Gauge-fixing and spacetime reconstruction in the Hamiltonian theory of cosmological perturbations

Boldrin Alice NCBJ

POTOR8 23/09/22

Based on: A.B., P. Małkiewicz, *Gauge-fixing and spacetime reconstruction in the Hamiltonian theory of cosmological perturbations*, arXiv:2206.06926

Content

Motivations

- **Perturbation theory** → Cosmological perturbation theory aims at examining the properties of primordial density fluctuations necessary to explain today's universe's structure and to clarify the origin and the evolutionary behavior of such density fluctuations
- Hamiltonian formalism
- Gauge theory

Motivations

- Perturbation theory
- **Hamiltonian formalism** →
- Gauge theory

The study of early stage of our Universe is more suited to a quantum description. The first step is to obtain the the Hamiltonian formalism of GR.

Motivations

- Perturbation theory
- Hamiltonian formalism
- **Gauge theory** →

The understanding of the gauge theory in GR or any other physical theory is essential to be able to distinguish between physical and unphysical degrees of freedom.

[R. Wald, general relativity]

ADM formalism

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

Spatial metric of a hypersurface

[R. Wald, general relativity]

ADM formalism

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

Spatial metric of a hypersurface

Einstein-Hilbert action

$$S_{gravity} = \frac{1}{2\kappa}\int R\sqrt{-g}\,d^4x$$

[R. Wald, general relativity]

ADM formalism

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

Spatial metric of a hypersurface

Einstein-Hilbert action

$$S_{gravity} = \frac{1}{2\kappa}\int R\sqrt{-g}\,d^4x$$

$$\delta S_{tot} = \frac{1}{16}\int \left(\pi^{ij}\frac{\delta g_{ij}}{\delta t} - N\mathcal{H}(g_{ij},\pi^{ij}) - N^i\mathcal{H}_i(g_{ij},\pi^{ij})\right)d^4x + \int \delta\mathcal{L}_{field}d^4x$$

[R. Wald, general relativity]

ADM formalism

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

Spatial metric of a hypersurface

Einstein-Hilbert action

$$S_{gravity} = \frac{1}{2\kappa}\int R\sqrt{-g}\,d^4x$$

$$\delta S_{tot} = \frac{1}{16}\int \left(\pi^{ij}\frac{\delta g_{ij}}{\delta t} - N\mathcal{H}(g_{ij}, \pi^{ij}) - N^i\mathcal{H}_i(g_{ij}, \pi^{ij})\right) d^4x + \int \delta\mathcal{L}_{field} d^4x$$

Lagrange multipliers

[R. Wald, general relativity]

ADM formalism

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

Spatial metric of a hypersurface

Einstein-Hilbert action $$S_{gravity} = \frac{1}{2\kappa}\int R\sqrt{-g}\,d^4x$$

$$\delta S_{tot} = \frac{1}{16}\int \left(\pi^{ij}\frac{\delta g_{ij}}{\delta t} - N\mathcal{H}(g_{ij},\pi^{ij}) - N^i\mathcal{H}_i(g_{ij},\pi^{ij})\right)d^4x + \int \delta\mathcal{L}_{field}d^4x$$

Constraints

[P. Dirac, Lectures in QM]

Types of constraints

First class constraints: quantities that have zero Poisson bracket with all the primary constraints

$$\{R, \phi_j\} \approx 0, \forall j = i, ..., m$$

[P. Dirac, Lectures in QM]

Types of constraints

Constraints that hold
Regardless of the eom.
Secondary constraints need
the Lagrange equations.

First class constraints: quantities that have zero Poisson bracket with all the primary constraints

$$\{R, \phi_j\} \approx 0, \forall j = i, ..., m$$

[P. Dirac, Lectures in QM]

Types of constraints

Constraints that hold Regardless of the eom. Secondary constraints need the Lagrange equations.

