \delta a_{\mathbf{p}} \frac{\partial a^\dagger_{\mathbf{p}}}{\partial \mathbf{p}^i} - \frac{\partial b^\dagger_{\mathbf{p}}}{\partial \mathbf{p}^i} \delta b_{\mathbf{p}} + \delta b^\dagger_{\mathbf{p}} \frac{\partial b_{\mathbf{p}}}{\partial \mathbf{p}^i} \right) \\
&+ \lambda^i \int d^3p\, \frac{\mathbf{p}_i}{2\omega_{\mathbf{p}}} \left(a_{\mathbf{p}} \delta a^\dagger_{\mathbf{p}} - \delta a_{\mathbf{p}} a^\dagger_{\mathbf{p}} + \delta b^\dagger_{\mathbf{p}} b_{\mathbf{p}} - b^\dagger_{\mathbf{p}} \delta b_{\mathbf{p}} \right)
\end{aligned}
$$

Apparent issue! Recall that $-\delta_\xi \lrcorner\, \Omega \stackrel{!}{=} \delta Q_\xi$

Undeformed case: boosts

Solution:

$$\frac{\partial a_{\mathbf{p}}}{\partial \mathbf{p}^i}\delta a^\dagger_{\mathbf{p}} - \delta a_{\mathbf{p}}\frac{\partial a^\dagger_{\mathbf{p}}}{\partial \mathbf{p}^i} = \frac{1}{2}\delta\left(\frac{\partial a_{\mathbf{p}}}{\partial \mathbf{p}^i}a^\dagger_{\mathbf{p}} - a_{\mathbf{p}}\frac{\partial a^\dagger_{\mathbf{p}}}{\partial \mathbf{p}^i}\right) + \frac{1}{2}\frac{\partial}{\partial \mathbf{p}^i}\left(a_{\mathbf{p}}\delta a^\dagger_{\mathbf{p}} - \delta a_{\mathbf{p}} a^\dagger_{\mathbf{p}}\right).$$

The second term on the RHS after integration by parts becomes

$$-\frac{\mathbf{p}_i}{2\omega_{\mathbf{p}}}\left(a_{\mathbf{p}}\delta a^\dagger_{\mathbf{p}} - \delta a_{\mathbf{p}} a^\dagger_{\mathbf{p}}\right)$$

which cancels with the second term in the previous expression.

Undeformed case: boosts

The final boost charge is given by

$$\mathcal{N}_i^{\kappa\to\infty} = \frac{1}{2}\int d^3p\,\omega_{\mathbf{p}}\left(\frac{\partial a_{\mathbf{p}}}{\partial \mathbf{p}^i}a_{\mathbf{p}}^{\dagger} - a_{\mathbf{p}}\frac{\partial a_{\mathbf{p}}^{\dagger}}{\partial \mathbf{p}^i} - \frac{\partial b_{\mathbf{p}}^{\dagger}}{\partial \mathbf{p}^i}b_{\mathbf{p}} + b_{\mathbf{p}}^{\dagger}\frac{\partial b_{\mathbf{p}}}{\partial \mathbf{p}^i}\right).$$

in accordance with the literature.

Deformed case: translations

We now start from the following transformations describing time translations.

$$\delta^T a_{\mathbf{p}} = i\epsilon^\mu p_\mu a_{\mathbf{p}}, \quad \delta^T a^\dagger_{\mathbf{p}} = i\epsilon^\mu S(p)_\mu a^\dagger_{\mathbf{p}},$$
$$\delta^T b^\dagger_{\mathbf{p}} = i\epsilon^\mu S(p)_\mu b^\dagger_{\mathbf{p}}, \quad \delta^T b_{\mathbf{p}} = i\epsilon^\mu p_\mu b_{\mathbf{p}}.$$

Notice the antipode! Therefore, a naive application of the previous procedure would not give a consistent result (no quantity Q_ξ such that $-\delta_\xi \lrcorner \Omega \overset{!}{=} \delta Q_\xi$). ‘Matching’ with the direct computation needed! We will need to introduce the antipode in the contraction of a vector field with a 2-form.

