Ascribing quantum system to Schwarzschild spacetime with naked singularity

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Ascribing quantum system to Schwarzschild spacetime with naked singularity

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I. Tools for quantization of spherically symmetric models of spacetime

$$\mathrm{d}s^2 = -A(t,r)\mathrm{d}t^2 + B(t,r)\mathrm{d}r^2 + C(t,r)\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)$$

where A, B and C are positive functions of t and r.
The configuration space:

$$T = \{(t,r) \mid (t,r) \in \mathbb{R}\times\mathbb{R}_+\}$$

where t and r are time and radial coordinate respectively.

Motivation

I. Tools for quantization of spherically symmetric models of spacetime

Classical picture

- Configuration space $(t, r) \in \mathbb{R} \times \mathbb{R}_+$;
- Observables which are functions of $f(t, r) \in \mathbb{R}$;

Scheme of quantization

1. Every point of configuration space has to be linked with projection operator in some carrier space $\mathcal{H}$:

$$(t, r) \to |t, r\rangle\langle t, r|$$

2. Quantization:

$$f(t, r) \to \hat{f} = \int d\mu(t, r)|t, r\rangle f(t, r)\langle t, r|$$

Motivation

II. Quantization of both temporal and spatial variables

- Equivalent treatment of spatial and time observables is required in relativistic physics.
- Because of direct correlation between spacetime points and appropriate projection operators the construction of time observable is natural in this quantization method.

III. Gravitational singularity in quantum description

- It is interesting to ask about quantum description of gravitational singularity. In what way does the quantum smearing affect existence of gravitational singularity?

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Affine coherent states quantization

Affine group structure

- The parametrization of the affine group $(p, q) \in \mathbb{R} \times \mathbb{R}_+$, with multiplication law

$$g(p_1, q_1) \cdot g(p_2, q_2) := g(p_1 + q_1 p_2, q_1 q_2) \in \mathrm{Aff}(\mathbb{R}),$$

- left invariant measure

$$d\mu(p, q) = \frac{1}{2\pi} dp \frac{dq}{q^2}$$

Every point of configuration space is uniquely identified with the corresponding group element

$$(t, r) \leftrightarrow g(\chi_1(t, r), \chi_2(t, r)) = g(p, q)$$

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Affine coherent states quantization

Building blocks for ACS quantization

- Carrier space $\mathcal{H}_x = L^2(\mathbb{R}_+, d\nu(x))$ where $d\nu(x) = \frac{dx}{x}$.
- Irreducible unitary representation of affine group on space $\mathcal{H}_x$

$$U(p,q)\Psi(x) = e^{ipx}\Psi(qx)$$

- Fiducial vector $\Phi_0(x) \in \mathcal{H}_x$:
  - **Coherent sates**

$$\langle x|g(p,q)\rangle = U(p,q)\Phi_0(x) = e^{ipx}\Phi_0(qx)\,,$$

Properties of the fiducial vector:

$$\langle\Phi_0|\Phi_0\rangle = \int_0^\infty d\nu(x)|\Phi_0(x)|^2 = 1$$

$$A_{\Phi_0} := \int_0^\infty \frac{dx}{x^2}|\Phi_0(x)|^2 < \infty$$

Affine coherent states quantization

The fiducial vector must be selected to provide resolution of unity

$$\frac{1}{A_{\Phi_0}} \int_{\mathrm{Aff}(\mathbb{R})} d\mu(p,q)\ |g(p,q)\rangle\langle g(p,q)| = \hat{\mathbf{1}}$$

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Affine coherent states quantization

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Affine coherent states quantization

Now we can formally map any observable $f : T \to \mathbb{R}$ into a symmetric operator $\hat{f} : \mathcal{H}_x \to \mathcal{H}_x$ as follows

$$\hat{f} := \frac{1}{A_{\Phi_0}} \int_{\mathrm{Aff}(\mathbb{R})} d\mu(p,q) |g(p,q)\rangle f(p,q) \langle g(p,q)|$$

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Affine coherent states quantization

Freedom in parametrization of affine group: $(t, r) \leftrightarrow g(\chi(t, r)) = g(p, q)$

"Center" of the group manifold

$$(t = 0, r = 1) \rightarrow g(a, b)$$

The element $g(a, b) \in \mathrm{Aff}(\mathbb{R})$ is called the "center" of the group manifold associated to this configuration space.

