Cosmological perturbations beyond the linear order

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1. The metric.
2. Specific solution to the second-order perturbation theory.
3. The gauge.

The metric.

The line element:

$$ds^2 = -dt^2 + a(t)^2\, c_{ij}(x^\mu, \lambda, k)\, dx^i\, dx^j\,, \qquad (x^\mu) = (t, x, y, z)$$

The metric.

The line element:

$$ds^2 = -dt^2 + a(t)^2\, c_{ij}(x^\mu, \lambda, k)\, dx^i\, dx^j\,, \qquad (x^\mu) = (t, x, y, z)$$

The meaning of the parameters λ and k:

$$\lim_{\lambda\to 0} c_{ij} = \frac{\delta_{ij}}{\left(1 + \frac{1}{4}k\,(x^2 + y^2 + z^2)\right)^2} \equiv \widetilde{c}_{ij}\,.$$

$$T^\mu{}_\nu = \rho\, U^\mu\, U_\nu \qquad (U^\mu) = (1, 0, 0, 0)$$

Two dimensional Taylor expansion.

Two dimensional truncated Taylor series:

$$\mathcal{T}_N[f(\lambda,k)] = \sum_{n=0}^{N} \frac{1}{n!} \sum_{m=0}^{n} \binom{n}{m} \frac{\partial^n f}{\partial \lambda^{n-m} \partial k^m}\bigg|_{\substack{\lambda=0\\k=0}} \lambda^{n-m}\, k^m$$

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Assumption: metric functions can be considered as a power series:

$$c_{ij}(\lambda, k, x^\mu) = \sum_{l=0}^{N} \sum_{m=0}^{l} c_{ij}^{(l-m,m)}(x^\mu)\, \lambda^{l-m}\, k^m\,, \qquad \lim_{\lambda\to 0} c_{ij} = \mathcal{T}_N[\widetilde{c}_{ij}]$$

Two dimensional Taylor expansion.

Two dimensional truncated Taylor series:

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Approximate Einstein equations:

$$\mathcal{T}_N[G^\mu{}_\nu(\lambda,k)] = 8\pi\, \rho\, U^\mu U_\nu - \Lambda\, \delta^\mu_\nu + 8\pi \Delta T^\mu{}_\nu$$

The first-order perturbations.

The form of the metric functions.

$$c_{ij}^{(1,0)} = \mathscr{A}_{10}(t)\frac{\partial^2}{\partial x^i \partial x^j} A_{10}(x,y,z) + B_{10}(x,y,z)\,\delta_{ij}$$

The first-order perturbations.

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The dust solution.

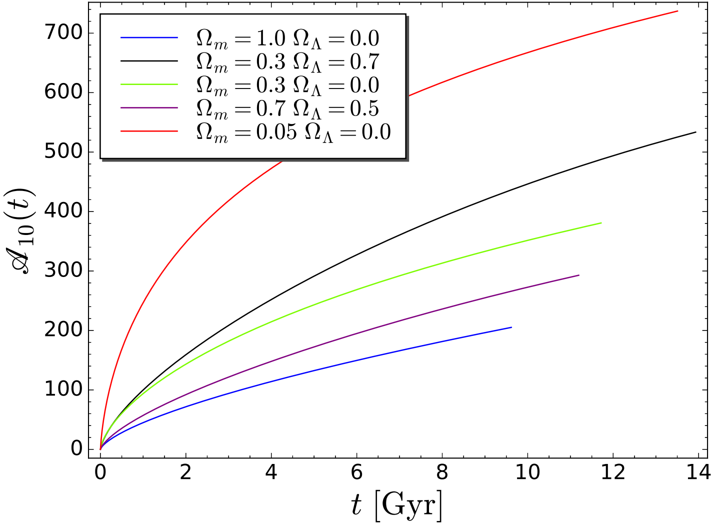
$$a^2\,\ddot{\mathscr{A}}_{10} + 3\,a\,\dot{a}\,\dot{\mathscr{A}}_{10} = \alpha$$

$$B_{10}(x, y, z) = \alpha\,A_{10}(x, y, z)$$

The first-order density.

$$\rho^{(1,0)} = \frac{a\,\dot{a}\,\dot{\mathscr{A}}_{10} - \alpha}{8\pi\,a^2}\left(\frac{\partial^2}{\partial x^2}A_{10} + \frac{\partial^2}{\partial y^2}A_{10} + \frac{\partial^2}{\partial z^2}A_{10}\right)$$

The first-order metric .

