Examining Quasinormal Mode Instability with the Pseudospectrum

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Quasinormal Modes

- Quasinormal modes (QNMs) $\leftrightarrow$ eigenmodes of a non self-adjoint operator
- Exponentially damped oscillation solutions to linearized equations
- E.g. Scalar field around a black hole: purely incoming(outgoing) at horizon(infinity)

$$\Psi(r_* \to -\infty) \sim e^{-i\omega_n(t+r_*)} \qquad \Psi(r_* \to \infty) \sim e^{-i\omega_n(t-r_*)}$$

- $\Re(\omega_n)$ sets the frequency while $\Im(\omega_n)$ sets the damping rate
- Present in many non-Hermitian systems

Black Hole Quasinormal Modes

- Consider a stationary, (3+1)-D spherically-symmetric black hole (BH) background

$$ds^2 = f(r)dt^2 - (f(r))^{-1}dr^2 - r^2 d\Omega^2, \quad f(r) = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}$$

plus minimally-coupled scalar field

$$\big(\partial_t^2 - \partial_{r_*}^2 + V(r_*)\big)\phi(t, r_*) = 0$$

- Pöschl-Teller potential gives $M\omega_0 = 0.1148 - 0.1148i$
- QNM spectrum encodes information about the background BH
- Schwarzschild [Chandrasekhar (1983), Ruffini (1973)], Reissner-Nordström [Gunter (1980)], Kerr [Leaver (1985)]
- Many more [Berti, Cardoso, Starinets (2009)]

Quasinormal Mode Instability

- Schwarzschild black hole spectrum is unstable under a class of small perturbations [Nollert & Price (1999)]
- Instability increases for higher overtones
- Overtones probe the near-horizon geometry (corrections to GR) [Konoplya & Zhidenko (2022)]
- Shifts in frequencies expected to be measurable in GW data [Jaramillo, Macedo, & Sheikh (2022)]
- Sensitive to the type of perturbation (IR/UV) [Jaramillo *et al.* (2021)]
- Pseudospectrum quantifies the stability of each QNM

Example: Scalar field on a fixed Schwarzschild background I

- Write scalar field EOM as an eigenvalue problem for non self-adjoint operator
- Hyperboloidal compactification

$$t = u - h(x), \quad r = g(x)$$

- Outgoing radiation propagates as a free wave at $\mathscr{I}^+$
- Boundary conditions become encoded in Sturm-Liouville operator [Jaramillo *et al.*, (2021)]

$$\left[\left(1 - (\frac{h'}{g'})^2\right)\partial_u^2 - \frac{2}{g'}(\frac{h'}{g'})\partial_u\partial_x - \frac{1}{g'}(\frac{h'}{g'})'\partial_u - \frac{1}{g'}\partial_x\left(\frac{1}{g'}\partial_x\right)\right]\phi = 0$$

Example: Scalar field on a fixed Schwarzschild background II

- First-order reduction in time via $\psi \equiv \partial\phi/\partial u$

$$\Phi = \begin{pmatrix}\phi \\ \psi\end{pmatrix} \qquad \partial_\tau \Phi = iL\Phi \qquad L = \frac{1}{i}\begin{pmatrix}0 & 1 \\ L_1 & L_2\end{pmatrix}$$

$$L_1 = \frac{1}{w(x)}\Big[\partial_x(p(x)\partial_x) - q(x)\Big]$$

$$L_2 = \frac{1}{w(x)}\Big[2\gamma(x)\partial_x + \partial_x\gamma(x)\Big]$$

- Sturm-Liouville functions w, p, q, γ related to compactification functions, must satisfy $w(x) > 0 \, \forall \, x \in (a, b)$, $p(a) = p(b) = 0$ [Hounkonnou *et al.*, (2005)]
- L_2 is non self-adjoint part

Example: Scalar field on a fixed Schwarzschild background III

- L is non-self-adjoint $\rightarrow$ system is non-conservative [Ashida *et al.*, 2020]
- Define the "energy" inner product from the total energy contained in a spatial slice associated with ϕ

Energy Inner Product

$$\langle \Phi_1, \Phi_2 \rangle_E \equiv \frac{1}{2} \int_a^b \left[w(x) \bar{\psi}_1 \psi_2 + p(x) \partial_x \bar{\phi}_2 \partial_x \phi_2 + V \bar{\phi}_1 \phi_2 \right] dx$$

- Fourier decomposition $\Phi \sim \Phi(x) e^{i\omega u}$ and solve the spectral problem $L\Phi = \omega\Phi$

The Pseudospectrum I

- Topographic maps of QNMs stability
- Unstable QNMs follow contour lines
- Given $A \in M_n(\mathbb{C})$, $\epsilon > 0$