Parametrization of "center" selection can be implemented in two ways:

$$(t, r) \leftrightarrow g(a, b) \cdot g(t, r) \Rightarrow \frac{1}{A_{\Phi_0}} \int_{\mathrm{Aff}(\mathbb{R})} d\mu(p, q)\; |g(a, b) \cdot g(p, q)\rangle\langle g(a, b) \cdot g(p, q)| = \\ = \hat{\mathbf{1}}$$

$$(t, r) \leftrightarrow g(t, r) \cdot g(a, b) \Rightarrow \frac{1}{A_{\Phi_0}} \int_{\mathrm{Aff}(\mathbb{R})} d\mu(p, q)\; |g(p, q) \cdot g(a, b)\rangle\langle g(p, q) \cdot g(a, b)| = \\ = \Delta(g(a, b)^{-1})\hat{\mathbf{1}}$$

Affine coherent states quantization

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$$(t, r) \leftrightarrow g(t, r) \cdot g(a, b) \;\Rightarrow \frac{1}{A_{\Phi_0}} \int_{\mathrm{Aff}(\mathbb{R})} d\lambda(p, q)\, |g(p, q) \cdot g(a, b)\rangle\langle g(p, q) \cdot g(a, b)| = \hat{\mathbf{1}}$$

$$d\lambda(p, q) = \Delta(g(a, b))d\mu(p, q)$$

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Affine coherent states quantization

The quantization in second case has a form:

$$\hat{f} = \frac{1}{A_{\Phi_0}} \int_{\mathrm{Aff}(\mathbb{R})} d\lambda(p,q)\, |g(p,q)\cdot g(a,b)\rangle\, f(p,q)\, \langle g(p,q)\cdot g(a,b)|\,.$$

After performing the right shift operation in the coherent states, one gets the expression for quantization of the function $f(t,r)$:

$$\hat{f} = \frac{1}{A_{\Phi_0}} \int_{\mathrm{Aff}(\mathbb{R})} d\mu(p,q)\, |g(p,q)\rangle\, f\left(p - \frac{a}{b}q, \frac{q}{b}\right)\, \langle g(p,q)|\,.$$

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Affine coherent states quantization

Quantization of elementary observables

$$\hat{t} = \frac{1}{A_\Phi} \int_{\mathrm{Aff}(\mathbb{R})} d\mu(p,q) |g(p,q)\rangle \left(p - \frac{a}{b} q \right) \langle g(p,q)| \,,$$

$$\hat{r} = \frac{1}{A_\Phi} \int_{\mathrm{Aff}(\mathbb{R})} d\mu(p,q) |g(p,q)\rangle \left(\frac{q}{b} \right) \langle g(p,q)| \,.$$

Consisteny conditions

Coherent state $|g(t,r)\rangle$ is quantum representation of spacetime point (t,r),

- The expectation value of an operator $\hat{t}$ in state $|g(t,r)\rangle$ should be equal to "classical" position in time

$$\langle g(t,r)|\hat{t}|g(t,r)\rangle = t \,.$$

- The expectation value of an operator $\hat{r}$ in state $|g(t,r)\rangle$ should be equal to "classical" position in space

$$\langle g(t,r)|\hat{r}|g(t,r)\rangle = r \,.$$

Variance

$$\mathrm{var}(\hat{A};\psi) := \langle\psi|\left(\hat{A} - \langle\psi|\hat{A}|\psi\rangle\right)^2|\psi\rangle = \langle\psi|\hat{A}^2|\psi\rangle - \langle\psi|\hat{A}|\psi\rangle^2$$

