Quantum chaos of the BKL scenario

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Based on:

[1] P. Goldstein and W.P.,
"Generic instability of the dynamics underlying the Belinski-Khalatnikov-Lifshitz scenario",
Eur. Phys. J. C (2022) 82: 216.

[2] A. Góźdź, A. Pędrak, and W.P.,
"Quantum dynamics corresponding to the classical BKL scenario", arXiv:2204.11274 [gr-qc].

[3] V. Belinski and W. P., collaboration since 2010.

OUTLINE

Introduction

- Friedmann model (1922)
  - assumes isotropy and homogeneity of space
  - solution includes gravitational singularity
  - commonly used in astrophysics and cosmology
- Lifshitz analysis (1946): isotropy is unstable in the evolution towards singularity

  E. M. Lifshitz, J. Phys., U. S. S. R. **10**, 116 (1946); E. M. Lifshitz and I. M. Khalatnikov, Adv. Phys. **12**, 185 (1963)
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- Dynamics of BVIII and BIX was analyzed to get insight into the dynamics of spacetime near the cosmological spacelike singularity

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- BKL conjecture states: general relativity implies the existence of generic solution that is singular
  - corresponds to non-zero measure subset of all initial data
  - is stable against perturbation of initial data
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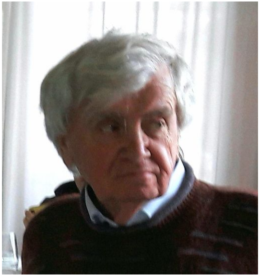
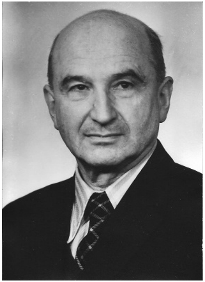


Vladimir Alekseevich Belinski

Isaak Markovich Khalatnikov



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  - the Lorenzian hyperbolic Kac-Moody algebra underlies asymptotic structure of spacetime near cosmological singularity

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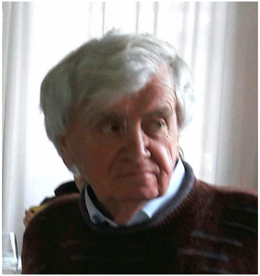

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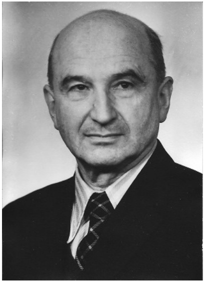

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BKL conjecture and singularity theorems

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- These theorems say little about the dynamics of gravitational field near singularities so that are of little usefulness in the context of finding corresponding quantum dynamics.
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Dynamics underlying BKL scenario

The asymptotic form (near the singularity) of the dynamical equations of general Bianchi VIII and IX models

Derived by V. Belinski, I. Khalatnikov, and M. Ryan in 1971; E. Czuchry and W. P., Phys. Rev. D **87**, 084021 (2013)

$$\frac{d^2 \ln a}{dt^2} = \frac{b}{a} - a^2, \quad \frac{d^2 \ln b}{dt^2} = a^2 - \frac{b}{a} + \frac{c}{b}, \quad \frac{d^2 \ln c}{dt^2} = a^2 - \frac{c}{b}, \tag{1}$$

where $a = a(t),\ b = b(t),\ c = c(t)$ are effective directional scale factors, and t is a monotonic function of proper time.

The solutions to (1) must satisfy the constraint

$$\frac{d \ln a}{dt}\frac{d \ln b}{dt} + \frac{d \ln a}{dt}\frac{d \ln c}{dt} + \frac{d \ln b}{dt}\frac{d \ln c}{dt} = a^2 + \frac{b}{a} + \frac{c}{b}. \tag{2}$$

Equations (1)-(2) present the essence of the dynamics underlying the BKL scenario.

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Equations (1)-(2) present the essence of the dynamics underlying the BKL scenario.

BKL scenario (cont)

BKL scenario has numerical support (via considering general BIX):

- Simulations of the dynamics near the singularity confirm the asymptotic dynamics

  C. Kiefer, N. Kwidzinski, and W.P., Eur. Phys. J. C (2018) 78:691

- The Kretschman curvature invariant diverge in the evolution towards the singularity

  N. Kwidzinski and W.P., Eur. Phys. J. C (2019) 79:199

BKL scenario (cont)

Remark!

