3d gravity, point particles and deformed symmetries

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[a]J. Kowalski-Glikman, G. Rosati & **T. T.**, *work in progress*
J. Kowalski-Glikman, J. Lukierski & **T. T.**, JHEP **09**, 096 (2020)
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Outline:

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1. Introduction

2. Chern-Simons theory coupled to particles
   - Local isometry groups and spinning conical defects
   - Effective particle actions and properties of particles

3. Deformations of symmetries in 3D gravity

Context and motivation

- Relative simplicity of 3D gravity allows it to be a testing ground
- Degrees of freedom are introduced by either a nontrivial topology or topological defects describing point particles[a]
- It can be formulated as the Chern-Simons gauge theory for an appropriate local isometry group[b]
- $\mathrm{AN}(2)$ group plays a distinguished role here but is it also true for the associated κ-Poincaré algebra[c]?
- (Fock-Rosly) quantization of the theory leads to a Hopf-algebraic deformation of the gauge group[d]
- Classification of Hopf algebras of 3D symmetries for any Λ has recently been completed[e]

[a] Staruszkiewicz, Acta Phys. Pol. **24**, 735 (1963); Deser & Jackiw, Ann. Phys. **153**, 405 (1984)
[b] Witten, NPB **311**, 46 (1988); Meusburger & Schroers, NPB **806**, 462 (2009)
[c] Lukierski, Nowicki, Ruegg, PLB **293**, 344 (1992)
[d] Fock & Rosly, AMS Transl. **191**, 67 (1999)
[e] Borowiec, Lukierski & Tolstoy, JHEP **1711**, 187 (2017)

Chern-Simons action of 3D gravity (with Λ)

Instead of the metric $g_{\alpha\beta}$, gravity can be described in terms of the vielbein $e_\alpha^{\ \mu}$ and spin connection $\omega_\alpha^{\ \mu\nu}$, defined as

$$e_\alpha^{\ \mu} e_\beta^{\ \nu} \eta_{\mu\nu} = g_{\alpha\beta}\,, \qquad \omega_\alpha^{\ \mu\nu} = e_\beta^{\ \mu} \partial_\alpha e^{\beta\nu} + e_\beta^{\ \mu} \Gamma^\beta_{\ \alpha\gamma} e^{\gamma\nu}\,. \qquad (1)$$

In (2+1)D, they neatly combine into a gauge field – with values in the local isometry algebra $\mathfrak{g}$ (3D Poincaré or (Anti-)de Sitter) – which is the Cartan connection

$$A = -\tfrac{1}{2}\epsilon^\mu_{\ \nu\sigma}\omega_\alpha^{\ \nu\sigma} J_\mu dx^\alpha + e_\alpha^{\ \mu} P_\mu dx^\alpha\,, \qquad (2)$$

where J_μ, P_μ are generators of $\mathfrak{g}$. Then, the Einstein-Hilbert action can be written as the Chern-Simons theory action

$$S = \frac{1}{16\pi G}\int \left(\langle dA \wedge A\rangle + \frac{1}{3}\langle A \wedge [A, A]\rangle\right) \qquad (3)$$

if the scalar product on $\mathfrak{g}$ is given by

$$\langle J_\mu, P_\nu\rangle = \eta_{\mu\nu}\,, \qquad \langle J_\mu, J_\nu\rangle = \langle P_\mu, P_\nu\rangle = 0\,. \qquad (4)$$

How to couple a conical defect

The massive point-particle spacetime interval looks as in vacuum:

$$ds^2 = (1 - \Lambda r^2)\, dt^2 - (1 - \Lambda r^2)^{-1} dr^2 - r^2 d\tilde{\phi}^2\,, \qquad (5)$$

when the polar angle is rescaled to $\tilde{\phi} := (1 - 4Gm)\,\phi$ (a conical defect). Similarly, spin $\neq 0$ introduces a jump in the time coordinate.