- Variance measures spread of $\hat{A}$ around its expectation value in particular quantum state $|\psi\rangle$.
- If $\hat{A}$ is self-adjoint, the variance of $\hat{A}$ is equal to 0, if and only if ψ is an eigenstate of $\hat{A}$

$$\left(\mathrm{var}(\hat{A};\psi) = 0\right) \Longleftrightarrow \left(\hat{A}\psi = \lambda\psi, \quad \lambda \in \mathbb{R}\right),$$

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ACS quantization-analysis of gravitational singularity

Quantum analysis of gravitational singularity

Let us assume that we have a set of curvature invariants $\mathcal{A}_n$, which after ACS quantization give a set of observables $\hat{\mathcal{A}}_n$.

The quantum gravitational singularity in the state $|\psi\rangle$ is reached, if the following two conditions are true

1. The expectation values of the curvature invariants quantum observables go to infinity in this state

$$\langle\psi|\hat{\mathcal{A}}_n|\psi\rangle \to \infty$$

2. The variances of the curvature invariants quantum observables go to 0

$$\mathrm{var}(\hat{\mathcal{A}}_n; \psi) \to 0$$

ACS quantization of the Schwarzschild spacetime

The Schwarzschild spacetime

- The metric in the Schwarzschild coordinates $(t, r, \theta, \phi) \in \mathbb{R} \times (0, \infty) \times S^2$ reads:

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{r_s}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right) ,$$

$r_s = 2M$.

- The Kretschmann scalar

$$\mathcal{K} := R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} = \frac{48 M^2}{r^6} ,$$

where parameter $M < 0$, so that it describes the naked singularity[a] and exhibits the gravitational singularity as $r \to 0$.

[a] P. Chruściel, *Geometry of Black Holes* (International Series of Monographs on Physics, 2020).

ACS quantization of the Schwarzschild spacetime

Using ACS quantization rules, the quantum Kretschmann observable can be written as

$$\hat{\mathcal{K}} = 48M^2 \langle \check{q} \rangle_0^6 \frac{1}{A_{\Phi_0}} \int_{\mathrm{Aff}(\mathbb{R})} d\mu(p,q) |g(p,q)\rangle \frac{1}{q^6} \langle g(p,q)| \, .$$

where
$\langle \check{q} \rangle_0 = \frac{1}{A_\Phi} \int_{\mathrm{Aff}(\mathbb{R})} d\mu(p,q) |\langle g(0,1)|g(p,q)\rangle|^2 q = \langle g(0,1)|\check{q}|g(0,1)\rangle$

The acting of Kretschmann operator can be describe as

$$\hat{\mathcal{K}}\Psi(x) = \int_{\mathbb{R}_+} d\nu(y) \, \mathrm{K}_{\mathcal{K}}(x,y) \, \Psi(y)$$

where the integral kernel has a form:

$$\mathrm{K}_{\mathcal{K}}(x,y) = \langle x|\hat{\mathcal{K}}|y\rangle = \mathcal{A} \, \delta(x-y) x^7 \quad \text{where} \quad \mathcal{A} = \frac{48M^2 \langle \check{q} \rangle_0^6}{A_{\Phi_0}} \int_{\mathbb{R}_+} \frac{dq}{q^8} |\Phi_0(q)|^2$$

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- **Expectation value:**

$$\langle g(t,r)|\hat{\mathcal{K}}|g(t,r)\rangle = 48M^2 \frac{\langle (q^{-6})\check{\ }\rangle_0}{\langle \check{q}\rangle_0^{-6}} \frac{1}{r^6}$$

- **Variance:**

$$\mathrm{var}(\hat{\mathcal{K}}; g(t,r)) = (48M^2)^2 \left(\langle ((q^{-6})\check{\ })^2\rangle_0 - \langle (q^{-6})\check{\ }\rangle_0^2\right) \cdot \cdot \langle \check{q}\rangle_0^{12} \frac{1}{r^{12}}$$

