Nonextensive Thermodynamics of Black Holes [1]

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1. Based on: arXiv:2208.04473, Phys.Rev.D 103 (2021) 2, 026021 and Phys.Rev.D 97 (2018) 4, 044029. In collaboration with I. Cimdiker, A. Alonso-Serrano, M. P. Dabrowski

Motivation

Whether Hawking temperature is thermodynamically appropriate to associate with nonextensive entropies other than Bekenstein entropy

Outline

Black Hole Thermodynamics

Bekenstein Entropy and Hawking Temperature

- Bekenstein (1973) and Hawking (1974)

$$S_{bh} = \frac{A}{4} \quad \text{(Motivated by Hawking area theorem)}$$

$$T_{bh} = \frac{\kappa}{2\pi} \quad \text{(Quantum effects on the horizon)}$$

- For a Schwarzschild black hole

$$A = 16\pi M^2, \quad \kappa = \frac{1}{2M}$$

Note : $c{=}G{=}\hbar{=}k_B{=}1$

Black Hole Thermodynamics

Laws of Black Hole Thermodynamics [Bardeen, Carter, Hawking (1973)]

- **Zeroth Law** : κ is always constant over the horizon of a stationary black hole.
  In thermodynamics, the zeroth Law states that the temperature is uniform everywhere in a system in thermal equilibrium.
- **First Law** : $dM = \frac{\kappa}{8\pi} dA \Longleftrightarrow dE = TdS$
- **Second Law** : It is basically Hawking's area theorem

$$dA \geq 0 \Longleftrightarrow dS \geq 0$$

- **Third Law** : The surface gravity of the horizon cannot be reduced to zero in a finite number of steps.

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Black Hole Thermodynamics

Basic principles of Gibbs thermodynamics

- The entropy is additive [2] : $S_{12} = S_1 + S_2$
  and S is extensive. i.e. $S \propto L^d$.
  Long range forces ignored. Since the size of the system is greater than the range of the force between the system's constituents
- The internal energy is additive : $U_{12} = U_1 + U_2$.
  Weakly interacting systems, thus ignoring interaction energy
- The temperatures obtained from the *transitivity relation* (**zeroth law**) and the *equilibrium condition* (**maximizing entropy**) are the same.

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Nonextensive Black Hole Entropy

Why we need nonextensive thermodynamics ?

- S_{bh} scales with area, hence it is nonextensive (*consider a black hole as a 3+1 dimensional object*)
- S_{bh} follows the following nonadditive rule [Abe et al., 2001] :

$$S_{bh12} = S_{bh1} + S_{bh2} + 2\sqrt{S_{bh1}}\sqrt{S_{bh2}}$$

Gibbs thermodynamics is not the appropriate choice

- **Long-range forces** play a significant role in several thermodynamic systems. For example, for self-gravitating systems, like black holes.
- We need nonextensive thermodynamics/statistics to study the black hole entropy. [Tsallis, 1988]

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Tsallis Nonextensive Entropy [Tsallis, 1988]

- Tsallis entropy generalizes the Gibbs-Shannon's entropy into

$$S_q = -\sum_i [p(i)]^q \ln_q p(i),$$

where $p(i)$ is the probability distribution defined on a set of microstates Ω

$$\ln_q p = \frac{p^{1-q} - 1}{1-q}$$

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Does zeroth law work in this case? [Abe et al 2001; Biro and Van, 2011; Czinner and Iguchi 2013-16; Ou and Chen, 2006; Nauenberg, 2003]

- Transitive relation does not work
- The temperature defined from the equilibrium condition is not equal to the absolute temperature.

$$\boxed{\delta S_{q12} = 0 \text{ with } \delta U_{q12} = \delta(U_{q1} + U_{q2}) = 0}$$

gives

$$\frac{\beta}{1+\lambda S_{q1}} = \frac{\beta}{1+\lambda S_{q2}} = \beta^*.$$

- Equilibrium condition implies

$$T_{eq} = \frac{1}{\beta^*} = (1+\lambda S_q)\frac{1}{\beta}.$$

$\beta = \frac{\partial S_{q1}}{\partial U_{q1}} = \frac{\partial S_{q2}}{\partial U_{q2}}$, and $\lambda = 1 - q$

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Nonextensive Black Hole Entropy

Equilibrium Entropy

- Using Clausius' relation

$$S_{eq} = \frac{1}{\lambda} \ln(1 + \lambda S_q).$$

- S_{eq} happens to be Rényi entropy [Rényi, 1959]

$$S_R = \frac{\ln \sum_i p^q(i)}{1 - q}.$$

- S_R or S_{eq} follows additive composition rule.