If spacetime has the topology $\mathbb{R} \times \mathcal{S}$, the field A may be expressed as $A = A_t dt + A_{\mathcal{S}}$ and the action of gravity with a particle is

$$S = \int dt\, L = \frac{1}{16\pi G} \int dt \int_{\mathcal{S}} \left\langle \dot{A}_{\mathcal{S}} \wedge A_{\mathcal{S}} \right\rangle - \int dt \left\langle c_0 h^{-1} \dot{h} \right\rangle + \int dt \int_{\mathcal{S}} \left\langle A_t \left(\frac{1}{8\pi G} F_{\mathcal{S}} - h c_0 h^{-1} \delta^2(\vec{x} - \vec{x}_*)\, dx^1 \wedge dx^2 \right) \right\rangle . \qquad (6)$$

Mass $\neq 0$ and spin of a particle are encoded by $\mathfrak{g} \ni c_0 = m\, J_0 + s\, P_0$, while a gauge group element h acting via $h c_0 h^{-1} = \mathbf{p} + \mathbf{j}$ determines its momentum $\mathbf{p} = p^\mu J_\mu$ and (generalized) angular momentum $\mathbf{j} = j^\mu P_\mu$.

Relating the gravitational and particle DOFs

A_t acts as a Lagrange multiplier imposing a constraint on the curvature of spatial connection $F_\mathcal{S} = dA_\mathcal{S} + [A_\mathcal{S}, A_\mathcal{S}]$:

$$F_\mathcal{S} = 8\pi G\, hc_0 h^{-1} \delta^2(\vec{x} - \vec{x}_*)\, dx^1 \wedge dx^2 \,. \tag{7}$$

From $F_\mathcal{S} = R_\mathcal{S} + T_\mathcal{S} + C_\mathcal{S}$ ($C_\mathcal{S}$ is the cosmological-constant term), it follows that the spatial Riemann curvature and torsion are given by

$$R_\mathcal{S} = -C_\mathcal{S} + 8\pi G\, \mathbf{p}\, \delta^2(\vec{x} - \vec{x}_*)\, dx^1 \wedge dx^2 \,,$$
$$T_\mathcal{S} = 8\pi G\, \mathbf{j}\, \delta^2(\vec{x} - \vec{x}_*)\, dx^1 \wedge dx^2 \,, \tag{8}$$

i.e. they vanish (on the constant background $R_\mathcal{S} = -C_\mathcal{S}$) everywhere except a singularity at the particle's worldline.
Alternatively, $\mathfrak{g}$ can be equipped with the scalar product

$$\langle J_\mu, P_\nu \rangle = 0 \,, \qquad \langle J_\mu, J_\nu \rangle = -\Lambda^{-1} \langle P_\mu, P_\nu \rangle = \eta_{\mu\nu} \,. \tag{9}$$

If our action is defined using it, **j** generates $R_\mathcal{S}$ and **p** generates $T_\mathcal{S}$.

Structure of the gauge algebra

- For any value of Λ, the brackets of $\mathfrak{g}$ have the form

$$[J_\mu, J_\nu] = \epsilon_{\mu\nu}{}^\sigma J_\sigma\,, \quad [J_\mu, P_\nu] = \epsilon_{\mu\nu}{}^\sigma P_\sigma\,, \quad [P_\mu, P_\nu] = -\Lambda\, \epsilon_{\mu\nu}{}^\sigma J_\sigma\,. \qquad (10)$$

- The identification $P_\mu \equiv \theta J_\mu$, $\theta^2 = -\Lambda$ allows to express each algebra as an extension of $\mathfrak{o}(2,1)$ over the ring $a + \theta b$, $a, b \in \mathbb{R}$.

- Introducing a vector $\mathbf{n} \in \mathbb{R}^{2,1}$ (timelike for de Sitter, lightlike for Poincaré or spacelike for anti-de Sitter), one may define

$$S_\mu := P_\mu + \epsilon_{\mu\nu}{}^\sigma n^\nu J_\sigma\,, \quad \mathbf{n}^2 = \Lambda\,, \qquad (11)$$

which are generators of the $\mathfrak{an}_\mathbf{n}(2)$ algebra (the same algebra as for κ-Minkowski space), while J_μ generate $\mathfrak{o}(2,1)$:

$$[J_\mu, J_\nu] = \epsilon_{\mu\nu}{}^\sigma J_\sigma\,, \qquad [J_\mu, S_\nu] = \epsilon_{\mu\nu}{}^\sigma S_\sigma + n_\nu J_\mu - \eta_{\mu\nu} n^\sigma J_\sigma\,,$$
$$[S_\mu, S_\nu] = n_\mu S_\nu - n_\nu S_\mu\,. \qquad (12)$$