- **Expectation value:**

$$\langle g(t,r)|\hat{\mathcal{K}}|g(t,r)\rangle = 48M^2 \frac{\langle (q^{-6})\check{}\ \rangle_0}{\langle \check{q}\rangle_0^{-6}} \frac{1}{r^6} \xrightarrow{r\to 0} \infty$$

- **Variance:**

$$\mathrm{var}(\hat{\mathcal{K}}; g(t,r)) = (48M^2)^2 \left(\langle ((q^{-6})\check{}\ )^2\rangle_0 - \langle (q^{-6})\check{}\ \rangle_0^2\right) \cdot \langle \check{q}\rangle_0^{12} \frac{1}{r^{12}} \xrightarrow{r\to 0} \infty$$

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ACS quantization of the Schwarzschild spacetime

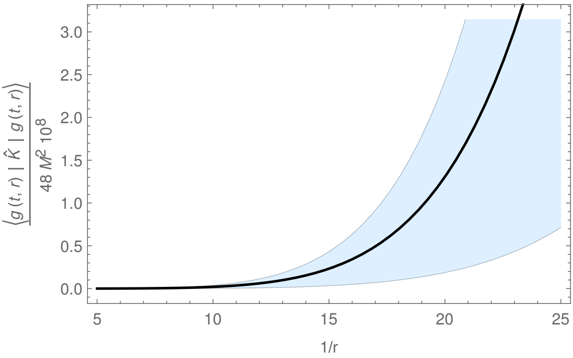

Figure: The $1/r$ dependence of the expectation value of the Kretschmann operator $\langle g(t,r)|\hat{\mathcal{K}}|g(t,r)\rangle$. The blue area defines the points for which distance from expected value is smaller than $\sqrt{\mathrm{var}(\hat{\mathcal{K}}; g(t,r))}$ (the distance is counted along fixed $1/r$ line). The fiducial vector is taken as $\Phi_0(x) = \frac{1}{\sqrt{(2n-1)!}} x^n e^{-\frac{x}{2}}$ with $n = 25$.

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ACS quantization of the Schwarzschild spacetime

$$\Psi_n(x) = N x^n \exp\left[i\tau_0 x - \frac{\gamma^2 x^2}{2}\right], \qquad N = 2\gamma^n/(n-1)!$$

$\Psi_n(x)$ form a set of functions which is dense in the Hilbert space $L^2(\mathbb{R}_+, d\nu(x))$.
The expectation values

$$\langle \Psi_n|\hat{t}|\Psi_n\rangle = \tau_0\,, \qquad \langle \Psi_n|\hat{r}|\Psi_n\rangle = \frac{1}{A_\Phi}\frac{\Gamma\left(n-\frac{1}{2}\right)}{(n-1)!}\,\gamma\,,$$

$$\langle \Psi_n|\hat{\mathcal{K}}|\Psi_n\rangle = \mathcal{A}\frac{(n+2)!}{(n-1)!}\,\frac{1}{\gamma^6}\,, \quad \mathrm{var}(\hat{\mathcal{K}};\Psi_n) = \mathcal{A}^2\left(\frac{(n+5)!}{(n-1)!} - \frac{(n+2)!^2}{(n-1)!^2}\right)\frac{1}{\gamma^{12}}$$

Also occures: $\langle \Psi_n|\hat{r}|\Psi_m\rangle \sim \gamma \quad \langle \Psi_n|\hat{\mathcal{K}}|\Psi_m\rangle \sim \frac{1}{\gamma^6} \quad \langle \Psi_n|\hat{\mathcal{K}}^2|\Psi_m\rangle \sim \frac{1}{\gamma^{12}} \quad n \neq m$

Which proves the same relation of $\langle\hat{\mathcal{K}}\rangle$, $\mathrm{var}(\hat{\mathcal{K}})$ in states which are any linear combination of Ψ_n.