- BKL scenario results from considering the dynamics of BVIII and BIX which include some contributions from matter fields (corresponding metrics are nondiagonal);
- dynamics of the mixmaster universe concerns vacuum BIX (corresponding metric is diagonal);
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Solution to the BKL scenario

We have found analytical solution to the dynamics (1)–(2):

P. Goldstein and W.P., Eur. Phys. J. C (2022) 82: 216

$$\tilde{a}(t) = \frac{3}{t - t_0}, \quad \tilde{b}(t) = \frac{30}{(t - t_0)^3}, \quad \tilde{c}(t) = \frac{120}{(t - t_0)^5}\,, \tag{3}$$

where $t > t_0$ and where t_0 is an arbitrary real number.
This is the only solution in the Painlevé sense.
The solution (3) is unstable against small perturbation:

$$a(t) = \tilde{a}(t) + \epsilon\alpha(t), \tag{4a}$$

$$b(t) = \tilde{b}(t) + \epsilon\beta(t), \tag{4b}$$

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Solution to the BKL scenario (cont)

Inserting (4) into (1)–(2) leads, in the first order in ϵ, to the following solution of the resulting equations:

$$\alpha(t) = \exp(-\theta/2)\big[K_1 \cos(\omega_1\theta+\varphi_1)+K_2 \cos(\omega_2\theta+\varphi_2)\big]+K_3 \exp(-2\theta), \tag{5a}$$

$$\beta(t) = \exp(-5\theta/2)\big[(4+6\sqrt{6})K_1 \cos(\omega_1\theta+\varphi_1) \tag{5b}$$

$$+ (4-6\sqrt{6})K_2 \cos(\omega_2\theta+\varphi_2)\big] + 30K_3 \exp(-4\theta), \tag{5c}$$

$$\gamma(t) = -4\exp(-9\theta/2)\big[(26+9\sqrt{6})K_1 \cos(\omega_1\theta+\varphi_1) \tag{5d}$$

$$+ (26-9\sqrt{6})K_2 \cos(\omega_2\theta+\varphi_2)\big] + 200K_3 \exp(-6\theta)\,, \tag{5e}$$

where $\theta = \ln(t - t_0)$. The two frequencies read

$$\omega_1 = \frac{1}{2}\sqrt{95-24\sqrt{6}}, \qquad \omega_2 = \frac{1}{2}\sqrt{95+24\sqrt{6}}\,, \tag{6}$$

where K_1, K_2, K_3, φ_1, and φ_2 are constants.

Chaotic phase of the BKL scenario

- The manifold $\mathcal{M}$ defined by $\{K_1, K_2, K_3, \varphi_1, \varphi_2\}$ is a submanifold of $\mathbb{R}^5$. Thus, (5) presents generic solution as the measure of $\mathcal{M}$ is nonzero.
- The relative perturbations α/a, β/b, and γ/c grow as $\exp(\frac{1}{2}\theta)$.
  - The multiplier $1/2$ plays the role of a Lyapunov exponent, describing the rate of their divergences.
  - Since it is positive, the evolution of the system towards the gravitational singularity ($\theta \to +\infty$) becomes chaotic.
- We expect that the solution to (1)–(2) for any initial condition is chaotic as well.
- Stochasticity results from strong nonlinearity of the dynamics and growing curvature of spacetime in the evolution towards singularity.

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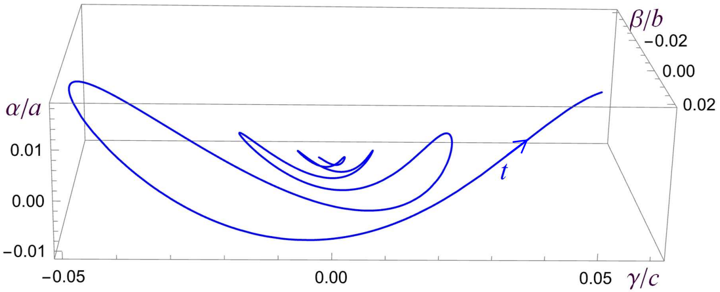

Figure: Linear instability of the special solution (3) for $K_1 = K_2 = 0.01$, $K_3 = 0$, $\varphi_1 = \varphi_2 = 0$ for the perturbations of the scale factors. The graph presents the parametric curve defined by the time dependence of α/a, β/b, and γ/c.

Quantization of the BKL scenario

In what follows, we quantize BKL scenario by making use of the coherent states quantization method

For details, see talks by Ola and Andrzej, or App. A.

We have already quantized Hamilton's dynamics of that scenario ignoring its chaotic phase

- quantum singularity turns into quantum bounce
- quantum evolution is unitary across quantum bounce

A. Góźdź, W.P., and G. Plewa, Eur. Phys. J. C **79**, 45 (2019); A. Góźdź and W.P., Eur. Phys. J. C **80**, 142 (2020)

Quantization of the chaotic phase of the BKL scenario:

- we do not quantize Hamilton's dynamics, but the solution to the BKL scenario
- we quantize both temporal and spatial variables to support general covariance of GR with respect to transformations of these variables

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Quantization of the BKL scenario (cont)

Outline of calculations

For details, see: A. Góźdź, A. Pędrak, and W.P., arXiv:2204.11274 [gr-qc].

- We calculate expectation values and variances (see, App B) of quantum observables corresponding to perturbed $\{a, b, c\}$ and unperturbed $\{\tilde{a}, \tilde{b}, \tilde{c}\}$ solutions to be compared.
- Expectation value of time operator is required to coincide with classical time.
- As quantum states, we take wave packets defined in the carrier (Hilbert) space of affine group representation.

