κ-deformed complex fields, (discrete) symmetries, and charges [1][2]

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[1] M. Arzano, **A. B.**, J. Kowalski-Glikman, G. Rosati, and J. Unger, κ-deformed complex fields and discrete symmetries. Phys.Rev.D, 103:106015

[2] **A.B.**, J. Kowalski-Glikman, and W. Wislicki, κ-deformed complex scalar field: conserved charges, symmetries and their impact on physical observables. Phys.Rev.D, 105:105004.

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Introduction, action, and fields

κ-Minkowski spacetime and momentum space picture

Non-commutative coordinates: $[\hat{x}^0, \hat{x}^i] = \frac{i}{\kappa}\hat{x}^i$ ($\mathfrak{an}(3)$ algebra)

Physical insight: $\left[\frac{1}{\kappa}\right] = L$. However, this κ-deformed theory is intended as an effective theory modelling quantum gravitational effects $\implies \frac{1}{\kappa} \approx l_p$. $\kappa \to \infty$ gives the "classical" limit.

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$$\hat{x}^0 = -\frac{i}{\kappa}\begin{pmatrix} 0 & \mathbf{0} & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 1 & \mathbf{0} & 0 \end{pmatrix} \quad \hat{\mathbf{x}} = \frac{i}{\kappa}\begin{pmatrix} 0 & \epsilon^T & 0 \\ \epsilon & \mathbf{0} & \epsilon \\ 0 & -\epsilon^T & 0 \end{pmatrix}$$

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$$\hat{e}_k = e^{ik_i\hat{x}^i}e^{ik_0\hat{x}^0} \in AN(3) \quad \leftarrow \quad \text{Time to the right + dim.ful}$$

$$e^{A\hat{x}} = \sum_{n=0}^{\infty}\frac{(A\hat{x})^n}{n!} \quad \leftarrow \quad \text{Definition of exp}$$

κ-Minkowski spacetime and momentum space picture

$$\hat{e}_k = \begin{pmatrix} \frac{\bar{p}_4}{\kappa} & \frac{\mathbf{k}}{\kappa} & \frac{p_0}{\kappa} \\ \frac{\mathbf{p}}{\kappa} & \mathbf{1} & \frac{\mathbf{p}}{\kappa} \\ \frac{\bar{p}_0}{\kappa} & -\frac{\mathbf{k}}{\kappa} & \frac{p_4}{\kappa} \end{pmatrix} \qquad \begin{aligned} p_0 &= \kappa \sinh \frac{k_0}{\kappa} + \frac{\mathbf{k}^2}{2\kappa} e^{k_0/\kappa} \\ p_i &= k_i e^{k_0/\kappa} \\ p_4 &= \kappa \cosh \frac{k_0}{\kappa} - \frac{\mathbf{k}^2}{2\kappa} e^{k_0/\kappa} \end{aligned}$$

Notice that $\hat{e}_k \Leftrightarrow (p_0, p_i, p_4)^T$ and if $\mathcal{O} = (0, \ldots, 0, \kappa)^T$ then

$$(p_0, p_i, p_4)^T = \hat{e}_k \mathcal{O}$$

$$-p_0^2 + \mathbf{p}^2 + p_4^2 = \kappa^2, \qquad p_4 > 0, \qquad p_+ := p_0 + p_4 > 0$$

κ-Minkowski spacetime and momentum space picture

Notice: both the k_A and the p_A can be interpreted as coordinates in (intrinsically curved) momentum space, and their sum is now non-trivial.

$$\hat{e}_k \hat{e}_l := \hat{e}_{k \oplus l} \quad \leftarrow \quad \text{Group property}$$

$$(k \oplus l)_0 = k_0 + l_0$$
$$(k \oplus l)_i = k_i + e^{-k_0/\kappa} l_i$$

$$(p \oplus q)_0 = \frac{p_0}{\kappa} q_+ + \frac{\mathbf{pq}}{p_+} + \frac{\kappa}{p_+} q_0$$
$$(p \oplus q)_i = \frac{\mathbf{p}_i}{\kappa} q_+ + \mathbf{q}_i$$
$$(p \oplus q)_4 = \frac{p_4}{\kappa} q_+ - \frac{\mathbf{pq}}{p_+} - \frac{\kappa}{p_+} q_0$$

For similar reasons, $-(.) \mapsto S(.)$ with $p \oplus S(p) = S(p) \oplus p = 0$.