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$$\langle\Psi_n|\hat{\mathcal{K}}|\Psi_n\rangle = \mathcal{A}\frac{(n+2)!}{(n-1)!}\,\frac{1}{\gamma^6}\,, \quad \mathrm{var}(\hat{\mathcal{K}};\Psi_n) = \mathcal{A}^2\left(\frac{(n+5)!}{(n-1)!} - \frac{(n+2)!^2}{(n-1)!^2}\right)\frac{1}{\gamma^{12}}$$

Also occures: $\langle\Psi_n|\hat{r}|\Psi_m\rangle \sim \gamma \quad \langle\Psi_n|\hat{\mathcal{K}}|\Psi_m\rangle \sim \frac{1}{\gamma^6} \quad \langle\Psi_n|\hat{\mathcal{K}}^2|\Psi_m\rangle \sim \frac{1}{\gamma^{12}} \quad n \neq m$
Which proves the same relation of $\langle\hat{\mathcal{K}}\rangle$, $\mathrm{var}(\hat{\mathcal{K}})$ in states which are any linear combination of Ψ_n.

In the ACS quantization of the Schwarzschild spacetime there is no such a quantum state which achieves gravitational singularity.

Summary

- The main assumption of the ACS quantization method is correspondence between space-time points and appropriate quantum projection operators. Its advantage is great simplicity which allows for qualitative analysis of gravitation models.
- The quantization of time variable on the same footing as spatial variables is novelty in the program of quantization of gravity. It seems to be fruitful and worth of being applied to more realistic models of spacetime.
- Making use of the ACS quantization of the Schwarzschild spacetime, we have found that the expectation value of the Kretschmann operator $\hat{\mathcal{K}}$ is singular and behaves like $1/r^6$ as in the classical case. However, its variance behaves like $1/r^{12}$. One can say that quantization smears the singularity, avoiding its localization in the region of the configuration space including the singularity.

Self-adjoint symmetric operators

$$\langle\Psi|\hat{f}|\Psi\rangle = \frac{1}{A_{\Phi_0}} \int_{\mathrm{Aff}(\mathbb{R})} d\mu(p,q) |\langle g(p,q)|\Psi\rangle|^2 f\left(\chi^{-1}\left(p - \frac{a}{b}q, \frac{q}{b}\right)\right)$$

is real for Ψ belonging to the domain of the operator $\hat{f}$. The operator $\hat{f}$ can be bounded by the following expression

$$\begin{aligned} \|\hat{f}|h\rangle\|^2 \leq \Bigg[& \int_{\mathrm{Aff}(\mathbb{R})} d\mu(p_1,q_1) \int_{\mathrm{Aff}(\mathbb{R})} d\mu(p_2,q_2) \\ & \left| f\left(\chi^{-1}\left(p_1 - \frac{a}{b}q_1, \frac{q_1}{b}\right)\right) \langle g(p_1,q_1)|g(p_2,q_2)\rangle \right. \\ & \left. f\left(\chi^{-1}\left(p_2 - \frac{a}{b}q_2, \frac{q_2}{b}\right)\right) \right| \Bigg] \| \, |h\rangle \, \|^2 , \end{aligned}$$

for all $|h\rangle$ in the domain of the operator $\hat{f}$. If the above integral contained in square bracket is finite the operator $\hat{f}$ is continuous in $L^2(\mathbb{R}_+, d\nu(x))$, i.e., $\hat{f}$ is a self-adjoint operator.