Structure of the gauge group

- Instead of 3D Lorentz group $\mathrm{SO}^{\uparrow}(2,1)$, we can use its double cover $\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SU}(1,1)$.
- For each gauge group G, $g \in G$ has the Iwasawa decomposition

$$g = \mathfrak{u}\,\mathfrak{s} \;\in\; \mathrm{SL}(2,\mathbb{R}) \bowtie \mathrm{AN}_{\mathbf{n}}(2)\,, \tag{13}$$

under the condition $s_3 + \frac{1}{2}\mathbf{n}\cdot\mathbf{s} > 0$, and/or

$$g = \mathfrak{r}\,\mathfrak{v} \;\in\; \mathrm{AN}_{\mathbf{n}}(2) \bowtie \mathrm{SL}(2,\mathbb{R})\,, \tag{14}$$

under the condition $r_3 - \frac{1}{2}\mathbf{n}\cdot\mathbf{r} > 0$. In the case of Poincaré group with $\mathbf{n} = \mathbf{0}$, $\mathrm{AN}_{\mathbf{n}}(2)$ becomes $\mathbb{R}^{2,1}$.

- The above conditions are given in the "quaternionic" parametrization:

$$\mathfrak{u} = u_3\mathbb{1} + u^\mu J_\mu\,, \quad u_3^2 = 1 - \tfrac{1}{4}\mathbf{u}^2\,,$$
$$\mathfrak{s} = s_3\mathbb{1} + s^\mu S_\mu\,, \quad s_3^2 = 1 + \tfrac{1}{4}(\mathbf{n}\cdot\mathbf{s})^2\,,$$

and analogously for $\mathfrak{v}$ and $\mathfrak{r}$.

Representations of the algebra and group

$\mathrm{SL}(2,\mathbb{R})$ can be expressed in a 2×2 representation of the algebra:

$$J_0 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

while the representation of $\mathrm{AN}_{\mathbf{n}}(2)$ is obtained using the above and the relation $S_\mu = \theta J_\mu + \epsilon_{\mu\nu}{}^{\sigma} n^\nu J_\sigma$.

It is less effective to use the exponential map $g = \exp(\xi^\mu J_\mu)\exp(\varepsilon^\mu S_\mu)$ and a 4×4 representation of $\mathrm{SO}^{\uparrow}(2,1) \bowtie \mathrm{AN}_{\mathbf{n}}(2)$, e.g. (for $\Lambda > 0$):

$$J_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad S_0 = \sqrt{\Lambda}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$J_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad S_1 = \sqrt{\Lambda}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$J_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad S_2 = \sqrt{\Lambda}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Alekseev-Malkin construction

Let us decompose (closed) space $\mathcal{S}$ into a disc $\mathcal{D}$ with the particle and an empty region $\mathcal{E}$, with the common boundary Γ. From the curvature constraint, we find that the connection on $\mathcal{E}$ has the form

$$A^{(\mathcal{E})}_{\mathcal{S}} = \gamma d\gamma^{-1}\,, \tag{15}$$

while on $\mathcal{D}$ (with coordinates $\rho \in [0,1]$, $\phi \in [0,2\pi)$) it is

$$A^{(\mathcal{D})}_{\mathcal{S}} = \bar\gamma\, 4G\, c_0 d\phi\, \bar\gamma^{-1} + \bar\gamma d\bar\gamma^{-1}\,, \quad \bar\gamma(\rho\!=\!0) = h\,, \tag{16}$$

with $\gamma, \bar\gamma \in G$. The continuity of $A_{\mathcal{S}}$ across Γ, i.e. $A^{(\mathcal{D})}_{\mathcal{S}}|_\Gamma = A^{(\mathcal{E})}_{\mathcal{S}}|_\Gamma$, leads to the sewing condition

$$\gamma^{-1}|_\Gamma = \alpha\, e^{4G\, c_0 \phi} \bar\gamma^{-1}|_\Gamma\,, \quad d\alpha = 0\,. \tag{17}$$

We solve it for γ, using Iwasawa decomposition $\gamma = \mathfrak{u}\,\mathfrak{s}$, $\bar\gamma = \bar{\mathfrak{u}}\,\bar{\mathfrak{s}}$.