First class constraints: quantities that have zero Poisson bracket with all the primary constraints

$$\{R, \phi_j\} \approx 0, \forall j = i, ..., m$$

Second class constraints: $\{R, \phi_j\} \not\approx 0, \forall j = 1, ..., m$

[K.A. Malik. D. Wands, Cosmological perturbation]

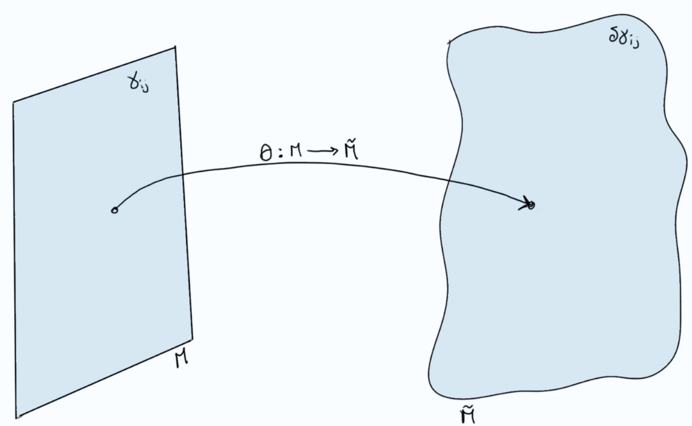
Gauge theory

The introduction of a homogeneous background model leads to an ambiguity in the choice of coordinates.

In general relativity there is, a priori, no preferred choice of coordinates.

“Choosing a set of coordinates in the inhomogeneous universe which will then be described by a homogeneous background plus perturbations amounts to assigning a mapping between spacetime points in the inhomogeneous universe and the homogeneous background model. “

The freedom in this choice is the gauge freedom, or gauge problem, in general relativistic perturbation theory.



[K. Nakamura, DOI: 10.1093/ptep/ptt006]

Gauge theory

Gauge invariance of first type

General relativity is a generally covariant theory:
it does not depend on the reference system

Gauge invariance of second type

Relation between physical manifold and the background spacetime:
Freedom in the map that relates the two manifolds

Only for perturbative theories

Cosmological perturbation theory

$$\mathbf{H} = N\mathcal{H}_0^{(0)} + \int_{\mathbf{T}^3} \left(N\mathcal{H}_0^{(2)} + \delta N \delta\mathcal{H}_0 + \delta N^i \delta\mathcal{H}_i \right) d^3x$$

Metric and conjugate momenta perturbations

$$\delta q_{ij} = q_{ij} - \bar{q}_{ij}$$
$$\delta \pi^{ij} = \pi^{ij} - \bar{\pi}^{ij}$$

Lapse and shift perturbations

$$N \to N + \delta N$$
$$N^i \to N^i + \delta N^i$$

The total hamiltonian defines a gauge system
If the dynamics is truncated at linear order.

FIRST CLASS CONSTRAINTS

$$\{\delta\mathcal{H}_i, \delta\mathcal{H}_j\} = 0, \quad \{\delta\mathcal{H}_i, \delta\mathcal{H}_0\} = 0$$

Cosmological perturbation theory

$$\mathbf{H} = N\mathcal{H}_0^{(0)} + \int_{\mathbf{T}^3} \left(N\mathcal{H}_0^{(2)} + \delta N \delta\mathcal{H}_0 + \delta N^i \delta\mathcal{H}_i \right) d^3x$$

Metric and conjugate momenta perturbations

$$\delta q_{ij} = q_{ij} - \bar{q}_{ij}$$
$$\delta \pi^{ij} = \pi^{ij} - \bar{\pi}^{ij}$$

Lapse and shift perturbations

$$N \to N + \delta N$$
$$N^i \to N^i + \delta N^i$$

The total hamiltonian defines a gauge system
If the dynamics is truncated at linear order.

FIRST CLASS CONSTRAINTS

Independent from the background spacetime model

$$\{\delta\mathcal{H}_i, \delta\mathcal{H}_j\} = 0, \quad \{\delta\mathcal{H}_i, \delta\mathcal{H}_0\} = 0$$

P. Dirac, *The Theory of Gravitation in Hamiltonian Form*, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, Vol. 246, No. 1246 (Aug. 19, 1958), pp. 333-343 (11 pages)

Cosmological perturbation theory

$$\mathbf{H} = N\mathcal{H}_0^{(0)} + \int_{\mathbf{T}^3} \left(N\mathcal{H}_0^{(2)} + \delta N \delta\mathcal{H}_0 + \delta N^i \delta\mathcal{H}_i \right) d^3x$$

Metric and conjugate momenta perturbations

$$\delta q_{ij} = q_{ij} - \bar{q}_{ij}$$
$$\delta\pi^{ij} = \pi^{ij} - \bar{\pi}^{ij}$$

Lapse and shift perturbations

$$N \to N + \delta N$$
$$N^i \to N^i + \delta N^i$$

The total hamiltonian defines a gauge system
If the dynamics is truncated at linear order.

