Electromagnetic and Gravitational Hopfion-like solutions in de Sitter spacetime

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Joint work with J. Jezierski and A. Grzela

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Motivation

- In Minkowski spacetime, Hopfions are finite energy, soliton-like family of solutions of Maxwell or linearized gravity theory

[1] I. Białynicki-Birula, Z. Białynicki-Birula *Uncertainty Relation for Photons* PRL **108**, 140401 (2012)

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- For quantum electrodynamics, pre-Hopfion solution treated like a photonic wave function saturates uncertainty relation for photons in three dimensions[1]:

$$\Delta r \Delta p \geq 4\hbar \tag{1}$$

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- Reduced data for pre-Hopfion is generated by imaginary shift in time for fundamental solution of wave equation:

$$\Phi_0 = (\mathbf{E} + \imath\mathbf{B}) \cdot \mathbf{r} = \frac{1}{(r^2 - (t - \imath)^2)} \tag{2}$$

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- Example of simply but highly non-trivial solution of Maxwell equations — limit of Kerr–Newman E-M field[2] for mass$\to 0$

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Plan of the talk

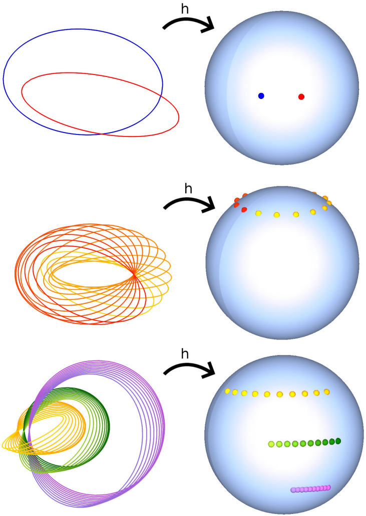
1. Brief introduction to Hopfions in Minkowski
   - Hopf fibration
   - Rañada's classical Hopfion

2. Hopfion-like solutions in de Sitter spacetime

Hopf fibration

Hopf fibration is a non-trivial principal bundle of a three-dimensional sphere:

$$S^3 \sim \mathbb{R}^3 \cup \{\infty\} \xrightarrow{h} \mathbb{C} \cup \{\infty\} \sim S^2 \quad (3)$$



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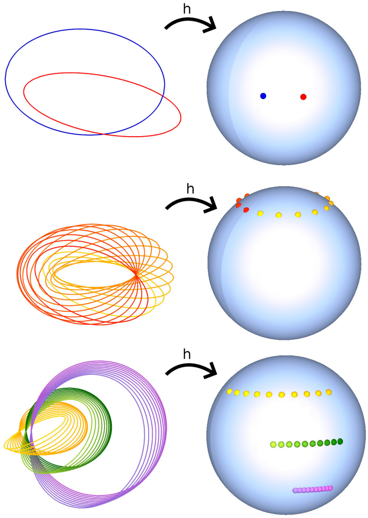

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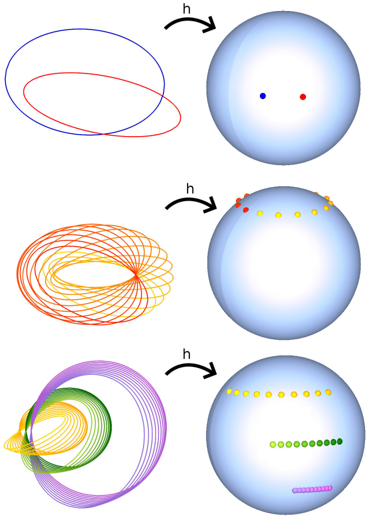
The mapping $h : \mathbb{R}^3 \cup \{\infty\} \xrightarrow{h} \mathbb{C} \cup \{\infty\}$ is given by:

$$h(x, y, z) = \frac{x + \imath z}{-y + \imath\left(\tilde{A} - 1\right)} \quad (4)$$

where $\tilde{A} = \frac{1}{2}(x^2 + y^2 + z^2 + 1)$.