κ-Minkowski spacetime and momentum space picture

Why p and not k? Using p, we can now work in a commutative spacetime.

In particular, using an object called Weyl map, one can send a group element $\hat{e}_k$ into a canonical plane wave e_p

$$\mathcal{W}(\hat{e}_k) = e_p \qquad e_p = e^{ip_\mu x^\mu} = e^{i(\omega t - \mathbf{px})}$$

$$\mathcal{W}(\hat{e}_{k\oplus l}) = e_{p(k)\oplus q(l)} = e_p \star e_q$$

This $\star$ product is in general non-commutative.

Because of the star product we have two possible orderings $\implies$ two possible actions.

$$S_1 = \int_{\mathbb{R}^4} d^4x \ (\partial^\mu \phi)^\dagger \star (\partial_\mu \phi) - m^2 \phi^\dagger \star \phi$$
$$S_2 = \int_{\mathbb{R}^4} d^4x \ \ (\partial_\mu \phi) \star (\partial^\mu \phi)^\dagger - m^2 \phi \star \phi^\dagger.$$

Therefore

$$S = \frac{1}{2} \int_{\mathbb{R}^4} d^4x \ [(\partial^\mu \phi)^\dagger \star (\partial_\mu \phi) + (\partial^\mu \phi) \star (\partial_\mu \phi)^\dagger \\ -m^2(\phi^\dagger \star \phi + \phi \star \phi^\dagger)]$$

Action, EOM, and fields

How to compute the EOM? Usually, one uses integration by parts. Here, however, the Leibniz rule for derivatives does not work!

$$i(p \oplus q)_\mu e_{p\oplus q} = \partial_\mu (e_p \star e_q) = (\partial_\mu e_p) \star e_q + e_p \star \partial_\mu e_q$$
$$= i(p+q) e_{p\oplus q}$$

Instead, more complicated rules need to be applied. Example:

$$\partial_0(\phi \star \psi) = \frac{1}{\kappa}(\partial_0 \phi) \star (\Delta_+ \psi) + \kappa(\Delta_+^{-1}\phi) \star (\partial_0 \psi) + i(\Delta_+^{-1}\partial_i \phi) \star (\partial_i \psi)$$

Action, EOM, and fields

After some computations, we obtain the following equations of motion

$$(\partial_\mu^\dagger (\partial^\mu)^\dagger - m^2)\phi^\dagger = 0 \qquad (\partial_\mu \partial^\mu - m^2)\phi = 0$$

The field satisfies the Klein-Gordon equations. (Notice: in momentum space, $\partial_\mu^\dagger (\partial^\mu)^\dagger \leftrightarrow S(p)_\mu S(p)^\mu = p_\mu p^\mu \leftrightarrow \partial_\mu \partial^\mu$)
Any complex scalar field satisfying these eom can be written as

$$\phi(x) = \int \frac{d^3 p}{\sqrt{2\omega_p}} \xi(p) a_{\mathbf{p}}\, e^{-i(\omega_p t - \mathbf{p}\mathbf{x})} + \int \frac{d^3 p^*}{\sqrt{2|\omega_p^*|}} \xi(p) b_{\mathbf{p}^*}^\dagger\, e^{i(S(\omega_p^*)t - S(\mathbf{p}^*)\mathbf{x})}$$

Properties of the fields under C, P, T

How do κ-deformed fields transform? P and T can consistently be defined as acting like in the undeformed case (they leave $[\hat{x}^0, \hat{x}^i] = \frac{i}{\kappa}\hat{x}^i$ invariant)

$$\mathcal{T}\phi(t,\mathbf{x})\mathcal{T}^{-1} = \phi(-t,\mathbf{x}) \quad \Longrightarrow \quad \mathcal{T}a_{\mathbf{p}}\mathcal{T}^{-1} = a_{-\mathbf{p}}$$
$$\mathcal{P}\phi(t,\mathbf{x})\mathcal{P}^{-1} = \phi(t,-\mathbf{x}) \quad \Longrightarrow \quad \mathcal{P}a_{\mathbf{p}}\mathcal{P}^{-1} = a_{-\mathbf{p}}$$