Effective particle Lagrangian

Consequently, we can express our Lagrangian as a boundary integral

$$L = \kappa \int_\Gamma \left\langle \partial_0 \left(\bar{\mathfrak{u}}^{-1}\mathfrak{u}\right) \mathfrak{u}^{-1}\bar{\mathfrak{u}} \left(d\bar{\mathfrak{s}}\,\bar{\mathfrak{s}}^{-1} - \bar{\mathfrak{s}}\,\frac{c_0}{2\pi\kappa} d\phi\,\bar{\mathfrak{s}}^{-1}\right) + \frac{c_0}{2\pi\kappa} d\phi\,\bar{\mathfrak{s}}^{-1}\dot{\bar{\mathfrak{s}}} \right\rangle,$$

where $\kappa \equiv \frac{1}{8\pi G}$, but the expression for $\mathfrak{u}$ is generally too complicated[a].

In the case of $\Lambda = 0$ (and $\mathbf{n} = \mathbf{0}$), we are able to integrate it over Γ and obtain the final Lagrangian

$$L = \kappa \left(\dot{\Pi}^{-1}\Pi\right)_\mu x^\mu + s\,\tfrac{1}{2}\epsilon_{0\mu}{}^{\nu}\dot{\Lambda}^\mu{}_\sigma(\bar{\mathfrak{u}}^{-1})\Lambda^\sigma{}_\nu(\bar{\mathfrak{u}})\,, \tag{18}$$

in terms of the particle's momentum $\Pi \equiv \bar{\mathfrak{u}}\, e^{\frac{m}{\kappa} J_0}\bar{\mathfrak{u}}^{-1} \in \mathrm{SL}(2,\mathbb{R})$ and position $\mathbf{x} \equiv \bar{\mathfrak{u}}\,\bar{\mathbf{s}}\,\bar{\mathfrak{u}}^{-1} \in \mathbb{R}^{2,1}$; moreover, $\Lambda^\mu{}_\nu(\bar{\mathfrak{u}})\, J_\mu := \bar{\mathfrak{u}}\, J_\nu \bar{\mathfrak{u}}^{-1}$ is a Lorentz transformation corresponding to $\mathfrak{u}$.

In the limit $G \to 0$, one recovers the free particle Lagrangian

$$L = p_\mu \dot{x}^\mu + s\,\tfrac{1}{2}\epsilon_{0\mu}{}^{\nu}\dot{\Lambda}^\mu{}_\sigma(\bar{\mathfrak{u}}^{-1})\Lambda^\sigma{}_\nu(\bar{\mathfrak{u}}) = \left\langle c_0 \alpha^{-1}\dot{\alpha}\right\rangle. \tag{19}$$

[a] **T. T.**, NPB **928**, 448 (2018)

Momenta of a gravitating particle

Parallel transport around the particle is described by the holonomy of connection $A_\mathcal{S}$ along the boundary Γ, which is a gauge group element

$$\mathcal{P}\, e^{\int_\Gamma A_\mathcal{S}} = \gamma(\phi=0)\,\gamma^{-1}(\phi=2\pi) = \Pi\left(\mathbb{1} + \tfrac{1}{\kappa}\Pi^{-1}\Upsilon\Pi\right)\,, \tag{20}$$

where $\Pi \in \mathrm{SL}(2,\mathbb{R})$, $\Upsilon \in \mathbb{R}^{2,1}$. The momentum manifold $\mathrm{SL}(2,\mathbb{R})$ is 3D anti-de Sitter space. Using the parametrization $\Pi = p_3\mathbb{1} + \frac{1}{\kappa}\,p^\mu J_\mu$, we uncover deformations of the mass shell condition

$$p_\mu p^\mu = 4\kappa^2 \sin^2 \tfrac{m}{2\kappa} \tag{21}$$

and angular momentum $\Upsilon = j^\mu P_\mu$,

$$j^\mu = p_3\, \epsilon^\mu{}_{\nu\sigma} x^\nu p^\sigma + \tfrac{1}{2\kappa}\left(x^\mu p_\nu p^\nu - x^\nu p_\nu p^\mu\right) + \tfrac{s}{m}\, p^\mu\,, \tag{22}$$

in contrast to the free particle case $j^\mu = \epsilon^\mu{}_{\nu\sigma} x^\nu p^\sigma + \frac{s}{m}\, p^\mu$. Variation of the action with respect to $\Lambda^\mu{}_\nu(\bar{\mathbf{u}})$ leads to the conservation law $\dot{j}_\mu = 0$.