FIRST CLASS CONSTRAINTS

Independent from the background spacetime model

$$\{\delta\mathcal{H}_i, \delta\mathcal{H}_j\} = 0, \quad \{\delta\mathcal{H}_i, \delta\mathcal{H}_0\} = 0$$

$$\{\mathcal{H}^{(0)} + \mathcal{H}_0^{(2)}, \delta\mathcal{H}_0\} = -ik^j \delta\mathcal{H}_j \approx 0, \quad \{\mathcal{H}^{(0)} + \mathcal{H}_0^{(2)}, \delta\mathcal{H}_i\} = 0$$

Dynamically stable

Gauge-fixing

Procedure to reduce to the physical phase space

It is usually a hard task to identify a set of gauge-invariant functions to parametrize the physical phase space.

To solve this we choose a set of functions to parametrize the gauge-orbits of the constraints.

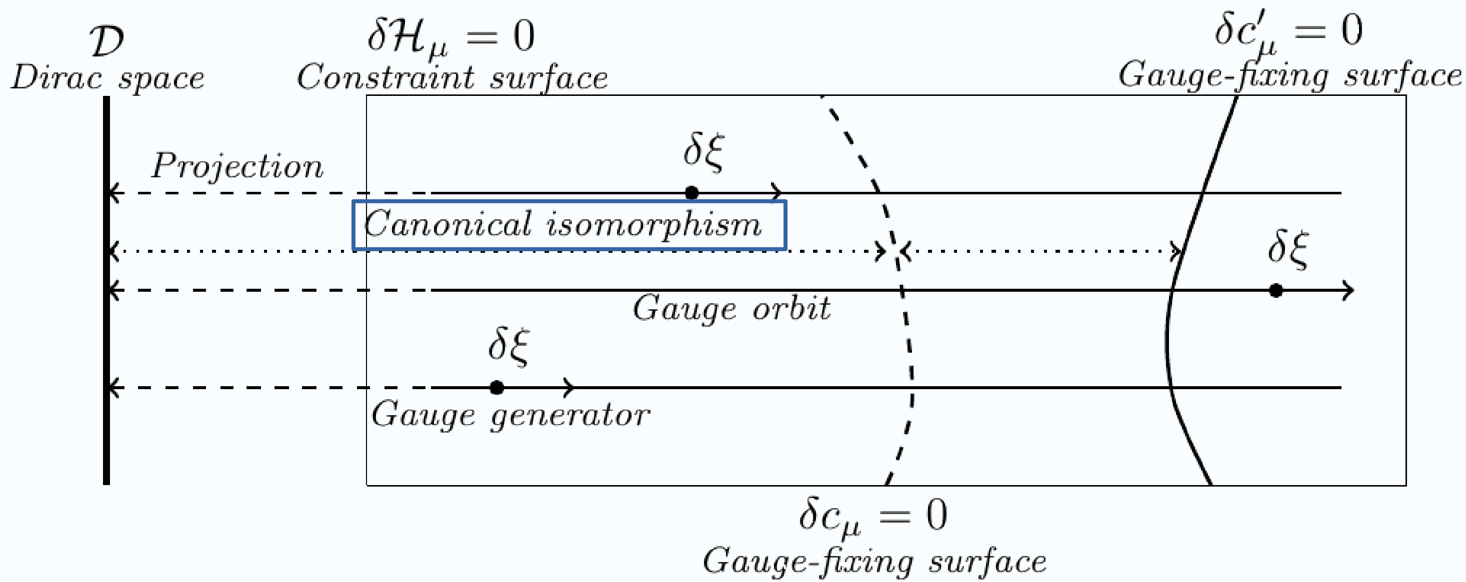
"Points along the same orbit are physically equivalent, so we can simply choose one to represent that specific orbit (physical state of the system)."