Because of the presence of the antipode S() in the fields and because of the form of the action, also C can be shown to behave like in the undeformed case (in its action on $a, a^\dagger, b, b^\dagger$) when acting on fields.

$$\mathcal{C}\phi(t,\mathbf{x})\mathcal{C}^{-1} = \phi(t,\mathbf{x})^\dagger \quad \Longrightarrow \quad \boxed{\mathcal{C}a_{\mathbf{p}}\mathcal{C}^{-1} = b_{\mathbf{p}^*}}$$

The action is manifestly invariant under C, P, T *and* under (deformed) Lorentz transformations.

We need the charges. How to get them?

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- Using the Noether theorem. However, difficult computations (recall integration by parts), so only the translational charges are easily obtainable in this way;

- More pragmatic approach: use the canonical formalism (Noether theorem) to compute translational charges, then switch to covariant phase space formalism for the others. Keep in mind, a kind of "matching" is necessary!

General charges, how to compute them, and properties

Geometric approach to conserved charges

How to compute a general charge of a given theory? As previously said, we will now concentrate on the more general case: geometric approach.

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Assuming that the charges come from a symmetry described by some continuous vector field ξ in spacetime, then

$$-\delta_{\xi} \lrcorner \Omega \stackrel{!}{=} \delta Q_{\xi}$$

where δ is the exterior derivative in phase space, Q_{ξ} is the charge associated to the vector ξ. $\delta_{\xi} A$ measures the infinitesimal variation of the object A in phase space due to the symmetry of the action along ξ in spacetime.

Deformed case: translations

We now start from the following transformations describing time translations.

$$\delta^T a_{\mathbf{p}} = i\epsilon^\mu p_\mu a_{\mathbf{p}}, \quad \delta^T a^\dagger_{\mathbf{p}} = i\epsilon^\mu S(p)_\mu a^\dagger_{\mathbf{p}},$$
$$\delta^T b^\dagger_{\mathbf{p}} = i\epsilon^\mu S(p)_\mu b^\dagger_{\mathbf{p}}, \quad \delta^T b_{\mathbf{p}} = i\epsilon^\mu p_\mu b_{\mathbf{p}}.$$

Notice the antipode. Therefore, a naive application of the previous procedure would not give a consistent result (no quantity Q_ξ such that $-\delta_\xi \lrcorner \Omega = \delta Q_\xi$). "Matching" with the direct computation needed! We will need to introduce the antipode in the contraction of a vector field with a 2-form.

Deformed case: translations

We postulate the following rule

$$\delta_\xi \lrcorner (\delta a^\dagger_{\mathbf{p}} \wedge \delta a_{\mathbf{p}}) = (\delta_\xi a^\dagger_{\mathbf{p}})\delta a_{\mathbf{p}} + \delta a^\dagger_{\mathbf{p}}[S(\delta_\xi) a_{\mathbf{p}}]$$

which solves the issue

$$\Omega^U = i \int d^3p\, \alpha\, (\delta a_{\mathbf{p}} \wedge \delta a^\dagger_{\mathbf{p}} - \delta b^\dagger_{\mathbf{p}^*} \wedge \delta b_{\mathbf{p}^*})$$

$$\begin{aligned} &- \delta^T \lrcorner\, \Omega \\ &= i \int d^3p\, \alpha\, (\delta^T a^\dagger_{\mathbf{p}} \delta a_{\mathbf{p}} + \delta a^\dagger_{\mathbf{p}} S(\delta^T) a_{\mathbf{p}} - \delta^T b_{\mathbf{p}} \delta b^\dagger_{\mathbf{p}} - \delta b_{\mathbf{p}} S(\delta^T) b^\dagger_{\mathbf{p}}) \\ &= -\epsilon^\mu \delta \left(\int d^3p\, \alpha\, [S(p)_\mu a^\dagger_{\mathbf{p}} \delta a_{\mathbf{p}} - p_\mu b^\dagger_{\mathbf{p}} \delta b_{\mathbf{p}} \right) \end{aligned}$$

$$\mathcal{P}_\mu = \int d^3p\, \alpha\, [-S(p)_\mu a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + p_\mu b^\dagger_{\mathbf{p}} b_{\mathbf{p}}],$$