Rañada's classical Hopfion

In 1990, Rañada proposed the following solution of Maxwell equations:

$$\mathbf{E}_R = \frac{1}{4\pi} \frac{\nabla\xi \times \nabla\bar{\xi}}{(1+\xi\bar{\xi})^2} \quad (5) \qquad \mathbf{B}_R = \frac{1}{4\pi} \frac{\nabla\eta \times \nabla\bar{\eta}}{(1+\eta\bar{\eta})^2} \quad (6)$$

where $\xi(t,x,y,z)$ and $\eta(t,x,y,z)$ are defined as

$$\xi = \frac{(Ax+ty)+\imath(Az+t(A-1))}{(tx-Ay)+\imath(A(A-1)-tz)} \quad (7) \qquad \eta = \frac{(Az+t(A-1))+\imath(tx-Ay)}{(ty+Ax)+\imath(A(A-1)-tz)} \quad (8)$$

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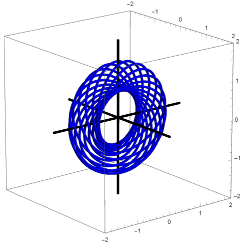
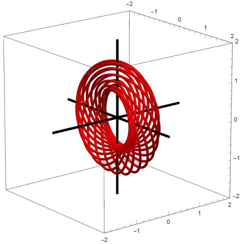
- The $\mathbf{E}$ field is tangential to lines of constant ξ.
- The helicities are preserved in time.

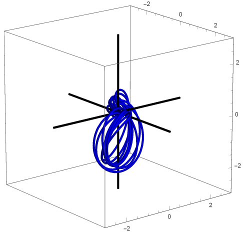
Field lines



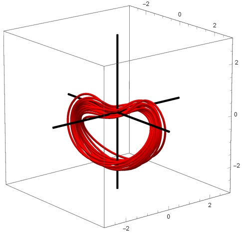
Hopfions (L=1), electric, t=0



Hopfions (L=1), magnetic, t=0



Hopfions (L=1), electric, t=1



Hopfions (L=1), magnetic, t=1

Topological invariants in physics

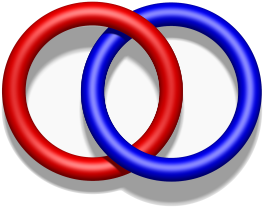
A topologically invariant configuration of field lines is to be linked and/or knotted.

A measure of linkedness is Gauss linking integral:

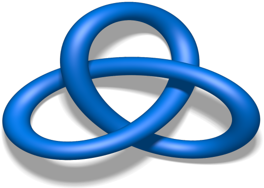
$$L(\mathbf{c}_1, \mathbf{c}_2) = \frac{1}{4\pi} \int \frac{\mathrm{d}\mathbf{c}_1}{\mathrm{d}t_1} \cdot \frac{\mathbf{c}_1 - \mathbf{c}_2}{\|\mathbf{c}_1 - \mathbf{c}_2\|^3} \times \frac{\mathrm{d}\mathbf{c}_2}{\mathrm{d}t_2} \tag{9}$$

where $\mathbf{c}_1(t_1), \mathbf{c}_2(t_2)$ are closed curves.

The self-linking number, $L(\mathbf{c}, \mathbf{c})$, is a measure of knotedness.



Linked, L=1.



Knotted

Topological invariant in physics

Physical analogue of Linking number are helicities[3]. The magnetic helicity:

$$h_m = \int_V dV\, \mathbf{A} \cdot \mathbf{B} \tag{10}$$

where $\mathbf{B} = \mathrm{rot}\, \mathbf{A}$.

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$$h_e = \int_V dV\, \mathbf{C} \cdot \mathbf{E} \tag{11}$$

where $\mathbf{E} = \mathrm{rot}\, \mathbf{C}$.

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Helicity as a Noether current

- Lagrangian for Maxwell theory

$$\mathcal{L}_{\mathrm{EM}} = -\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} = \frac{1}{2}\left(E^2 - B^2\right) \quad (12)$$

where $E^2 = E_i E^i, B^2 = B_i B^i$.

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- The Euler-Lagrange equation together with $d^2 A = 0$

$$\partial_\beta F^{\alpha\beta} = 0 \quad \partial_\beta (*F)^{\alpha\beta} = 0 \tag{13}$$

  is invariant under the duality reflection $F_{\alpha\beta} \to (*F)_{\alpha\beta}$.

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- Following[4], we analyze the duality symmetric Lagrangian

$$\mathcal{L}_{\mathrm{EM-ds}} = -\frac{1}{8}\left(F_{\alpha\beta} F^{\alpha\beta} + G_{\alpha\beta} G^{\alpha\beta}\right) \tag{14}$$

where we impose duality constraint $G_{\alpha\beta} = (*F)_{\alpha\beta}$.