[A. Dapor, J. Lewandowski, J. Puchta, "QFT on quantum spacetime: a compatible classical framework"]

This is equivalent to choosing a "slice" in the constraint surface which mirrors the physical phase space.

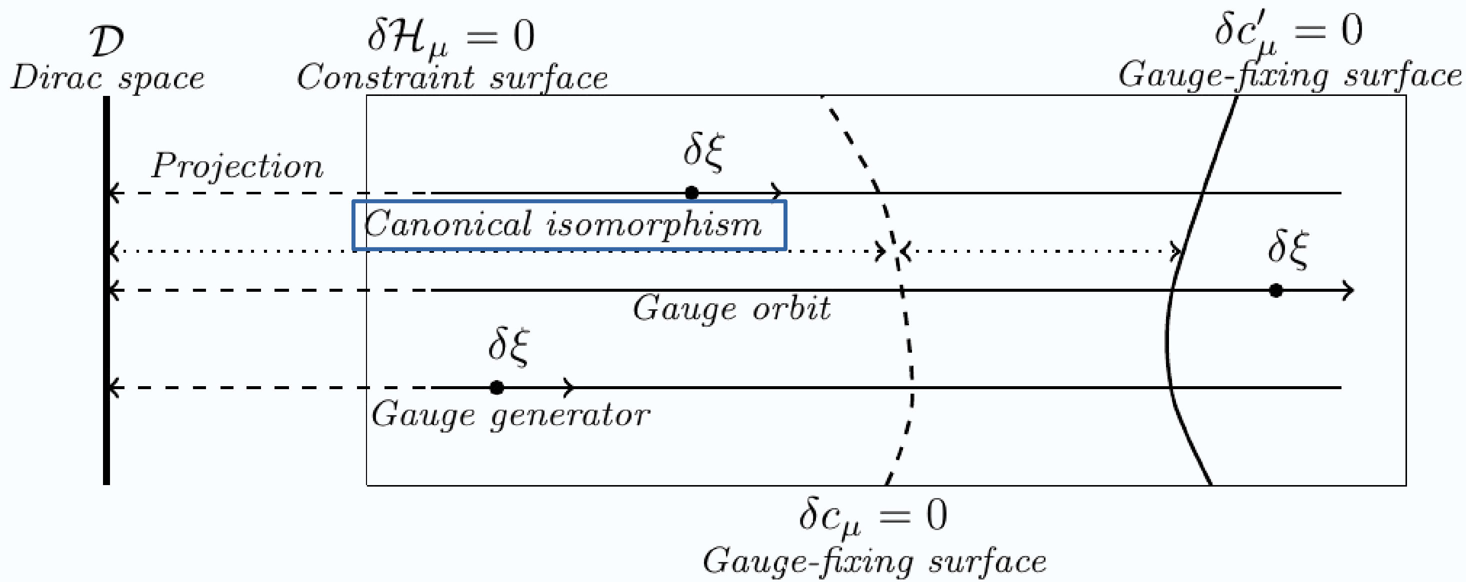
It is parametrized by the remaining variables, which thus represent the physical degrees of freedom of the system.

Dirac Procedure



Dirac Procedure

The **Dirac observables** provide a parametrization of the space of the gauge orbits in the constraint surface whereas the **physical variables** provide a parameterization of a particular gauge-fixing surface



There exists a one-to-one relation between the Dirac observables and the physical variables

$$\delta D_I + \xi_I^\mu \delta c_\mu + \zeta_I^\mu \delta\mathcal{H}_\mu = \delta O_I^{phys}(\delta q^{phys}, \delta\pi_{phys})$$

Space time reconstruction

The gauge-fixing conditions, the constaints and the Dirac observables
determine the geometry of the hypersurface

To reconstruct the full space-time we need the value of the **lapse** and **shif**

$$\{\delta c_\nu, \mathbf{H}\} = 0 \implies \frac{\delta N^\mu}{N} = -\{\delta c_\nu, \delta \mathcal{H}_\mu\}^{-1} \left(\{\delta c_\nu, \mathcal{H}^{(0)}\} + \{\delta c_\nu, \mathcal{H}^{(2)}\} \right)$$