$$S_{R12} = S_{R1} + S_{R2}$$

- *S_R is an equilibrium entropy associated with T_{eq} for the nonextensive case.*

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Applications to Black Holes

Rényi Black Hole Entropy and Temperature [Biro, Czinner, Iguchi (2013-16) ; Alonso-Serrano, Dabrowski, HG (2021)]

Assume that $S_q = S_{bh} = 4\pi r_h^2$; where $r_h = 2M$, then

$$S_R = \frac{1}{\lambda}\ln(1 + \pi\lambda r_h^2), \quad T_R = \frac{1}{4\pi r_h} + \frac{\lambda r_h}{4} = T_{bh} + T_\lambda$$

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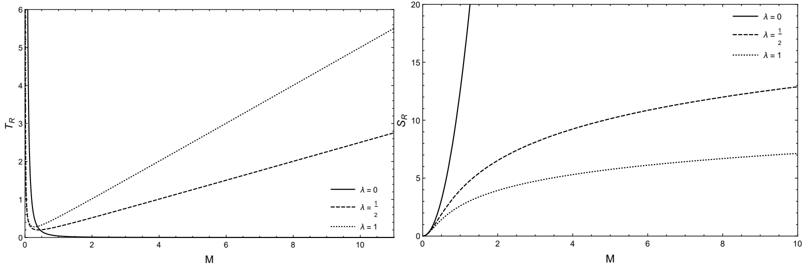


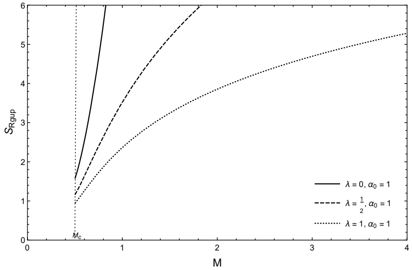
This means that S_R is the equilibrium entropy associated the equilibrium temperature T_R in the nonextensive setup

Applications to Black Holes

Quantum Gravity Corrections [Alonso-Serrano, Dabrowski, HG (2018-21)]

$$S_{Rgup} = \frac{k_B}{\lambda} \ln \left\{ 1 + \pi\lambda \frac{M^2}{m_p^2} \left(2 + f(M)\right) - \pi\lambda \frac{\alpha_0}{2} \ln \left[\frac{M}{M_0} \left(2 + f(M)\right) \right] \right\}$$

where, $f(M) = \sqrt{4 - \alpha_0 \frac{m_p^2}{M^2}}$



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$$T_{Rgup} = T_{gup} + T_\lambda \left\{ 1 + \frac{\alpha_0 m_p^2}{2M^2\left(2 + f(M)\right)} \ln\left[\frac{M}{M_0}\left(2 + f(M)\right)\right]\right\},$$

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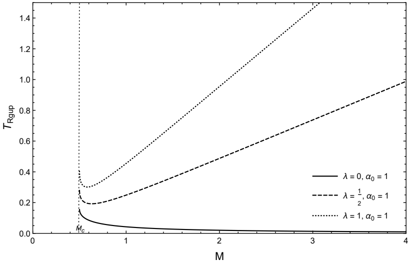

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Sparsity of Hawking and Rényi Radiation [Alonso-Serrano, Dabrowski, HG (2018-21)]

$$\eta = C\left[\frac{\lambda^2_{\text{thermal}}}{A_{eff}}\right]_{Rgup} =$$

$$g(M) \times \left[\frac{\pi\lambda}{m_p^2}\left\{2 + (-2 + f(M))\ln\left[\frac{M}{M_0}(2 + f(M))\right]\right\} - \frac{2}{\alpha_0 m_p^2}(-2 + f(M))\right]^{-2}$$

$$g(M) = \frac{256\pi^3}{27M^2\left[16M^2 - \alpha_0 m_p^2 \ln\left(\frac{M^2}{M_0^2}\right)\right]}, \; \lambda_{thermal} = \frac{2\pi}{T}, \; A_{eff} = \frac{27}{4}A$$

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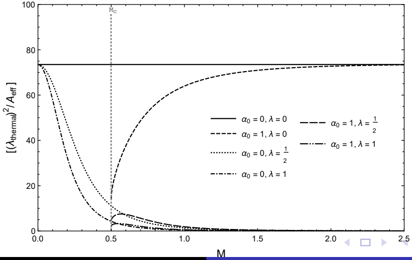

Tsallis-Cirto Black Hole Entropy

Can we make black entropy extensive and additive ? [Tsallis and Cirto, 2013]

- For a general d dimensional system, the Gibbs free energy G

  $$G = U - TS + pV - \mu N,$$

  S, V N (extensive) scaling with L^d
  T, p, μ (intensive) scaling with L^θ, and the energies, G, U scaling with L^ϵ. It follows that $\epsilon = \theta + d$.