Alekseev-Malkin construction for multiple particles

In the n-particle case, the action $S = \int dt\ L$ is given by the Lagrangian

$$L = \frac{1}{16\pi G} \int_{\mathcal{S}} \left\langle \dot{A}_{\mathcal{S}} \wedge A_{\mathcal{S}} \right\rangle - \sum_{i=1}^{n} \left\langle c_{(i)} h_i^{-1} \dot{h}_i \right\rangle + \int_{\mathcal{S}} \left\langle A_0 \left(\frac{1}{8\pi G} F_{\mathcal{S}} - \sum_{i=1}^{n} h_i c_{(i)} h_i^{-1} \delta^2(\vec{x} - \vec{x}_i)\, dx^1 \wedge dx^2 \right) \right\rangle , \quad (23)$$

where $c_{(i)} = m_{(i)} J_0 + s_{(i)} P_0$. Dividing space $\mathcal{S}$ into n particle discs $\mathcal{D}_i$ and the empty polygon $\mathcal{E}$, with the common boundary $\Gamma = \bigcup_i \Gamma_i$, we can follow the single particle example for $\Lambda = 0$ and derive the analogous (modulo an “interaction term”) effective Lagrangian

$$L = \sum_{i=1}^{n} \left(\kappa \left(\dot{\Pi}_i^{-1} \Pi_i \right)_\mu (\mathbf{x}_i)^\mu + s_{(i)} \tfrac{1}{2} \epsilon_{0\mu}{}^{\nu} \dot{\Lambda}^\mu{}_\sigma (\bar{\mathfrak{u}}_i^{-1}) \Lambda^\sigma{}_\nu (\bar{\mathfrak{u}}_i) - \left(\partial_0 (\Pi_{i-1}^{-1} \ldots \Pi_1^{-1}) \Pi_1 \ldots \Pi_{i-1} \right)_\mu (\Upsilon_i)^\mu \right) . \quad (24)$$

Properties of multiple gravitating particles

The holonomy obtained by circumventing $j \leq n$ particles is given by

$$\mathcal{P}\, e^{\int_{\Gamma(j)} A_S} = \gamma(\phi_1 = 0)\, \gamma^{-1}(\phi_j = 2\pi) = \Pi_1 \ldots \Pi_j \cdot \left(\mathbb{1} + \tfrac{1}{\kappa}\Pi_j^{-1} \ldots \Pi_1^{-1} \Upsilon_1 \Pi_1 \ldots \Pi_j + \ldots + \tfrac{1}{\kappa}\Pi_j^{-1} \Upsilon_j \Pi_j\right). \quad (25)$$

It is not invariant under permutations of particles $(g_i, g_{i+1}) \to (g_{i+1}, g_i)$ but under their right- or left-handed braids (here $g_i \equiv \Pi_i(\mathbb{1} + \Upsilon_i)$)

$$(g_i, g_{i+1}) \to (g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}),$$
$$(g_i, g_{i+1}) \to (g_i g_{i+1} g_i^{-1}, g_i). \quad (26)$$

However, we still obtain the free-particle equations of motion

$$\dot{x}^\mu_{(i)} = \lambda_{(i)} \cos \frac{m_{(i)}}{2\kappa}\, p^\mu_{(i)}, \qquad \dot{p}^\mu_{(i)} = 0, \quad (27)$$

where $\lambda_{(i)}$ are Lagrange multipliers.

Limit of pseudo-Carroll particles

The theory with Poincaré gauge group is also recovered in the limit $\mathrm{SL}(2,\mathbb{R}) \bowtie \mathrm{AN}_{\mathbf{n}}(2) \to \mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^{2,1}$.
On the other hand, the limit $\mathrm{SL}(2,\mathbb{R}) \bowtie \mathrm{AN}_{\mathbf{n}}(2) \to \mathbb{R}^{2,1} \rtimes \mathrm{AN}_{\mathbf{n}}(2)$ for de Sitter leads to the effective particle Lagrangian[a][b]

$$L = \kappa \left(\Pi\dot{\Pi}^{-1}\right)_{\mu} x^{\mu} + s \left(\bar{\mathfrak{s}}^{-1}\dot{\bar{\mathfrak{s}}}\right)_{0} , \qquad (28)$$

with $\Pi \equiv \bar{\mathfrak{s}}\, e^{\frac{m}{\kappa} P_0} \bar{\mathfrak{s}}^{-1} \in \mathrm{AN}(2)$ and $\mathbf{x} \equiv \bar{\mathfrak{u}} \in \mathbb{R}^{2,1}$. The parametrization $\Pi = e^{\frac{1}{\kappa} p^a S_a} e^{\frac{1}{\kappa} p^0 S_0}$ allows us to obtain the mass shell condition $p_0 = m$ and the equations of motion of a Carroll particle

$$\dot{x}^0 = \lambda\, m , \qquad \dot{x}^a = 0 , \qquad \dot{p}^\mu = 0 . \qquad (29)$$

The generalization to multiple particles works as in the Poincaré case.