Space time reconstruction

The gauge-fixing conditions, the constaints and the Dirac observables
determine the geometry of the hypersurface

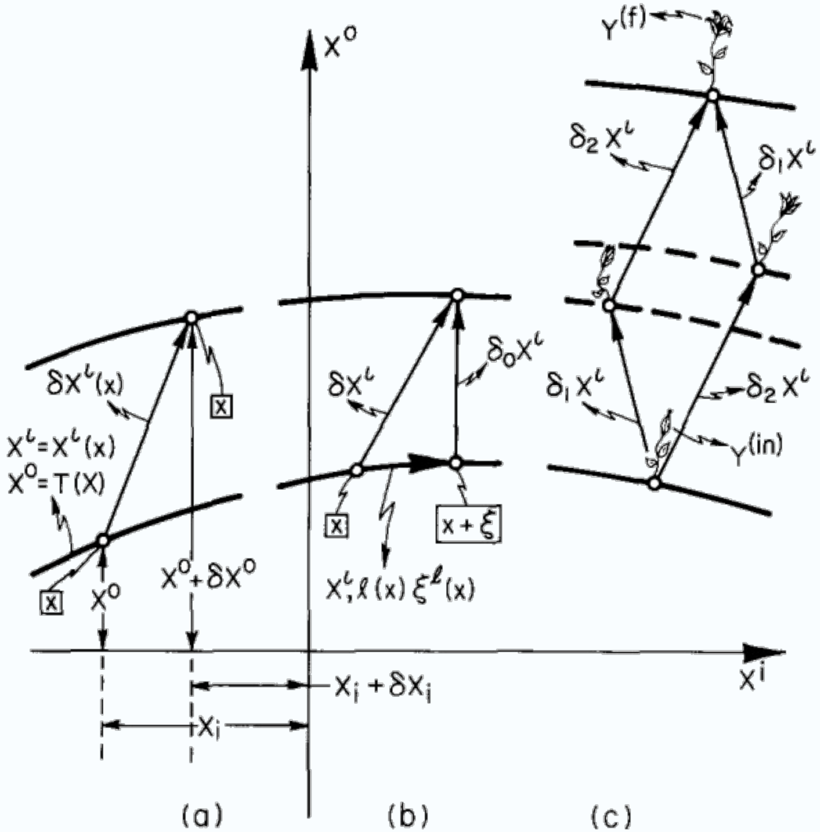
To reconstruct the full space-time we need the value of the **lapse** and **shif**

$$\{\delta c_\nu, \mathbf{H}\} = 0 \implies \frac{\delta N^\mu}{N} = -\{\delta c_\nu, \delta \mathcal{H}_\mu\}^{-1} \left(\{\delta c_\nu, \mathcal{H}^{(0)}\} + \{\delta c_\nu, \mathcal{H}^{(2)}\}\right)$$

$$\mathbf{H} = N\mathcal{H}_0^{(0)} + \int_{\mathbf{T}^3} \left(N\mathcal{H}_0^{(2)} + \delta N \delta\mathcal{H}_0 + \delta N^i \delta\mathcal{H}_i\right) d^3x$$

Can we improve the spacetime reconstruction?

Kuchař decomposition



[K. Kuchar, "A Bubble-Time Canonical Formalism for Geometrodynamics"]

Kuchař decomposition

It is a special parametrization of the kinematical phase space with the constraints chosen as canonical variables.

$$(\delta q_{ij}, \delta\pi^{ij}) \to (\delta\mathcal{H}_\mu, \delta c^\mu, \delta Q_I, \delta P^I)$$

New Hamiltonian $\mathbf{H}_K = \mathbf{H}_{ADM} + \mathbf{K}$

$$\mathbf{H}_K = N \int \left(\underbrace{\mathcal{H}^{(2)}_{phys}(\delta Q_I, \delta P^I)}_{\text{physical part}} + \underbrace{(\lambda_1^{\mu I}\delta Q_I + \lambda^{\mu}_{2I}\delta P^I + \lambda_3^{\mu\nu}\delta\mathcal{H}_\nu + \lambda^{\mu}_{4\nu}\delta c^\nu + \frac{\delta N^\mu}{N})\delta\mathcal{H}_\mu}_{\text{weakly vanishing part}} \right) d^3x,$$