Deformed case: translations

We postulate the following rule

$$\delta_\xi \lrcorner (\delta a^\dagger_{\mathbf{p}} \wedge \delta a_{\mathbf{p}}) = (\delta_\xi a^\dagger_{\mathbf{p}})\delta a_{\mathbf{p}} + \delta a^\dagger_{\mathbf{p}}[S(\delta_\xi) a_{\mathbf{p}}]$$

which solves the issue!

$$\begin{aligned} & - \delta^T \lrcorner \Omega \\ & = i \int d^3p\, \alpha\, (\delta^T a^\dagger_{\mathbf{p}} \delta a_{\mathbf{p}} + \delta a^\dagger_{\mathbf{p}} S(\delta^T) a_{\mathbf{p}} - \delta^T b_{\mathbf{p}} \delta b^\dagger_{\mathbf{p}} - \delta b_{\mathbf{p}} S(\delta^T) b^\dagger_{\mathbf{p}}) \\ & = -\epsilon^\mu \delta \left(\int d^3p\, \alpha\, [S(p)_\mu a^\dagger_{\mathbf{p}} \delta a_{\mathbf{p}} - p_\mu b^\dagger_{\mathbf{p}} \delta b_{\mathbf{p}} \right) \end{aligned}$$

$$\mathcal{P}_\mu = \int d^3p\, \alpha\, [-S(p)_\mu a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + p_\mu b^\dagger_{\mathbf{p}} b_{\mathbf{p}}],$$

Deformed case: boosts

In the case of boosts, we need to assume the following transformations for the creation/annihilation transformations

$$\delta^B a_{\mathbf{p}} = -i\lambda^j \, \omega_{\mathbf{p}} \left[\frac{\partial}{\partial \mathbf{p}^j} + \frac{1}{2} \frac{1}{\omega_{\mathbf{p}}} \frac{\partial [\omega_{\mathbf{p}} S(\alpha)]}{\partial \mathbf{p}^j} \right] a_{\mathbf{p}},$$

$$\delta^B a^\dagger_{\mathbf{p}} = -i\lambda^j \, S(\omega_{\mathbf{p}}) \left[\frac{\partial}{\partial S(\mathbf{p})^j} + \frac{1}{2} \frac{1}{S(\omega_{\mathbf{p}})} \frac{\partial [S(\omega_{\mathbf{p}})\alpha]}{\partial S(\mathbf{p})^j} \right] a^\dagger_{\mathbf{p}},$$

$$\delta^B b_{\mathbf{p}} = -i\lambda^j \, \omega_{\mathbf{p}} \left[\frac{\partial}{\partial \mathbf{p}^j} + \frac{1}{2} \frac{1}{\omega_{\mathbf{p}}} \frac{\partial [\omega_{\mathbf{p}} \alpha]}{\partial \mathbf{p}^j} \right] b_{\mathbf{p}},$$

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Charges

For the boosts, we obtain the charge

$$\mathcal{N}_i = -\frac{1}{2}\int d^3p\,\alpha\left\{S(\omega_p)\left[\frac{\partial a^\dagger_{\mathbf{p}}}{\partial S(\mathbf{p})^i}a_{\mathbf{p}} - a^\dagger_{\mathbf{p}}\frac{\partial a_{\mathbf{p}}}{\partial S(\mathbf{p})^i}\right]\right.$$
$$\left. + \omega_p\left[b_{\mathbf{p}}\frac{\partial b^\dagger_{\mathbf{p}}}{\partial \mathbf{p}^i} - \frac{\partial b_{\mathbf{p}}}{\partial \mathbf{p}^i}b^\dagger_{\mathbf{p}}\right]\right\}.$$

Notice: all the deformed charges satisfy the undeformed Poincaré algebra (checked by direct tedious computations).

However, the transformations of the creation/annihilation operators related to the above boost charge now correspond to a **non-trivial** transformation of the field!

If we translate the creation/annihilation operators transformations in terms of the field we get (at first order in $1/\kappa$)

$$\delta^B \phi(x) = i\lambda_i \, x^i \frac{\partial}{\partial t}\, \phi(x) - i\lambda_i \int \frac{d^3p}{\sqrt{2\omega_{\mathbf{p}}}} \left\{ \frac{\mathbf{p}_i}{\kappa} \left(\frac{m^2}{4\omega_{\mathbf{p}}^2} - \frac{1}{2} \right) a_{\mathbf{p}} e^{-i(\omega_{\mathbf{p}} t - \mathbf{p}\mathbf{x})} + \frac{\mathbf{p}_i}{\kappa} \left(-\frac{m^2}{4\omega_{\mathbf{p}}^2} - 1 \right) b^\dagger_{\mathbf{p}} e^{-i(S(\omega_{\mathbf{p}})t - S(\mathbf{p})\mathbf{x})} \right\},$$

Analogous relation for $\phi^\dagger$, which means that particles and antiparticles receive an additional shift under boost.

