Integral quantization and quantum time

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Collaboration

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Introduction

There are as many quantization methods as quantum physicists :)

Physical degrees of freedom of a physical system:

- Phase space is of dimension 2N, N=number of degrees of physical degrees of freedom.
- Configuration space is of dimension N (classical and quantum description).

- To treat time on the same footing as other coordinates it should be considered as an additional physical degree of freedom.
- The "configuration space"/"phase space" should contain time as a variable.

Some Quantizations 1/4

Arbitrary choice of quantization methods !

(C) Old canonical quantization

$\Theta : CM \to QM$ (CM=classical model, QM=Quantum model):

$$\Theta(\{f, g\}) = \frac{1}{i\hbar}[\Theta(f), \Theta(g)]$$

- There is no reasonable quantization map satisfying the above identity exactly for all classical functions f, g (Groenewold theorem).
- Problem of the operators ordering.

Some Quantizations 2/4

(S) Star product quantization:

$$f \star g = f \cdot g + (i\hbar) C_1(f,g) + \sum_{n=2}^{\infty} (i\hbar)^n C_n(f,g)$$

$$C_1(f,g) = \frac{1}{2}\{f,g\}$$

$$\Theta(f) \cdot \Theta(g) = \Theta(f \star g)$$

- Different choices of the $\star$-products cover very large class of quantization methods.
  The mapping $Q : CM \to QM$ is given by
  $\Theta(f) \cdot \Theta(g) = \Theta(f \star g)$
- Groenewold objection and the ordering problem still valid.

https://arxiv.org/abs/quant-ph/0208163v1, Dirac, von Neuman, Groenewold, Gerstenhaber,...

Some Quantizations 3/4

(O) Huge amount of quantization methods, only a few of them:

..., perturbatively quantized gravity, geometrodynamical canonical quantization, the Wheeler-Dewitt equation, path integrals, the Euclidean path integral approach of Hawking, Penrose twistor theory, string theory, asymptotically safe gravity, causal dynamical triangulation, emergent gravity, loop quantum gravity, ...

Some Quantizations 4/4

(D) Deformation of a measure as a quantization method:

Overcomplete sets, coherent states, POVM, different kinds of integral quantizations (Berezin, Klauder, Gazeau, Piechocki, Bergeron, ...).

State space of a physical system

CONSTRUCTION of the STATE SPACE

- The Quantum Motions Algebras QMA(G) described below allow to construct directly quantum models which configuration/phase space can be parametrized by the group of motions G.
- As a quantization approach QMA(G) belongs to the class (D= Deformation of a measure) of the quantization methods.

AG., A. Pędrak, in preparation.

QMA(G)= QMA'(G)+QMA"(G)

$S \in \mathrm{QMA(G)}$ is a pair of two complex functions $\mathrm{G} \to \mathbb{C}$: the function $f \in L^1(\mathrm{G}, d\mu(g))$ and the discrete function $\check{\tau} \in l^1(\mathrm{G})$:

$$S := f + \check{\tau} \equiv \check{\tau} + f, \quad S(g) := f(g) \dot{+} \check{\tau}(g)$$
$$(S_1 + S_2)(g) := (f_1 + f_2)(g) \dot{+} (\check{\tau}_1 + \check{\tau}_2)(g)$$

The symbol $+$ is interpreted as additive operation in QMA(G) and in QM means "quantum interference" of both elements.
$\dot{+}$ separate numerical values of $L^1(\mathrm{G}, d\mu(g))$ and $l^1(\mathrm{G})$ functions.

Multiplication

The multiplication in QMA(G) is defined as the convolution.

QMA(G) algebra is isomorphic to a subalgebra of measures on G.