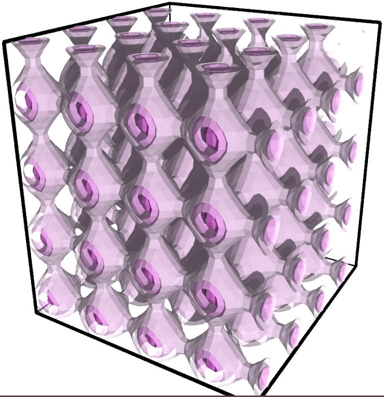


The second-order perturbations.

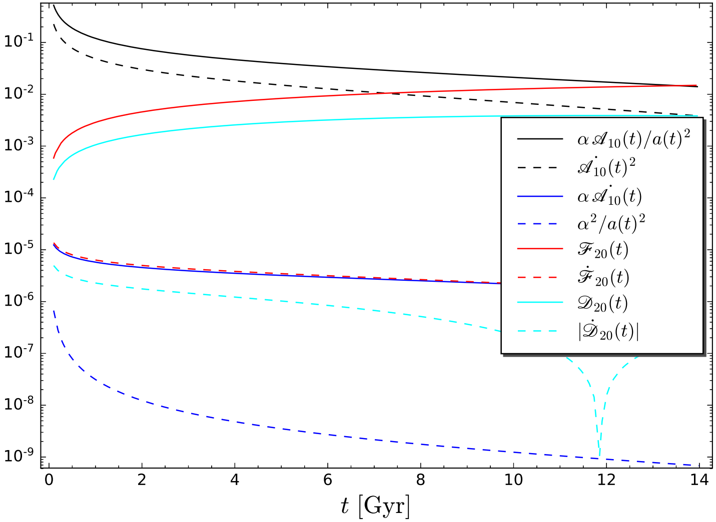
Simplification:

$$A_{10}(x, y, z) = C_{10}(x) + C_{10}(y) + C_{10}(z)$$

$$B_{10}(x, y, z) = D_{10}(x) + D_{10}(y) + D_{10}(z)$$



The second-order energy-momentum tensor elements.



The second-order approximate solution.

The metric functions:

$$c_{ij}^{(2,0)} = \left(\mathscr{A}_{20}(t)\,C_{20}(x^i) + \mathscr{D}_{20}(t)(D_{20}(x) + D_{20}(y) + D_{20}(z))\right)\delta_{ij} +$$
$$+\mathscr{F}_{20}(t)\,F_{20}(x^i, x^j)\,(1 - \delta_{ij})$$

The second-order approximate solution.

The metric functions:

$$c_{ij}^{(2,0)} = \left(\mathscr{A}_{20}(t) C_{20}(x^i) + \mathscr{D}_{20}(t)(D_{20}(x) + D_{20}(y) + D_{20}(z))\right) \delta_{ij} +$$
$$+\mathscr{F}_{20}(t)\, F_{20}(x^i, x^j)\,(1 - \delta_{ij})$$