Relation between CPT and boost charges

Although the undeformed Poincaré algebra is satisfied, non trivial relations arise! One can show that

$$[\mathcal{N}_i, C] \neq 0 \qquad [\mathcal{N}_i, C] \xrightarrow{\kappa \to \infty} 0$$

This is due to $p \neq -S(p)$, and translates (for example) into a **difference of decay times** for particles and antiparticles in a boosted frame.

$$\mathcal{P}_{\text{part}}(t) = \frac{\Gamma E}{M} \exp\left(-\Gamma \frac{E}{M} t\right)$$

$$\mathcal{P}_{\text{apart}}(t) = \Gamma\left(\frac{E}{M} - \frac{\mathbf{p}^2}{\kappa M}\right) \exp\left[-\Gamma\left(\frac{E}{M} - \frac{\mathbf{p}^2}{\kappa M}\right) t\right]$$

where $\mathcal{P}$ = decay probability density function, and $\Gamma = 1/\tau$

Relation between CPT and boost charges

The Greenberg's theorem relates CPT invariance with Lorentz invariance of a theory. According to the theorem,

$$\text{Lorentz invariance} \;\Leftrightarrow \text{CPT invariance}$$

In our case, the action is manifestly invariant under deformed Lorentz invariance and CPT transformations. However, CPT symmetry is broken in a more subtle way (see previous slide).

Greenberg's theorem does not work: $p \neq -S(p)$.

Conclusion and future works

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- Immediate generalization: higher spins.

Thank you

One particle states

We now have all the tools to show that indeed particles and antiparticles behave differently.

How to see it?

- Since we have the translation charges (i.e. the operators $\mathcal{P}_\mu$), we can apply them to the a-particle and b-particle states and get their eigenvalues. We will see that they are different;

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How to see it?

- Since we have the translation charges (i.e. the operators $\mathcal{P}_\mu$), we can apply them to the a-particle and b-particle states and get their eigenvalues. We will see that they are different;
- Use $\mathcal{C}$ to link the a-particle to the b-particle state. We will see that C switches a particle with its antiparticle with *different* momentum.

One particle states

Define the vacuum by $a_{\mathbf{p}}|0\rangle = b_{\mathbf{p}*}|0\rangle = 0$. We then define one-particle and one-antiparticle state by

$$a^{\dagger}_{\mathbf{p}}|0\rangle := |\mathbf{p}\rangle_a \qquad b^{\dagger}_{\mathbf{p}*}|0\rangle := |\mathbf{p}\rangle_b$$

Now we want to know $\mathcal{P}_\mu|\mathbf{p}\rangle_a$ and $\mathcal{P}_\mu|\mathbf{p}\rangle_b$.

$$\left[a_{\mathbf{p}}, a^{\dagger}_{\mathbf{q}}\right] = \frac{1}{\alpha}\delta(\mathbf{p}-\mathbf{q})$$

$$\begin{aligned}\mathcal{P}_i|\mathbf{q}\rangle_a &= \int d^3p\alpha\left\{-a^{\dagger}_{\mathbf{p}}\, a_{\mathbf{p}}\, S(\mathbf{p})_i + b^{\dagger}_{\mathbf{p}^*}\, b_{\mathbf{p}^*}\,\mathbf{p}_i\right\} a^{\dagger}_{\mathbf{q}}|0\rangle \\ &= \int d^3p\alpha\left\{-a^{\dagger}_{\mathbf{p}}\,\frac{1}{\alpha}\delta(\mathbf{p}-\mathbf{q})S(\mathbf{p})_i + a^{\dagger}_{\mathbf{p}}\, a^{\dagger}_{\mathbf{q}} a_{\mathbf{p}}\,\mathbf{p}_i\right\}|0\rangle + 0 \\ &= -S(\mathbf{q})_i|\mathbf{q}\rangle_a\end{aligned}$$

One particle states

Doing the same thing for all $\mathcal{P}_\mu$ we have

$$\mathcal{P}_i|\mathbf{p}\rangle_a = -S(\mathbf{p})_i|\mathbf{p}\rangle_a \qquad \mathcal{P}_i|\mathbf{p}\rangle_b = \mathbf{p}_i|\mathbf{p}\rangle_b$$

$$\mathcal{P}_0|\mathbf{p}\rangle_a = -S(\omega_p)|\mathbf{p}\rangle_a \qquad \mathcal{P}_0|\mathbf{p}\rangle_b = \omega_p|\mathbf{p}\rangle_b$$

Notice: $\mathbf{p} \neq -S(\mathbf{p})$ and $\omega_p \neq -S(\omega_p)$, but $p_\mu p^\mu = m^2$ and $S(p)_\mu S(p)^\mu = m^2$, so a-particle and b-particle have same mass.

One particle states

We can use $\mathcal{C}$ to relate $|\mathbf{p}\rangle_a$ and $|\mathbf{p}\rangle_b$

$$\mathcal{C}|\mathbf{p}\rangle_b = \mathcal{C}b^\dagger_{\mathbf{p}*}\mathcal{C}^{-1}\mathcal{C}|0\rangle = a^\dagger_{\mathbf{p}}|0\rangle = |\mathbf{p}\rangle_a$$

Very easy steps due to the simplicity of the C transformation of our deformed field!

Therefore C (and CPT) transforms a particle into an anti-particle with different momentum, and vice versa.