Gauge dependent

Kuchař decomposition

It is a special parametrization of the kinematical phase space with the constraints chosen as canonical variables.

$$(\delta q_{ij}, \delta\pi^{ij}) \to (\delta\mathcal{H}_\mu, \delta c^\mu, \delta Q_I, \delta P^I)$$

New Hamiltonian $\mathbf{H}_K = \mathbf{H}_{ADM} + \mathbf{K}$

$$\mathbf{H}_K = N\int\left(\underbrace{\mathcal{H}^{(2)}_{phys}(\delta Q_I, \delta P^I)}_{\text{physical part}} + \underbrace{(\lambda_1^{\mu I}\delta Q_I + \lambda^\mu_{2I}\delta P^I + \lambda_3^{\mu\nu}\delta\mathcal{H}_\nu + \lambda^\mu_{4\nu}\delta c^\nu + \frac{\delta N^\mu}{N})\delta\mathcal{H}_\mu}_{\text{weakly vanishing part}}\right)d^3x,$$

Depends on the constraints algebra

Gauge trasformation

The Kuchař decomposition provides a class of parametrizations of the kinematical phase space.

$$\mathcal{K} : (\delta\mathcal{H}_\mu, \delta c^\mu, \delta Q_I, \delta P^I) \mapsto (\delta\tilde{\mathcal{H}}_\mu, \delta\tilde{c}^\mu, \delta\tilde{Q}_I, \delta\tilde{P}^I)$$

$\delta\tilde{\mathcal{H}}_\mu = \delta\mathcal{H}_\mu$ the constraint functions are preserved by the map.

$$\{\delta\mathcal{H}_\nu, \delta\tilde{c}^\mu - \delta c^\mu\} = 0$$

Background dependent $\quad \delta\tilde{c}^\mu = \delta c^\mu + \alpha^\mu{}_I \delta P^I + \beta^{\mu I}\delta Q_I + \gamma^{\mu\nu}\delta\mathcal{H}_\nu$

Space-time reconstruction

$$\left.\frac{\delta \tilde{N}^\mu}{N}\right|_{\delta\tilde{c}^\mu=0} - \left.\frac{\delta N^\mu}{N}\right|_{\delta c^\mu=0} \approx \left(\lambda^\mu_{4\nu}\beta^{\nu I} - \dot{\beta}^{\mu I} - \frac{\partial^2 \mathcal{H}^{(2)}_{phys}}{\partial\delta Q_I \partial\delta P^J}\beta^{\mu J} + \frac{\partial^2 \mathcal{H}^{(2)}_{phys}}{\partial\delta Q_I \partial\delta Q_J}\alpha^\mu{}_J\right)\delta Q_I$$
$$- \left(\lambda^\mu_{4\nu}\alpha^\nu{}_I - \dot{\alpha}^\mu{}_I + \frac{\partial^2 \mathcal{H}^{(2)}_{phys}}{\partial\delta P^I \partial\delta Q_J}\alpha^\mu{}_J - \frac{\partial^2 \mathcal{H}^{(2)}_{phys}}{\partial\delta P^I \partial\delta P^J}\beta^{\mu J}\right)\delta P^I.$$

The tranfosmation depends on:

⟶ the physical Hamiltonian $\mathcal{H}^{(2)}_{phys}$

Space-time reconstruction

$$\left.\frac{\delta\tilde{N}^\mu}{N}\right|_{\delta\tilde{c}^\mu=0} - \left.\frac{\delta N^\mu}{N}\right|_{\delta c^\mu=0} \approx \left(\lambda^\mu_{4\nu}\beta^{\nu I} - \dot{\beta}^{\mu I} - \frac{\partial^2\mathcal{H}^{(2)}_{phys}}{\partial\delta Q_I\partial\delta P^J}\beta^{\mu J} + \frac{\partial^2\mathcal{H}^{(2)}_{phys}}{\partial\delta Q_I\partial\delta Q_J}\alpha^\mu{}_J\right)\delta Q_I$$

$$-\left(\lambda^\mu_{4\nu}\alpha^\nu{}_I - \dot{\alpha}^\mu{}_I + \frac{\partial^2\mathcal{H}^{(2)}_{phys}}{\partial\delta P^I\partial\delta Q_J}\alpha^\mu{}_J - \frac{\partial^2\mathcal{H}^{(2)}_{phys}}{\partial\delta P^I\partial\delta P^J}\beta^{\mu J}\right)\delta P^I.$$