The solution:

$$\mathscr{F}_{20}(t) = \frac{1}{4} a^2\, \dot{\mathscr{A}}_{10}^2 + \alpha\, \mathscr{A}_{10}$$

$$\frac{\partial^2}{\partial w \partial v} F_{20}(v, w) = -\frac{\mathrm{d}^2}{\mathrm{d}v^2} C_{10}(v)\, \frac{\mathrm{d}^2}{\mathrm{d}w^2} C_{10}(w)\,, \quad v, w = x, y, z$$

$$\mathscr{D}_{20}(t) = a^2\, \dot{\mathscr{A}}_{10}^2$$

$$\frac{\mathrm{d}^2}{\mathrm{d}w^2} D_{20}(w) = -\frac{1}{2} \left(\frac{\mathrm{d}^2}{\mathrm{d}w^2} C_{10}(w)\right)^2$$

$$a^2\, \ddot{\mathscr{A}}_{20} + 3\, a\, \dot{a}\, \dot{\mathscr{A}}_{20} = \alpha\, \mathscr{A}_{10}$$

$$C_{20}(w) = -\frac{1}{4}\, \frac{\mathrm{d}}{\mathrm{d}w} C_{10}(w)\, \frac{\mathrm{d}^3}{\mathrm{d}w^3} C_{10}(w)$$

The second-order approximate solution.

The metric functions:

$$c_{ij}^{(1,1)} = \left(\mathscr{A}_{11}(t)C_{11}(x^i) + \mathscr{D}_{11}(t)(D_{11}(x) + D_{11}(y) + D_{11}(z))\right)\delta_{ij} +$$

$$+\mathscr{F}_{11}(t)\,F_{11}(x^i, x^j)\,(1 - \delta_{ij})$$

The second-order approximate solution.

The metric functions:

$$c_{ij}^{(1,1)} = \left(\mathscr{A}_{11}(t)\,C_{11}(x^i) + \mathscr{D}_{11}(t)(D_{11}(x) + D_{11}(y) + D_{11}(z))\right)\delta_{ij} +$$

$$+\mathscr{F}_{11}(t)\,F_{11}(x^i, x^j)\,(1-\delta_{ij})$$

The solution:

$$\mathscr{F}_{11}(t) = \mathscr{A}_{10}$$

$$F_{11}(v,w) = \frac{1}{2}\left(v\,\frac{\mathrm{d}}{\mathrm{d}w}C_{10}(w) + w\,\frac{\mathrm{d}}{\mathrm{d}v}C_{10}(v)\right)\,, \quad v,w = x,y,z$$

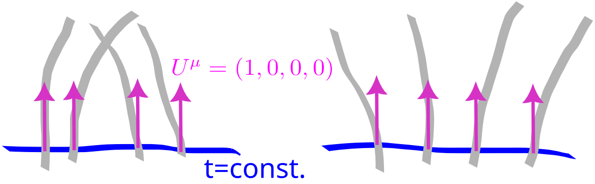
$$\mathscr{D}_{11}(t) = \mathscr{A}_{10}$$

$$D_{11}(w) = C_{10}(w) - \frac{1}{2}w\,\frac{\mathrm{d}}{\mathrm{d}w}C_{10}(w)$$

$$a^2\,\ddot{\mathscr{A}}_{11} + 3\,a\,\dot{a}\,\dot{\mathscr{A}}_{11} = \mathscr{A}_{10}$$

$$C_{11}(w) = \frac{\mathrm{d}^2}{\mathrm{d}w^2}C_{10}(w)$$

The synchronous comoving gauge.



The metric:

$$ds^2 = -dt^2 + a(t)^2\, c_{ij}(x^\mu, \lambda, k)\, dx^i\, dx^j\,, \qquad (x^\mu) = (t, x, y, z)$$

Possible coordinate transormations:

$$x^i \mapsto x^{i'} = x^i + \lambda\, [\chi^{(1,0)}]^i(x^j) + k\; [\chi^{(0,1)}]^i(x^j) + \ldots$$

$$\rho(t, \chi^i(x^j)) \approx \rho(t, x^j) + \lambda\, \frac{\partial \rho}{\partial x^m}\, [\chi^{(1,0)}]^m(x^j) + k\, \frac{\partial \rho}{\partial x^m}\, [\chi^{(0,1)}]^m(x^j)\,.$$

Thank you for your attention.