QMA(G)

The composition rules:

$$
\begin{aligned}
(f_1 \circ f_2)(g) &:= \int_G d\mu(h) f_1(h) f_2(h^{-1}g) \\
(\check{\tau}_1 \circ \check{\tau}_2)(g) &:= \sum_{h \in \mathrm{G}} \check{\tau}_1(h) \check{\tau}_2(h^{-1}g) \\
(\check{\tau} \circ f)(g) &:= \sum_{h \in \mathrm{G}} \check{\tau}(h) f(h^{-1}g) \\
(f \circ \check{\tau})(g) &:= \sum_{h \in \mathrm{G}} \Delta_G(h^{-1}) f(gh^{-1}) \check{\tau}(h)
\end{aligned}
$$

The involution operation is defined as

$$
\begin{aligned}
f^{\sharp}(g) &:= \Delta_G(g^{-1}) f(g^{-1})^{\star} \\
\check{\tau}^{\sharp}(g) &:= \check{\tau}(g^{-1})^{\star}
\end{aligned}
$$

Wave packets realization of QMA(G)

$$\hat{S} = \int_G d\mu(g) f(g) T(g) + \sum_{g \in G} \tau(g) T(g) \ ,$$

$T(g)$ represents here a unitary representation (action) of the group G in the state space $\mathcal{K}_{phys}$.
The discrete sum over the group ensures (it is important in physical applications) that the state from which the quantum state space is generated also belongs to this state space, though it is not always needed.

State space, GNS construction

Elementary probability amplitude (positive function, metastate):

$$\begin{aligned}
&\langle \rho ; \rangle : \mathrm{G} \to \mathbb{C} \\
&\langle \rho ; g \rangle = \mathrm{Tr}(\rho g) \text{ where } \rho \text{ is the density operator}
\end{aligned}$$

Extension to the full algebra

$$\langle \rho ; S \rangle = \langle \rho ; f + \check{\tau} \rangle = \int_{\mathrm{G}} d\mu(g) f(g) \langle \rho ; g \rangle + \sum_{g \in \mathrm{G}} \check{\tau}(g) \langle \rho ; g \rangle$$

The module of zero elements:

$$I_\rho := \left\{ z \in \mathrm{QMA}(G) \colon \left\langle \rho ; R^\sharp \circ z \right\rangle = 0, \text{ for all } R \in \mathrm{QMA}(G) \right\}$$

State space, GNS construction

Physical state space: $\mathcal{K} = \mathrm{QMA(G)}/I_\rho$ (quotient space) with the scalar product

$$\langle S_1|S_2\rangle := \left\langle \rho; S_1^\sharp \circ S_2 \right\rangle$$
$$\langle g_1|g_2\rangle = \left\langle \rho; g_1^\sharp \circ g_2 \right\rangle \;,$$

E.g.,

$$\langle f_2|f_1\rangle = \int_{\mathrm{G}} d\mu(g_2) \int_{\mathrm{G}} d\mu(g_1) f_2(g_2)^\star \left\langle \rho; g_2^{-1} g_1 \right\rangle f_1(g_1)$$

Every state $|S\rangle = S|e_G\rangle$ can be generated from the single state $|e_G\rangle$, where $e_G \in \mathrm{G}$ is the group neutral element.

The set of states $|g\rangle = g|e_G\rangle$, $g \in \mathrm{G}$ is overcomplete.

State space, interpretation (?)

Quantum spacetime states

- The vector $|e_G\rangle$ is the cyclic (fiducial) vector in the algebra QMA(G).
- The above means that every state of the spacetime belonging the state space $\mathcal{K}$ can be written as $|S\rangle = S|e_G\rangle$, where $S \in \mathrm{QMA(G)}$.
- The scalar product $\langle S_2|S_1\rangle$ represents the transition probability amplitude between quantum spacetime states $|S_1\rangle \to |S_2\rangle$.

State space, interpretation

Special spacetime states

- (H) The set of states $\{|g\rangle = g|e_G\rangle\}$ represents the degenerate vacuum state of the spacetime (or in general, the quantum representation of the configuration/phase space).
- The states $|g\rangle = g|e_G\rangle$ can be seen as the "quantum spacetime points".
- Every spacetime state can be expressed as a generalized linear combinations of the quantum spacetime points $|g\rangle$.
- Every unitary transformation mapping $\{|g\rangle = g|e_G\rangle\}$ into a new overcomplete set gives another equivalent set of the vacuum spacetime states (probably much larger set of transformations leads to the same conclusion).