[a] J. Kowalski-Glikman & **T. T.**, PLB **737**, 267 (2014)
[b] **T. T.**, NPB **928**, 448 (2018)

Lie bialgebras and classical r-matrices

A Lie bialgebra is a Lie algebra $(\mathfrak{g}, [,])$ equipped with a cobracket

$$\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}\,, \qquad (\mathfrak{g}, \delta) \ \text{is a Lie coalgebra}\,,$$
$$\forall_{x,y\in\mathfrak{g}} : \delta([x, y]) = [x \otimes 1 + 1 \otimes x, \delta(y)] + [\delta(x), y \otimes 1 + 1 \otimes y]\,. \quad (30)$$

The structure of a coboundary Lie bialgebra is determined by

$$r \in \mathfrak{g} \otimes \mathfrak{g}\,, \qquad \forall_{x\in\mathfrak{g}} : \delta(x) = [x \otimes 1 + 1 \otimes x, r]\,. \quad (31)$$

Such a Lie bialgebra is called quasitriangular if $r = r_A + r_S$ satisfies

$$[[r, r]] = 0\,, \qquad [x \otimes 1 + 1 \otimes x, r_S] = 0\,, \qquad r_A = r - r_{21} \in \mathfrak{g} \wedge \mathfrak{g} \quad (32)$$

and triangular if additionally the symmetric part of r vanishes, $r_S = 0$. $[[,]]$ is called Schouten bracket, $[[r, r]] = 0$ the classical Yang-Baxter equation and r a classical r-matrix.

Poisson structure via the Fock-Rosly construction

$\mathfrak{g}$ equipped with r becomes the Lie algebra of a Poisson-Lie group of spacetime symmetries, dual to the particle phase space. At the same time, r determines the Hopf-algebraic deformation of $\mathfrak{g}$, providing the quantization of the theory. The consistency with 3D gravity requires

$$r = r_A + r_S\,, \quad r_S = \alpha\,(P_\mu \otimes J^\mu + J^\mu \otimes P^\mu) + \beta\,(\Lambda\, J^\mu \otimes J_\mu - P^\mu \otimes P_\mu)\,, \quad \alpha, \beta \in \mathbb{R}\,, \tag{33}$$

where r_S corresponds to the generalized form of the inner product in Chern-Simons action ($\beta = 0$ in the standard case), while r satisfies the homogeneous Yang-Baxter equation, hence r_A:

$$\begin{aligned}[[r_A, r_A]] &= -[[r_S, r_S]] \\ &= -(\alpha^2 - \Lambda\beta^2)\left(\Lambda\, J_0 \wedge J_1 \wedge J_2 + \tfrac{1}{2}\,\epsilon^{\mu\nu\sigma}\, J_\mu \wedge P_\nu \wedge P_\sigma\right) \\ &\quad - 2\alpha\beta\left(\tfrac{1}{2}\,\Lambda\,\epsilon^{\mu\nu\sigma}\, J_\mu \wedge J_\nu \wedge P_\sigma + P_0 \wedge P_1 \wedge P_2\right)\,. \end{aligned} \tag{34}$$

We call such a r_A to be FR-compatible and classify all of them in J. Kowalski-Glikman, J. Lukierski & **T. T.**, JHEP **09**, 096 (2020).

r-matrices of 3D (A)dS algebra relevant for gravity

Calculating the Schouten bracket $[[r_A, r_A]]$, we find that r-matrices are:

	FR-compatible $\forall\alpha,\beta$	FR-compatible for $\beta=0$	FR-compatible for $\alpha,\beta\neq 0$
$\mathfrak{o}(3,1)$	r_{III}, r^a_{III}	r_{IV}, r^a_{IV}	
$\dot{\mathfrak{o}}(2,2)$	r_{III}	r_{IV}, r^a_{IV}	r_V
$\dot{\mathfrak{o}}'(2,2)$			r_{III}