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- The Lagrangian $\mathcal{L}_{\mathrm{EM-ds}} = -\frac{1}{8}\left(F_{\alpha\beta}F^{\alpha\beta} + G_{\alpha\beta}G^{\alpha\beta}\right)$ is invariant under acting U(1) symmetry

$$F_{\alpha\beta} \to F_{\alpha\beta}\cos\theta + G_{\alpha\beta}\sin\theta \quad G_{\alpha\beta} \to G_{\alpha\beta}\cos\theta - F_{\alpha\beta}\sin\theta \tag{15}$$

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- Corresponding Noether current, known as helicity current

$$J^{\alpha}_{\mathcal{H}} = \frac{\partial\mathcal{L}_{\mathrm{EM-ds}}}{\partial\left(\partial_\alpha A_\beta\right)}C_\beta - \frac{\partial\mathcal{L}_{\mathrm{EM-ds}}}{\partial\left(\partial_\alpha C_\beta\right)}A_\beta = \frac{1}{2}\left(G^{\alpha\beta}A_\beta - F^{\alpha\beta}C_\beta\right) \tag{16}$$

where the helicity density $\mathcal{H}$ and helicity flux density–the spin density $\boldsymbol{J}_{\mathcal{H}}$ are, in transverse gauge

$$J^0_{\mathcal{H}} \equiv \mathcal{H} = \frac{1}{2}(\boldsymbol{A}\cdot\boldsymbol{B} - \boldsymbol{C}\cdot\boldsymbol{E}) \quad \boldsymbol{J}_{\mathcal{H}} = \frac{1}{2}(\boldsymbol{E}\times\boldsymbol{A} + \boldsymbol{B}\times\boldsymbol{C}) \tag{17}$$

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- Provided the equation of motion holds, the helicity current is conserved $\partial_\alpha J^\alpha_{\mathcal{H}} = 0$,

$$\dot{\mathcal{H}} + \nabla\cdot\boldsymbol{S} = 0 \tag{18}$$

Conformal flatness of de Sitter spacetime

De Sitter spacetime equipped with the metric

$$g = -(1-k^2r^2)dt^2 + \frac{dr^2}{1-k^2r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad (19)$$

can be locally transformed in a conformally flat metric in Cartesian coordinates

$$g_{dS} = \frac{4}{k^2(1+S^2)^2}\left(-dT^2 + dX^2 + dY^2 + dZ^2\right) \qquad (20)$$

where $S^2 = -T^2 + X^2 + Y^2 + Z^2$. Exact form of the transform reads

$$\begin{aligned} T &= \frac{\sqrt{1-k^2r^2}\sinh kt}{1-\sqrt{1-k^2r^2}\cosh kt}, \quad & X &= \frac{kr\sin\theta\cos\phi}{1-\sqrt{1-k^2r^2}\cosh kt}, \\ Y &= \frac{kr\sin\theta\sin\phi}{1-\sqrt{1-k^2r^2}\cosh kt}, \quad & Z &= \frac{kr\cos\theta}{1-\sqrt{1-k^2r^2}\cosh kt} \end{aligned} \qquad (21)$$

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- Conformally transformed classical Hopfion for t=0 reads

$$Z^r = \frac{4e^{i\phi}\sin\theta s(r)}{\left(\left(1-\frac{4}{k^2}\right)s(r)+\frac{4}{k^2}+1\right)^2} \tag{22}$$

$$Z^\theta = \frac{4k^4e^{i\phi}\left(\cos\theta\left(k^2\left(s(r)+1\right)+4\left(s(r)-1\right)\right)-4ik^2rs^3(r)\right)}{r\left(k^2\left(s(r)+1\right)-4s(r)+4\right)^3} \tag{23}$$

$$Z^\phi = -\frac{4k^4e^{i\phi}\left(-4k^2r\cos\theta s^3(r)-i\left(k^2\left(s(r)+1\right)+4\left(s(r)-1\right)\right)\right)}{r\sin\theta\left(k^2\left(s(r)+1\right)-4s(r)+4\right)^3} \tag{24}$$