COHERENT STATE QUANTIZATION

$\mathrm{QMA}'(\mathrm{G}) \subset \mathrm{QMA}(\mathrm{G})$

If the metastate $\langle \rho; \rangle$ leads to an irreducible representation, then

$$\frac{1}{A_\phi}\int_{\mathrm{G}} d\mu(g)|g\rangle\langle g| = \hat{\mathbb{1}}$$

$$M(\Omega) := \frac{1}{A_\phi}\int_{\Omega} d\mu(g)|g\rangle\langle g|,\ \text{where } \Omega \subset \mathrm{G},$$ imprimitivity, covariance

If the metastate $\langle \rho; \rangle$ do not leads to an irrep:

A) This can be interpreted as lack of important properties of the spacetime points vacuum states. One can to extend the group of motion G and the corresponding metastate $\langle \rho; \rangle$ to larger group to obtain irreducible representation.

B) The metastate can be decomposed into irreducible components and create decomposition of "unity" for every component independently.

C) The decomposition of unity can be postponed.

CS quantization

Let $\chi : \mathrm{CCS} \to \mathrm{G}$, one-to-one mapping,
where CCS =Classic Configuration Space/ Phase Space.
CS quantization of the classic function $f(q) = f(\chi^{-1}(g))$:

$$\hat{f} := \lim \sum_k f(\chi^{-1}(g_k)) M(\Omega_k) = \frac{1}{A_\phi} \int_{\mathrm{G}} d\mu(g) |g\rangle f(\chi^{-1}(g)) \langle g|$$

where $g_k \in \Omega_k \subset \mathrm{G}$ and $\bigcup_k \Omega_k = G$.

The quantum operator $\hat{f}$ modifies amplitudes of decomposition of the spacetime state Ψ into vacuum spacetime points giving a new state Ψ' of the spacetime

$$|\Psi'\rangle = \hat{f}|\Psi\rangle = \frac{1}{A_\phi} \int_{\mathrm{G}} d\mu(g) \langle g|\Psi\rangle f(\chi^{-1}(g)) |g\rangle$$

CS quantization

1st non-uniqueness

The CS quantization is dependent on parametrization of the group G, i.e., on a choice of the mapping $\chi : \mathrm{CCS} \rightarrow \mathrm{G}$.

A.G., W. Piechocki, T. Schmitz, Eur. Phys. J. Plus(2021) 136:18

2nd non-uniqueness

There is a freedom in fixing the element $\chi(q) = e_G$ of the map χ from the classic configuration space to the group parameters space.

A.G., A. Pędrak, W. Piechocki, Class. Quantum Grav. 39 (2022) 145005

Time as an observable

QUANTUM TIME

Time as an observable

Time

Quantum Gravity and also Quantization of Gravity require time to be considered on the same footing as other observables.

Quantum time

Quantum time should be a part of the spacetime position quantum observable.

Possible solution:

Projection evolution, PEv

A.G., M. Góźdź and A. Pędrak, Projection evolution of quantum states, arXiv:1910.11198v2 [quant-ph]

Projection evolution

The changes principle:

The evolution of a system is a random process caused by the spontaneous changes in the Universe.

The projection evolution operators at the evolution step τ_n are defined as a family of transformations:
$\digamma(\tau_n; \nu, \cdot) : \mathcal{T}_1^+(\mathcal{K}(\tau_{n-1})) \to \mathcal{T}^+(\mathcal{K}(\tau_n))$, where
$\mathcal{T}^+(\mathcal{K}(\tau_n))$ is the quantum state space at the evolution step τ_n.

τ_n enumerates subsequent changes of quantum states –
it is a global parameter – it is not TIME !

Quantum time

PEv approach requires to treat time as a quantum observable, i.e., quantum time is considered on the same footing as other degrees of freedom.

Projection evolution, chooser

The generalized Lüders projection postulate is proposed as the principle for the evolution (chooser):

$$\rho(\tau_n;\nu_n) = \frac{\text{₣}(\tau_n;\nu_n,\rho(\tau_{n-1};\nu_{n-1}))}{\mathrm{Tr}\left(\text{₣}(\tau_n;\nu_n,\rho(\tau_{n-1};\nu_{n-1}))\right)}.$$

Probability distribution

The probability distribution for the chooser is given by the quantum mechanical transition probability from the previous to the next state.