Example – FR-compatible r-matrices of dS algebra:

$$
\begin{aligned}
r_{III}(\gamma-\bar\gamma,\gamma+\bar\gamma,\eta;\Lambda) &= \tfrac{1}{2}(\gamma-\bar\gamma)\left(J_1\wedge J_2-\Lambda^{-1}P_1\wedge P_2\right)\\
&\quad+\Lambda^{-1/2}\tfrac{1}{2}(\gamma+\bar\gamma)\,(J_1\wedge P_2-J_2\wedge P_1)+\Lambda^{-1/2}\frac{\eta}{2}\,J_0\wedge P_0\,,\\
r^a_{III}(\gamma-\bar\gamma,\gamma+\bar\gamma,\eta;\Lambda) &= \Lambda^{-1/2}\tfrac{1}{2}(\gamma-\bar\gamma)\,(J_0\wedge P_2-J_2\wedge P_0)\\
&\quad+\tfrac{1}{2}(\gamma+\bar\gamma)\left(J_0\wedge J_2-\Lambda^{-1}P_0\wedge P_2\right)+\Lambda^{-1/2}\frac{\eta}{2}\,J_1\wedge P_1\,,\\
r_{IV}(\gamma,\varsigma;\Lambda) &= \gamma\left(J_1\wedge J_2-\Lambda^{-1/2}J_0\wedge P_0-\Lambda^{-1}P_1\wedge P_2\right)\\
&\quad+\frac{\varsigma}{2}\left(J_1-\Lambda^{-1/2}P_2\right)\wedge\left(J_2+\Lambda^{-1/2}P_1\right),\\
r^a_{IV}(\gamma,\varsigma;\Lambda) &= \Lambda^{-1/2}\gamma\,(J_0\wedge P_1-J_1\wedge P_0-J_2\wedge P_2)\\
&\quad+\Lambda^{-1/2}\frac{\varsigma}{2}\,(J_0-J_1)\wedge(P_0-P_1)\,. \qquad (35)
\end{aligned}
$$

To be compared with P. K. Osei & B. J. Schroers, CQG **35**, 075006 (2018).

r-matrices of (A)dS algebra in the $\Lambda \to 0$ limit

Quantum IW contractions of r-matrices of (A)dS algebra lead to the following r-matrices of 3D Poincaré algebra:

r-matrix automorphism class[a]	$\mathfrak{o}(3,1)\downarrow$	$\dot{\mathfrak{o}}(2,2)\downarrow$	$\dot{\mathfrak{o}}'(2,2)\downarrow$
$r_1 = \chi\,(J_0 + J_1) \wedge J_2$	r_I^b	r_I^a	
$\hat{r}_2 = \hat{\gamma}\,(J_0 \wedge \mathcal{P}_2 - J_2 \wedge \mathcal{P}_0) + \frac{1}{2}\hat{\eta}\, J_1 \wedge \mathcal{P}_1$	$\hat{r}_{III}^a$	$\hat{r}_{III}$	
$\hat{r}_3 = \hat{\gamma}\,(J_1 \wedge \mathcal{P}_2 - J_2 \wedge \mathcal{P}_1) + \frac{1}{2}\hat{\eta}\, J_0 \wedge \mathcal{P}_0$	$\hat{r}_{III}$		$\hat{r}_{III}$
$\hat{r}_4 = \frac{1}{\sqrt{2}}\hat{\chi}\,(J_+ \wedge \mathcal{P}_1 - J_1 \wedge \mathcal{P}_+) - \hat{\varsigma}\, J_+ \wedge \mathcal{P}_+$	$\hat{r}_{II}^a$	$\hat{r}_{II}^a$	
$\hat{r}_5 = \frac{1}{2}\hat{\tilde{\chi}} J_1 \wedge (\mathcal{P}_0 + \mathcal{P}_2)$	$\hat{r}_I^a$	$\hat{r}_V$	
$\hat{r}_6 = \hat{\gamma}\,(J_0 \wedge \mathcal{P}_2 - J_2 \wedge \mathcal{P}_0 - J_1 \wedge \mathcal{P}_1) - \hat{\varsigma}\, J_+ \wedge \mathcal{P}_+$	$\hat{r}_{IV}^a$	$\hat{r}_{IV}^a$	
$\hat{r}_7 = \hat{\gamma}\,(J_0 \wedge \mathcal{P}_0 - J_1 \wedge \mathcal{P}_1 - J_2 \wedge \mathcal{P}_2)$	$\hat{r}_{IV}$	$\hat{r}_{IV}$	