Pseudospectrum (Perturbative)

$$\sigma^{\epsilon}(A) = \{\lambda \in \mathbb{C}, \exists\ \delta A \in M_n(\mathbb{C}), ||\delta A|| < \epsilon : \lambda \in \sigma(A + \delta A)\}$$

Pseudospectrum (Resolvant)

$$\sigma^{\epsilon}(A) = \{\lambda \in \mathbb{C} : ||R_A(\lambda)|| = ||(\mathbb{I}\lambda - A)^{-1}|| > 1/\epsilon\}$$

- Stable if $\mathrm{dist}(\sigma^{\epsilon}(A), \sigma(A)) \sim \epsilon$, unstable if $\mathrm{dist}(\sigma^{\epsilon}(A), \sigma(A)) \ggg \epsilon$

The Pseudospectrum II

- Add perturbations to the potential term in the spectral equation

$$\delta L = \frac{1}{i}\begin{pmatrix} 0 & 0 \\ \delta L_1 & 0 \end{pmatrix}, \quad \delta L_1 = \delta V(x)$$

- Random perturbations, infrared/ultraviolet perturbations
- Contour lines give the height of $||R_L(\omega)||_E$
- Stability characterized by steep descents near eigenvalues

The Pseudospectrum III

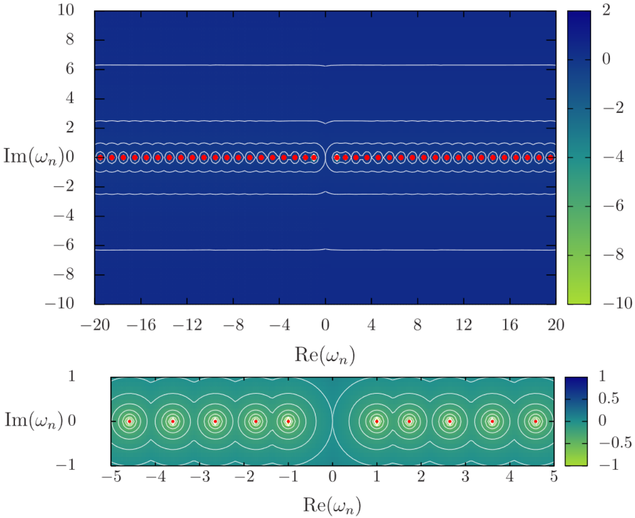
Jaramillo *et al.* (2021)

- Schwarzschild background metric, $\Box\phi = 0$, Pöschl-Teller potential, hyperboloidal compactification [Bizoń & Mach (2017)]

$$t = \tau - \frac{1}{2} \ln\left(1 - x^2\right)$$
$$\bar{x} = \tanh^{-1}(x)$$

- Self-adjoint case, $L_2 = 0$ (stable)

The Pseudospectrum III



[Jaramillo et al. (2021)]

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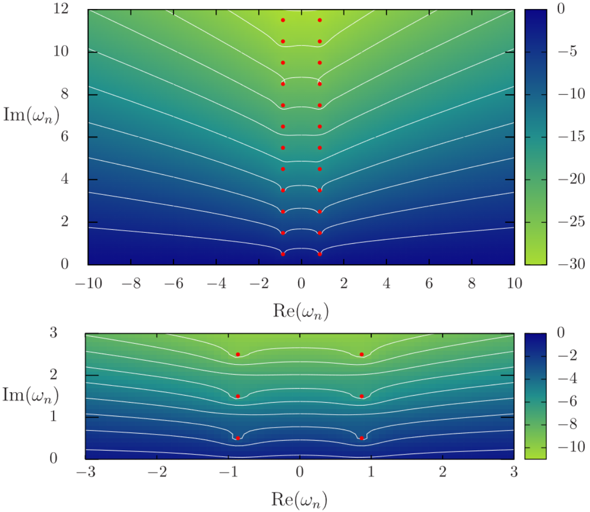
$$t = \tau - \frac{1}{2}\ln\left(1 - x^2\right)$$
$$\bar{x} = \tanh^{-1}(x)$$

- Self-adjoint case, $L_2 = 0$ (stable)
- Non self-adjoint case, $L_2 \neq 0$ (unstable)

The Pseudospectrum III



[Jaramillo et al. (2021)]

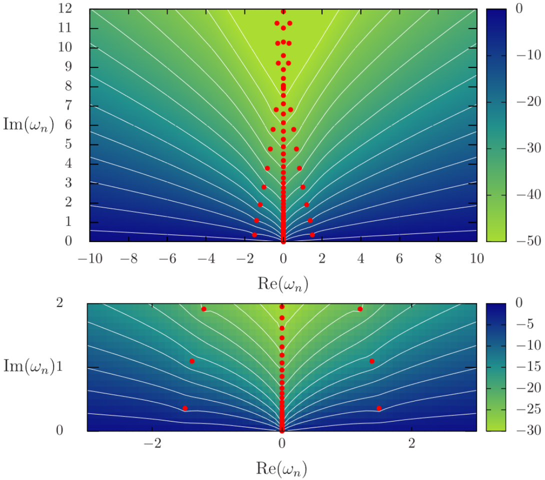
The Pseudospectrum IV

Jaramillo *et al.* (2021)