Deformed case: boosts

In the case of boosts, we need to assume the following transformations for the creation/annihilation transformations

$$\delta^B a_{\mathbf{p}} = -i\lambda^j\,\omega_{\mathbf{p}}\left[\frac{\partial}{\partial \mathbf{p}^j} + \frac{1}{2}\frac{1}{\omega_{\mathbf{p}}}\frac{\partial[\omega_{\mathbf{p}} S(\alpha)]}{\partial \mathbf{p}^j}\right] a_{\mathbf{p}},$$

$$\delta^B a^\dagger_{\mathbf{p}} = -i\lambda^j\,S(\omega_{\mathbf{p}})\left[\frac{\partial}{\partial S(\mathbf{p})^j} + \frac{1}{2}\frac{1}{S(\omega_{\mathbf{p}})}\frac{\partial[S(\omega_{\mathbf{p}})\alpha]}{\partial S(\mathbf{p})^j}\right] a^\dagger_{\mathbf{p}},$$

$$\delta^B b_{\mathbf{p}} = -i\lambda^j\,\omega_{\mathbf{p}}\left[\frac{\partial}{\partial \mathbf{p}^j} + \frac{1}{2}\frac{1}{\omega_{\mathbf{p}}}\frac{\partial[\omega_{\mathbf{p}}\alpha]}{\partial \mathbf{p}^j}\right] b_{\mathbf{p}},$$

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Charges

For the boosts, we obtain the charge

$$\mathcal{N}_i = -\frac{1}{2}\int d^3p\,\alpha\left\{S(\omega_p)\left[\frac{\partial a^\dagger_{\mathbf{p}}}{\partial S(\mathbf{p})^i}a_{\mathbf{p}} - a^\dagger_{\mathbf{p}}\frac{\partial a_{\mathbf{p}}}{\partial S(\mathbf{p})^i}\right]\right.$$
$$\left. + \omega_p\left[b_{\mathbf{p}}\frac{\partial b^\dagger_{\mathbf{p}}}{\partial \mathbf{p}^i} - \frac{\partial b_{\mathbf{p}}}{\partial \mathbf{p}^i}b^\dagger_{\mathbf{p}}\right]\right\}.$$

Notice: all the deformed charges satisfy the undeformed Poincaré algebra (checked by direct tedious computations).

However, the transformations of the creation/annihilation operators related to the above boost charge now correspond to a non-trivial transformation of the field.

Charges

If we translate the creation/annihilation operators transformations in terms of the field we get (at first order in $1/\kappa$)

$$\delta^B \phi(x) = i\lambda_i\, x^i \frac{\partial}{\partial t}\,\phi(x) - i\lambda_i \int \frac{d^3p}{\sqrt{2\omega_{\mathbf{p}}}} \left\{ \frac{\mathbf{p}_i}{\kappa}\left(\frac{m^2}{4\omega_{\mathbf{p}}^2} - \frac{1}{2}\right) a_{\mathbf{p}} e^{-i(\omega_{\mathbf{p}}t - \mathbf{p}\mathbf{x})} + \frac{\mathbf{p}_i}{\kappa}\left(-\frac{m^2}{4\omega_{\mathbf{p}}^2} - 1\right) b^\dagger_{\mathbf{p}} e^{-i(S(\omega_{\mathbf{p}})t - S(\mathbf{p})\mathbf{x})} \right\},$$

Analogous relation for $\phi^\dagger$, which means that particles and antiparticles receive an additional shift under boost.

Relation between CPT and boost charges

One has therefore a set of charges (translations, rotations, boosts) which leave the action invariant, together with discrete symmetries C, P, T which do the same. All of these have been explicitly computed.
However, non trivial relations between them arise! One can show that

$$[N_i, C] \neq 0$$

This translate (for example) into a difference of decay times for particles and antiparticles in a boosted frame.