Variance

Let $\hat{A}$ is self-adjoint operator on some dense subspace $\mathcal{S}$ of the Hilbert space $\mathcal{H}_x$.

$$\left(\mathrm{var}(\hat{A};\psi)=0\right) \iff \left(\hat{A}\psi=\lambda\psi, \quad \lambda\in\mathbb{R}\right),$$

The statement is implied by properties of the scalar product, norm and the operator itself:

$$\mathrm{var}(\hat{A};\psi)=\langle(\hat{A}-\langle\psi|\hat{A}|\psi\rangle)\psi\,|\,(\hat{A}-\langle\psi|\hat{A}|\psi\rangle)\psi\rangle=\|(\hat{A}-\langle\psi|\hat{A}|\psi\rangle)\psi\|^2 .$$

Thus,

$$\left(\mathrm{var}(\hat{A};\psi)=0\right)\Rightarrow\left((\hat{A}-\langle\psi|\hat{A}|\psi\rangle)\psi=0\right)\Rightarrow\left(\hat{A}\psi=\langle\psi|\hat{A}|\psi\rangle\psi\right).$$

On the other hand, if $\hat{A}\psi=\lambda\psi$, we have

$$\mathrm{var}(\hat{A};\psi)=\langle\psi|\hat{A}^2|\psi\rangle-\langle\psi|\hat{A}|\psi\rangle^2=\lambda^2\langle\psi|\psi\rangle-\lambda^2\langle\psi|\psi\rangle^2=0\,,$$

as ψ is a normalized vector.

Eigenproblem of Kretschmann operator

$$\int_{\mathbb{R}_+} d\nu(y)\, \mathrm{K}_{\mathcal{K}}(x,y)\, \psi_k^{(\mathcal{K})}(y) = k\, \psi_k^{(\mathcal{K})}(x)\,,$$

where the integral kernel

$$\mathrm{K}_{\mathcal{K}}(x,y) = \langle x|\hat{\mathcal{K}}|y\rangle = \mathcal{A}\,\delta(x-y)x^7\,, \qquad \mathcal{A} = \frac{48\,M^2}{A_{\Phi_0}}\langle \check{q}\rangle_0^6 \left[\int_{\mathbb{R}_+} \frac{dq}{q^8}\,|\Phi_0(q)|^2\right]$$

It must be noticed that the condition $\mathcal{A} < \infty$ requires an appropriate behavior of the fiducial vector at x equal to zero and infinity.
Direct calculations lead to the following generalized eigenfunctions

$$\psi_k^{(\mathcal{K})}(x) = \delta\left(x^6 - \frac{k}{\mathcal{A}}\right), \quad 0 < k < \infty\,,$$

and the positive spectrum $0 < k < \infty$ of the Kretschmann operator.

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Affine coherent states quantization-singularity existence

Uncertainty principle

By using variance one can also construct the uncertainty principle[a]

$$\mathrm{var}(\hat{A};\psi)\mathrm{var}(\hat{B};\psi) \geq \frac{1}{4}\left|\langle\Psi|[\hat{A},\hat{B}]|\Psi\rangle\right|^2$$

[a] H. P. Robertson, "The Uncertainty Principle", Phys. Rev. **34**, 163 (1929).

For operators t and r one gets

$$\mathrm{var}(\hat{t};g(t,r))\,\mathrm{var}(\hat{r};g(t,r)) \geq \frac{\langle i[\check{p},\check{q}]\rangle_0^2}{4\langle\check{q}\rangle_0^2}\,r^4\,,$$

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According to ACS quantization, for group manifold center equal to (a, b) one gets:

$$\langle g(t,r)|\hat{t}|g(t,r)\rangle = t + \left(\langle\check{p}\rangle_0 - \frac{a}{b}\langle\check{q}\rangle_0\right) r$$

$$\langle g(t,r)|\hat{r}|g(t,r)\rangle = \frac{\langle\check{q}\rangle_0}{b} r$$

where

$$\langle\check{f}\rangle_0 = \frac{1}{A_\Phi}\int_{\mathrm{Aff}(\mathbb{R})} d\mu(p,q)|\langle g(0,1)|g(p,q)\rangle|^2 f(p,q) = \langle g(0,1)|\check{f}|g(0,1)\rangle$$

Assuming $\langle\check{p}\rangle_0 - \frac{a}{b}\langle\check{q}\rangle_0 = 0$ and $b = \langle\check{q}\rangle_0$, i.e. $a = \langle\check{p}\rangle_0$, the self consistency conditions become fulfilled.

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