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Stochastic aspects of quantum evolution

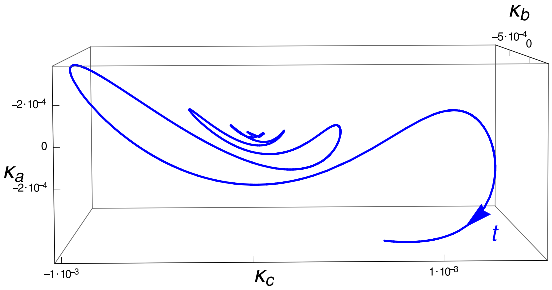

Figure: Parametric curve of relative quantum perturbations

$$\kappa_k := \frac{var(\hat{\xi}_k;\Psi_{pert}) - var(\hat{\xi}_k;\Psi_{unpert})}{var(\hat{\xi}_k;\Psi_{unpert})}, \qquad k = a, b, c \tag{7}$$

where $\hat{\xi}_a := \hat{a}$, $\hat{\xi}_b := \hat{b}$, $\hat{\xi}_c := \hat{c}$.

Conclusions

- The relative quantum perturbations grow as the system evolves towards the singularity.
- The quantum randomness amplifies the deterministic classical chaos.
- Hypothesis: In the region corresponding to the neighbourhood of the classical singularity the dynamics, both classical and quantum, enter the stochastic phase.
- Variances describe quantum smearing of observables;
  as calculated variances are always non-zero, the probability of obtaining divergencies corresponding to gravitational singularity is equal to zero.

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Thank you!

Essence of ACS quantization

The affine configuration space Π is a half-plane:

$$\Pi := \{(q,p) \in \mathbb{R} \times \mathbb{R}_+\}, \quad \mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\}.$$

Π can be identified with the affine group Aff($\mathbb{R}$).
This group has UIR realized in the Hilbert space $L^2(\mathbb{R}_+, d\nu(x))$, where $d\nu(x) = dx/x$, defined by

$$U(q,p)\psi(x) = e^{iqx}\psi(px).$$

This enables defining the continuous family of affine coherent states $|q,p\rangle \in L^2(\mathbb{R}_+, d\nu(x))$ as follows

$$|q,p\rangle = U(q,p)|\phi\rangle \ ,$$

where $|\phi\rangle \in L^2(\mathbb{R}_+, d\nu(x))$, is the so-called fiducial vector, which is a free parameter of ACS quantization scheme.

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This group has UIR realized in the Hilbert space $L^2(\mathbb{R}_+, d\nu(x))$, where $d\nu(x) = dx/x$, defined by

$$U(q,p)\psi(x) = e^{iqx}\psi(px).$$

This enables defining the continuous family of affine coherent states $|q,p\rangle \in L^2(\mathbb{R}_+, d\nu(x))$ as follows

$$|q,p\rangle = U(q,p)|\phi\rangle \ ,$$

where $|\phi\rangle \in L^2(\mathbb{R}_+, d\nu(x))$, is the so-called fiducial vector, which is a free parameter of ACS quantization scheme.

Essence of ACS quantization

The affine configuration space Π is a half-plane:

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Essence of ACS quantization (cont

The irreducibility of the representation leads (due to Schur' lemma) to the resolution of the unity in $L^2(\mathbb{R}_+, d\nu(x))$:

$$\int_\Pi d\mu(q,p)|q,p\rangle\langle q,p| = A_\phi \, \mathbb{I} \,, \tag{8}$$

where $d\mu(q,p) := dq\, dp/p^2$ is the left invariant measure on Π, and where $A_\phi := \int_0^\infty |\phi(x)|^2 \frac{dx}{x^2} < \infty$ is a constant.

Using (8), enables quantization of any observable $f : \Pi \to \mathbb{R}$

$$f \longrightarrow \hat{f} = \frac{1}{A_\phi} \int_\Pi d\mu(q,p)|q,p\rangle f(q,p) \langle q,p| \,. \tag{9}$$

The operator $\hat{f}$ is symmetric (Hermitian) by construction.
No ordering ambiguity occurs (disaster of canonical quantization).

A. Góźdź, W.P., and T. Schmitz, Eur. Phys. J. Plus (2021) 136:18

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Variance of quantum observable

Variance is a stochastic deviation from expectation value of quantum observable; it determines the value of smearing of quantum observable.
The variance is the average of the squared differences from the mean. In the quantum state labelled by ψ, the variance is defined to be

$$var(\hat{A};\psi) := \langle(\hat{A} - \langle\hat{A};\psi\rangle)^2;\psi\rangle = \langle\hat{A}^2;\psi\rangle - \langle\hat{A};\psi\rangle^2 , \qquad (10)$$

where $\langle\hat{B};\psi\rangle := \langle\psi|\hat{B}|\psi\rangle$.
If $\hat{A}$ is a self-adjoint operator, we have the important statement:

$$\Big(var(\hat{A};\psi) = 0\Big) \Longleftrightarrow \Big(\hat{A}\psi = \lambda\psi, \quad \lambda \in \mathbb{R}\Big) , \qquad (11)$$

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