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- We have a well defined theory $\Longrightarrow$ propagator and n-point functions.

- Issue of the loop computations: non-planar vertices are difficult to evaluate. Possible solution: use group field theory formalism and braiding (see yesterday talk by T. Trześniewski in a different context).

- Immediate generalization: higher spin.

Thank you

One particle states

We now have all the tools to show that indeed particles and antiparticles behave differently.

How to see it?

- Since we have the translation charges (i.e. the operators $\mathcal{P}_\mu$), we can apply them to the a-particle and b-particle states and get their eigenvalues. We will see that they are different;

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How to see it?

- Since we have the translation charges (i.e. the operators $\mathcal{P}_\mu$), we can apply them to the a-particle and b-particle states and get their eigenvalues. We will see that they are different;
- Use $\mathcal{C}$ to link the a-particle to the b-particle state. We will see that C switches a particle with its antiparticle with *different* momentum.

Define the vacuum by $a_{\mathbf{p}}|0\rangle = b_{\mathbf{p}*}|0\rangle = 0$. We then define one-particle and one-antiparticle state by

$$a^\dagger_{\mathbf{p}}|0\rangle := |\mathbf{p}\rangle_a \qquad b^\dagger_{\mathbf{p}*}|0\rangle := |\mathbf{p}\rangle_b$$

Now we want to know $\mathcal{P}_\mu|\mathbf{p}\rangle_a$ and $\mathcal{P}_\mu|\mathbf{p}\rangle_b$.

One particle states

$$\left[a_{\mathbf{p}}, a^{\dagger}_{\mathbf{q}}\right] = \frac{1}{\alpha}\delta(\mathbf{p}-\mathbf{q})$$

$$\begin{aligned}
\mathcal{P}_i|\mathbf{q}\rangle_a &= \int d^3p\alpha \left\{-a^{\dagger}_{\mathbf{p}}\, a_{\mathbf{p}}\, S(\mathbf{p})_i + b^{\dagger}_{\mathbf{p}^*}\, b_{\mathbf{p}^*}\, \mathbf{p}_i\right\} a^{\dagger}_{\mathbf{q}}|0\rangle \\
&= \int d^3p\alpha \left\{-a^{\dagger}_{\mathbf{p}}\, \frac{1}{\alpha}\delta(\mathbf{p}-\mathbf{q})S(\mathbf{p})_i + a^{\dagger}_{\mathbf{p}}\, a^{\dagger}_{\mathbf{q}} a_{\mathbf{p}}\, \mathbf{p}_i\right\} |0\rangle + 0 \\
&= -S(\mathbf{q})_i|\mathbf{q}\rangle_a
\end{aligned}$$

One particle states

Doing the same thing for all $\mathcal{P}_\mu$ we have

$$\mathcal{P}_i|\mathbf{p}\rangle_a = -S(\mathbf{p})_i|\mathbf{p}\rangle_a \qquad \mathcal{P}_i|\mathbf{p}\rangle_b = \mathbf{p}_i|\mathbf{p}\rangle_b$$

$$\mathcal{P}_0|\mathbf{p}\rangle_a = -S(\omega_p)|\mathbf{p}\rangle_a \qquad \mathcal{P}_0|\mathbf{p}\rangle_b = \omega_p|\mathbf{p}\rangle_b$$

Notice: $\mathbf{p} \neq -S(\mathbf{p})$ and $\omega_p \neq -S(\omega_p)$, but $p_\mu p^\mu = m^2$ and $S(p)_\mu S(p)^\mu = m^2$, so a-particle and b-particle have same mass.

One particle states

We can use $\mathcal{C}$ to relate $|\mathbf{p}\rangle_a$ and $|\mathbf{p}\rangle_b$

$$\mathcal{C}|\mathbf{p}\rangle_b = \mathcal{C}b^\dagger_{\mathbf{p}*}\mathcal{C}^{-1}\mathcal{C}|0\rangle = a^\dagger_{\mathbf{p}}|0\rangle = |\mathbf{p}\rangle_a$$

Very easy steps due to the simplicity of the C transformation of our deformed field!

Therefore C (and CPT) transforms a particle into an anti-particle with different momentum, and vice versa.