- For Schwarzschild black holes, M scales with L. Since $\epsilon = 1$ for this case, hence, $\theta = 1 - d$.
  Consider black hole as two dimensional object and, hence S_{bh} scales with L^2. This indicates that the temperature scales with L^{-1}, which is true for $T_{bh} \propto 1/M$.

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Tsallis and Cirto's proposal

- Consider black holes as three-dimensional objects, based on the Legendre structure, the definitions of entropy and temperature alter for black holes.
- Tsallis proposed a new type of black hole entropy to address this issue, and it is defined as follows :

$$S_T = k_B \left(\frac{S_{bh}}{k_B} \right)^{\delta},$$

where $\delta > 0$ and its composition rule is given by

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Tsallis and Cirto's proposal (2013)

- In this context, S_T is nonadditive and S_{bh} is additive
- For $\delta = 3/2$, $S_T \propto r_h^3$, hence extensive
- The corresponding temperature

$$T_\delta = \frac{T_{bh}}{\delta}\left(\frac{S_{bh}}{k_B}\right)^{1-\delta}.$$

  For $\delta = 3/2$, $T_\delta \propto 1/M^2$, for the case of Schwarzschild black hole.
- Note that S_T and T_δ $(\delta = 3/2)$ satisfies Legendre structure for $(3+1)$ dimensional Schwarzschild black hole.

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Equilibrium condition [Cimdiker, Dabrowski, HG (2022)]

- $\delta S_{T12} = 0$ with $\delta U_{12} = \delta(U_1 + U_2) = 0$, then the equilibrium condition gives

$$(S_{T1})^{\frac{1-\delta}{\delta}} \frac{\partial S_{T1}}{\partial U_1} = (S_{T2})^{\frac{1-\delta}{\delta}} \frac{\partial S_{T2}}{\partial U_2} = \beta^*,$$

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- In the context of the above composition rule, the equilibrium entropy for this case is also additive.

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- In the context of the above composition rule, the equilibrium entropy for this case is also additive.

Tsallis-Cirto Black Hole Entropy

Equilibrium condition [Cimdiker, Dabrowski, HG (2022)]

- $\delta S_{T12} = 0$ with $\delta U_{12} = \delta(U_1 + U_2) = 0$, then the equilibrium condition gives

$$(S_{T1})^{\frac{1-\delta}{\delta}} \frac{\partial S_{T1}}{\partial U_1} = (S_{T2})^{\frac{1-\delta}{\delta}} \frac{\partial S_{T2}}{\partial U_2} = \beta^*,$$

- The equilibrium temperature

$$T_{eq} = \frac{1}{\beta^*} = T_\delta (S_T)^{\frac{\delta-1}{\delta}} = \frac{T_{bh}}{\delta}$$

- Corresponding equilibrium entropy

$$S_{eq} = \delta (S_T)^{1/\delta} = \delta S_{bh}.$$

- In the context of the above composition rule, the equilibrium entropy for this case is also additive.

Thermodynamic inconsistencies

Inconsistent quantities

- Hawking temperature vs. Bekenstein entropy

$$dE = T_{bh} dS_{bh} \rightarrow E = M$$

- Hawking temperature vs. Rényi entropy

$$dE_R = T_{bh} dS_R \longrightarrow E_R = M - \frac{4\pi(1-q)M^2}{3}.$$

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Thermodynamic inconsistencies

Inconsistent quantities

- Hawking temperature vs. Tsallis-Cirto entropy

$$dE_T = T_{bh} dS_T \to E_T \neq M$$

But

$$dE_T = T_T dS_T \longrightarrow E_T = M$$

Summary and Conclusions

- Due to Bekenstein entropy's nonextensive nature, nonextensive thermodynamics is the right approach for studying black thermodynamics.
- In the nonextensive setup, temperature is defined from the equilibrium condition rather than the transitive relation
- Rényi entropy is the equilibrium entropy associated with the equilibrium temperature in the Tsallis nonextensive setup.
- Other than Bekenstein entropy, Hawking temperature is not the appropriate choice to associate with nonextensive entropies.

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For more details : arXiv: 2208.04473

Thank you !