This probability for pure quantum states is determined by the appropriate probability amplitudes in the form of scalar products. The transition probability among mixed states remains an open problem.

₣ are some quantum operations, e.g., K. Krauss: States, Effects and Operations: Fundamental Notions of Quantum Theory, Springer Verlag 1983.

Main ingredients

Elementary quantum observables

Let us assume the classical spacetime position 4-vector in the CCS is represented by a set of functions

$$x^\mu = x^\mu(q), \text{ where } \mu = 0, 1, 2, 3.$$

Within the CS quantization the spacetime position four vector operator is

$$\hat{x}^\mu := \frac{1}{A_\phi} \int_{\mathrm{G}} d\mu(g) |g\rangle x^\mu(\chi^{-1}(g)) \langle g|$$

Other quantized observables

$$\hat{f} := \lim \sum_k f(\chi^{-1}(g_k)) M(\Omega_k) = \frac{1}{A_\phi} \int_{\mathrm{G}} d\mu(g) |g\rangle f(\chi^{-1}(g)) \langle g|$$

Main ingredients

Spectrum and eigenvectors of the observable A.

Probability related to the measure $M_A(\Omega)$ of the observable A

$$\mathrm{Prob}\,(\rho; A, \Omega) = \mathrm{Tr}(M_A(\Omega)\rho)$$

Expectation value of an observable A

$$\langle \rho; A \rangle = \mathrm{Tr}(\hat{A}\rho)$$

Variance of an observable A

$$\mathrm{var}(\rho; A) = \mathrm{Tr}((\hat{A} - \langle \rho; A \rangle)^2 \rho)$$

Simplifications

PHENOMENOLOGY

Semiquantal approach

Heuristic method

To not consider the full PEv approach which requires a construction of the evolution operators one can try, in the first approximation, to search for the states of the spacetime calculating expectation values of quantized classical observables and compare them to classical solutions.

Semiquantal approach

- Assume, we have a set of characteristic classical spacetime observables:
  - spacetime positions x^μ and their quantized versions $\hat{x}^\mu$;
  - $\xi_l = \phi_l(\{x^\mu\})$ and their quantized versions $\hat{\xi}_l$;

  as constraints determining construction of required spacetime states.
- One needs to choose a family of trial states $|\Psi_\eta\rangle$. The set of parameters η enumerate this family of states.
- To determine required approximate states, i.e., to find η, one needs to solve the following system of equations

$$\langle \Psi_\eta; \hat{x}^\mu \rangle = \langle \Psi_\eta | \hat{x}^\mu | \Psi_\eta \rangle = x^\mu$$

$$\left\langle \Psi_\eta; \hat{\xi}_l \right\rangle = \langle \Psi_\eta | \hat{\xi}_l | \Psi_\eta \rangle = \phi_l(\{x^\mu\})$$

Note that $\phi_l(\{x^\mu\}) = \phi_l(\langle \Psi_\eta; \hat{x}^\mu \rangle)$

Semiquantal approach

cont.

- Having states $|\Psi_\eta\rangle$ one can calculate
  - required probability distributions
  - expectation values
  - variances

  of remaining observables .

Remarks:

- Constraints are reproduced exactly, other quantum observables can differ from their classic counterparts.
- Calculated variances of constraints show smearing of quantum observables around their classical values.
- Nonzero variances prevent the corresponding quantum observables to be singular in the points where the classical observables have singularities.

SUMMARY

- (The group of motions G) + (Elementary probability amplitude function $\langle \rho(\tau); g\rangle$) creates (the τ dependent state space $\mathcal{K}_\tau$ which correspond to evolving spacetime).
- Quantum evolution is a stochastic process (PEv).
- Background independence – in general one has $\{|\tau; g\rangle\}$.
- Treating the quantum time on the same footing as the other observables (PEv).
- Deformation of quantum measure as the quantization method (CS).
- A possibility of construction of approximate spacetime states reproducing, to some extend, classical properties of the spacetime.
- Possibility of avoiding of singularities in quantized description of the spacetime.

Problems

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