(as well as the irrelevant cases $\sim P_\mu \wedge P_\nu$), where $J_+ \equiv \frac{1}{\sqrt{2}}(J_0 + J_2)$, $P_+ \equiv \frac{1}{\sqrt{2}}(P_0 + P_2)$. Only $\hat{r}_2$, $\hat{r}_6$ and $\hat{r}_7$ are relevant for 3D gravity, i.e.

$$
\begin{aligned}
[[r_1, r_1]] = [[\hat{r}_4, \hat{r}_4]] = [[\hat{r}_5, \hat{r}_5]] &= 0\,,\\
[[\hat{r}_3, \hat{r}_3]] &= \hat{\gamma}^2 \epsilon^{\mu\nu\sigma} J_\mu \wedge \mathcal{P}_\nu \wedge \mathcal{P}_\sigma\,,\\
[[\hat{r}_2, \hat{r}_2]] = [[\hat{r}_6, \hat{r}_6]] = [[\hat{r}_7, \hat{r}_7]] &= -\hat{\gamma}^2 \epsilon^{\mu\nu\sigma} J_\mu \wedge \mathcal{P}_\nu \wedge \mathcal{P}_\sigma\,. \qquad (36)
\end{aligned}
$$

[a] P. Stachura, J. Phys. A: Math. Gen. **31**, 4555 (1998)

Abridged definition of the Hopf algebra

A Hopf algebra A is the vector space over a field K, equipped with a product (e.g. a Lie bracket) $\nabla : A \otimes A \to A$, satisfying the associativity

$$\nabla \circ (\nabla \otimes \mathrm{id}) = \nabla \circ (\mathrm{id} \otimes \nabla)\,; \tag{37}$$

a coproduct $\Delta : A \to A \otimes A$, satisfying the coassociativity

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta\,; \tag{38}$$

and an antipode $S : A \to A$, satisfying the relation

$$\nabla \circ (S \otimes \mathrm{id}) \circ \Delta = \nabla \circ (\mathrm{id} \otimes S) \circ \Delta = \mathbb{1}\,. \tag{39}$$

The tensor product of a pair of algebra representations (ρ_1, V_1), (ρ_2, V_2) (where $\rho_{1,2} : A \to \mathrm{GL}(V_{1,2})$) is given by $(\rho, V_1 \otimes V_2)$, such that

$$\rho(a)(v_1 \otimes v_2) = (\rho_1 \otimes \rho_2)(\Delta(a))(v_1 \otimes v_2)\,, \tag{40}$$

where $a \in A$, $v_{1,2} \in V_{1,2}$.

Example – the Hopf algebra corresponding to r_{III}

Denoting $H_0 \equiv H$, $H_1 \equiv \bar{H}$, $E_{0\pm} \equiv E_\pm$, $E_{1\pm} \equiv \bar{E}_\pm$ and $q_0 \equiv e^{\gamma/2}$, $q_1 \equiv e^{\bar{\gamma}/2}$, $\theta \equiv e^{\eta/4}$, we write down the deformed brackets

$$[H_k, E_{k\pm}] = E_{k\pm}\,, \qquad [E_{k+}, E_{k-}] = \frac{q_k^{2H_k} - q_k^{-2H_k}}{q_k - q_k^{-1}}\,, \tag{41}$$

where $k = 0, 1$. In the limit $q_k \to 1$ it reduces to $[E_{k+}, E_{k-}] = 2H_k$. Meanwhile, the coproducts have the form

$$\Delta(H_k) = H_k \otimes 1 + 1 \otimes H_k\,,$$
$$\Delta(E_{k\pm}) = E_{k\pm} \otimes q_k^{H_k} \theta^{\mp(-1)^k H_{k+1}} + \theta^{\pm(-1)^k H_{k+1}} q_k^{-H_k} \otimes E_{k\pm} \tag{42}$$

and antipodes

$$S(H_k) = -H_k\,, \qquad S(E_{k\pm}) = -q_k^{\pm 1} E_{k\pm}\,. \tag{43}$$

The dual of the subalgebra of translations are spacetime coordinates

$$[X_0, X_a] = 2\gamma\, X_a\,, \qquad [X_a, X_b] = 0\,, \quad a, b = 1, 2\,. \tag{44}$$