The tranfosmation depends on:

the **physical Hamiltonian** $\mathcal{H}^{(2)}_{phys}$

⟶ the **background-dependent parameters** $\alpha_{\mu I},\ \beta^{\mu I}$

Space-time reconstruction

$$\left.\frac{\delta \tilde{N}^\mu}{N}\right|_{\delta\tilde{c}^\mu=0} - \left.\frac{\delta N^\mu}{N}\right|_{\delta c^\mu=0} \approx \left(\lambda^\mu_{4\nu}\beta^{\nu I} - \dot{\beta}^{\mu I} - \frac{\partial^2 \mathcal{H}^{(2)}_{phys}}{\partial\delta Q_I \partial\delta P^J}\beta^{\mu J} + \frac{\partial^2 \mathcal{H}^{(2)}_{phys}}{\partial\delta Q_I \partial\delta Q_J}\alpha^\mu{}_J\right)\delta Q_I$$
$$- \left(\lambda^\mu_{4\nu}\alpha^\nu{}_I - \dot{\alpha}^\mu{}_I + \frac{\partial^2 \mathcal{H}^{(2)}_{phys}}{\partial\delta P^I \partial\delta Q_J}\alpha^\mu{}_J - \frac{\partial^2 \mathcal{H}^{(2)}_{phys}}{\partial\delta P^I \partial\delta P^J}\beta^{\mu J}\right)\delta P^I.$$

The tranfosmation depends on:

- the physical Hamiltonian $\mathcal{H}^{(2)}_{phys}$
- the background-dependent parameters $\alpha_{\mu I},\ \beta^{\mu I}$
- → the coefficient $\lambda^\mu_{4\nu}$

[K.A. Malik. D. Wands, Cosmological perturbation]

Partial Gauge-fixing

Some of the gauge-fixing conditions are replaced by conditions on the lapse and shift.

Example in GR:

Synchronous Gauge

$$\delta N = \delta N_i = 0$$

Partial Gauge-fixing condition: the proper time for observers at fixed spatial coordinates coincides with cosmic time in the background.

[K.A. Malik. D. Wands, Cosmological perturbation]

Partial Gauge-fixing

Some of the gauge-fixing conditions are replaced by conditions on the lapse and shift.

Example in GR:

Synchronous Gauge

$$\delta N = \delta N_i = 0$$

Partial Gauge-fixing condition: the proper time for observers at fixed spatial coordinates coincides with cosmic time in the background.

Spatially flat Gauge

$$\delta R = 0$$

Gauge-fixing condition defined purely by local metric quantities, the curvature on each space-like sheet is set to zero.

Partial Gauge-fixing

Some of the gauge-fixing conditions are replaced by conditions on the lapse and shift.

Example in electrodynamics:

Lorenz Gauge

$$\partial_\mu A^\mu = 0$$

Partial gauge-fixing condition
on the temporal component of the four potential A^0
that plays a role of the Lagrange multiplier analogously
to the lapse and shifts in the present theory.

Coulomb Gauge

$$\nabla \vec{A} = 0$$

Gauge-fixing condition on the kinematical phase space
made of the spatial components of the 4-potential
and its conjugate momenta $(\vec{A}, \vec{\pi})$

Conclusions

→ The Dirac method can be used to derive the gauge-invariant variables and their Hamiltonian dynamics.

→ The Kuchař decomposition allows for an explicit representaion of the space of all the possible gauge-fixing conditions, and the spacetime reconstruction becomes significantly easier.

→ This application is completely general and can be applied to more complex background such as the anisotropic ones.

→ We obtain an in-depth comprehension of the gauge-fixing precudure, space-time reconstruction which are essential in the definition of the Hamiltonian formalism for our system. This will be essential it in the future for addressing the key conceptual problems in quantum gravity and cosmology.

Thank you