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- Weak k limit

$$\mathbf{Z} = \mathbf{Z}_{\text{Minkowski}} + O(k^2) \tag{25}$$

Near cosmological horizon analysis

$$
\begin{aligned}
Z^r &= \frac{4\sqrt{2k}e^{i\phi}\sin\theta}{\left(1+\frac{4}{k^2}\right)^2}\sqrt{\frac{1}{k}-r}+O\left(\frac{1}{k}-r\right)\\
Z^\theta &= \frac{4k^5\left(k^2-4\right)e^{i\phi}\cos\theta}{\left(k^2+4\right)^3}+O\left(\sqrt{\frac{1}{k}-r}\right)\\
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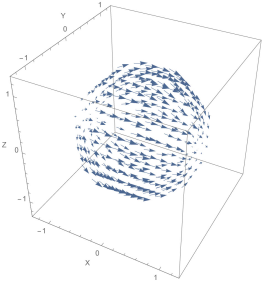
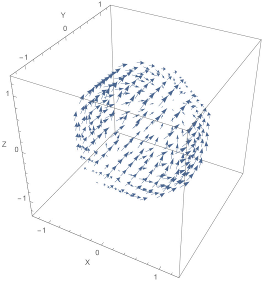
- The fields flow from one pole of the sphere to the other along the OX (electric) and OY (magnetic) axes.



Electric field



Magnetic field

Field lines

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- We obtain a straight section with a parametrization

$$r'(\tau) = \frac{4\sqrt{1 - k^2 r(\tau)^2}}{\left(\left(1 - \frac{4}{k^2}\right)\sqrt{1 - k^2 r(\tau)^2} + \frac{4}{k^2} + 1\right)^2} \tag{26}$$

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- For all x: $E_y(x, 0, 0) = E_z(x, 0, 0) = 0$
- We obtain a straight section with a parametrization

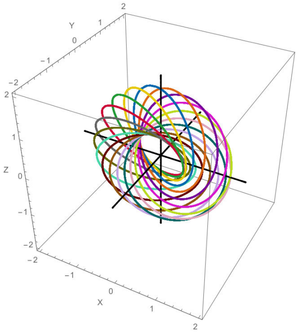
$$r'(\tau) = \frac{4\sqrt{1 - k^2 r(\tau)^2}}{\left(\left(1 - \frac{4}{k^2}\right)\sqrt{1 - k^2 r(\tau)^2} + \frac{4}{k^2} + 1\right)^2} \quad (26)$$

- It reach cosmological horizon $r = 1/k$ in finite time

$$\tau_H = \frac{(8 + 3\pi)k^4 + 8\pi k^2 + 16(3\pi - 8)}{16k^5} \quad (27)$$

Integral curves of B field, $t = 0$ — numeric analysis

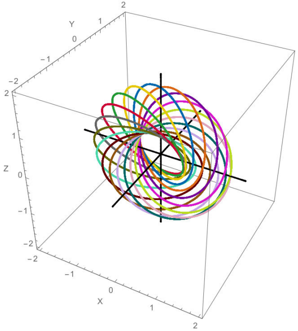
- Part of Integral curves behaves like in the Minkowski case (Hopf fibration structure) – fig. (a)



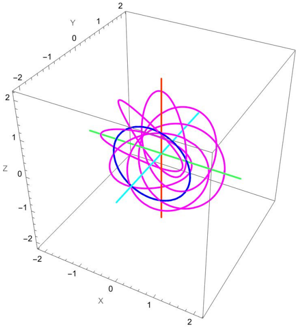
(a) $k = 10^{-6}$

Integral curves of B field, $t = 0$ — numeric analysis

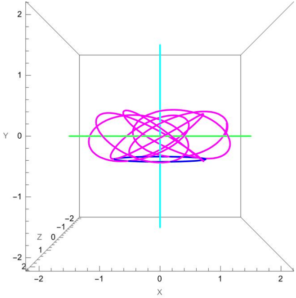
- Part of Integral curves behaves like in the Minkowski case (Hopf fibration structure) – fig. (a)
- Group of closed field lines which cross XZ plane many times – fig. (b)



(a) $k = 10^{-6}$



(b.1) $k = 0.32$



(b.2) $k = 0.32$

Summary

- Hopfions are real (finite energy) soliton-like family of solutions of Maxwell (linearized gravity) theory

[5] T. S, J. Jezierski *Simple description of generalized electromagnetic and gravitational hopfions*, (2018) Class. Quantum Grav. **35** 245010.

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  - Field lines which reach cosmological horizon in finite time parameter
  - Closed field lines for which linking number is greater than one. Various cases in point of view of topological classes
- In our paper[5], presented construction enables one to obtain corresponding solution for Linearized Gravity

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1 W. T. Irvine, D. Bouwmeester *Linked and knotted beams of light* Nature Physics, **4** 716-720, (2008).

2 M. Arrayás, D. Bouwmeester, J. L. Trueba *Knots in electromagnetism* Physics Reports 667 (2017)

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3 A. Grzela *Magical hopfions - topologically non-trivial solutions of Maxwell equations*, (2022) Bachelor thesis (in polish)

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Thank You!