- Stability of Schwarzschild $\ell = 2$ gravitational modes, Regge-Wheeler potential, hyperboloidal compactification [Ansorg & Macedo (2016)]

$$t = \tau - \frac{1}{2}\left(\ln x + \ln(1-x) + 1/x\right)$$
$$\bar{x} = \frac{1}{2}\left(1/x + \ln(1-x) - \ln x\right)$$

The Pseudospectrum IV



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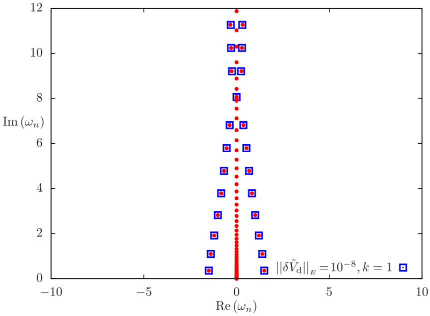
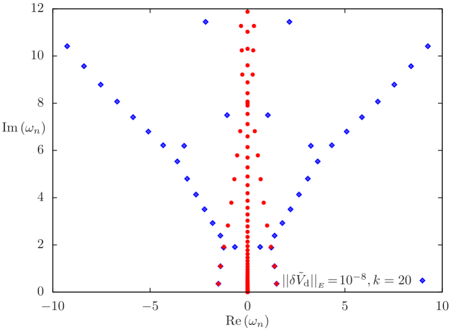
$$\bar{x} = \frac{1}{2}\left(1/x + \ln(1-x) - \ln x\right)$$

- Instabilities similar to Pöschl-Teller
- Stable against infrared perturbations, unstable against ultraviolet perturbations

The Pseudospectrum IV



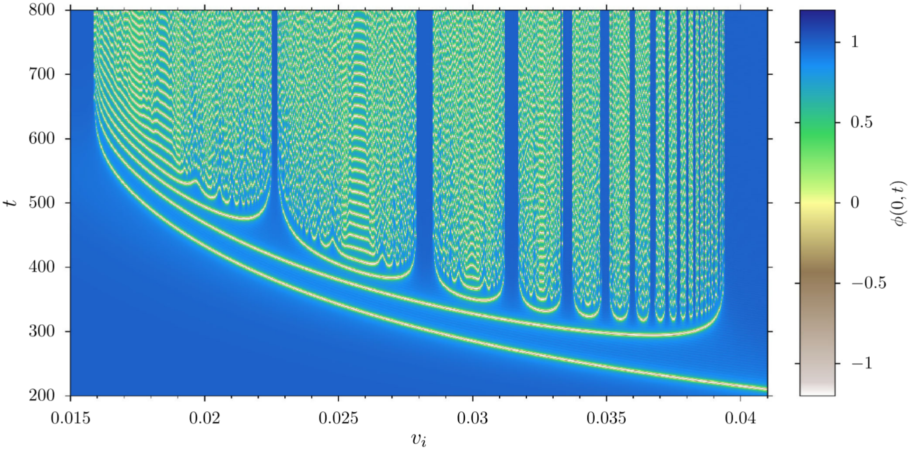


[Jaramillo *et al.* (2021)]

Solitons I

- Solitons: localized solutions that carry topological charge and appear in a wide range of physical systems
- Kinks: minimal-energy solutions in (1+1)-D that interpolate between vacuua
- e.g. Yang-Mills soliton, ϕ^4, sine-Gordon
- Highly non-trivial dynamics [Wereszczynski *et al.* (2021); Alonso-Izquierdo *et al.* (2021); Romańczukiewicz (2005)]
- Quasinormal modes can mediate resonant scattering of kink/anti-kink systems when internal modes are not present [Dorey & Romańczukiewicz (2018)]

Solitons I



[Adam *et al.* (2021)]

Solitons II

- How do soliton QNMs change due to different perturbations?
- QNM stability related to choice of compactification? Frequency dependant?
- Connection to staccato radiation? [Dorey *et al.* (2020)]

Solitons III

- (4+1)-D equivariant Yang-Mills potential $f(t, r)$
- $r \in [1, \infty)$ with Dirichlet condition $f(t, 1) = 0$
- Static, minimum-energy solution half-kink, $Q(r)$, $\mathcal{E}(Q(r)) = 2/3$

$$0 = \Box f + 2f\left(1 - f^2\right)/r^2, \quad Q(r) = \frac{r^2 - 1}{r^2 + 1}$$

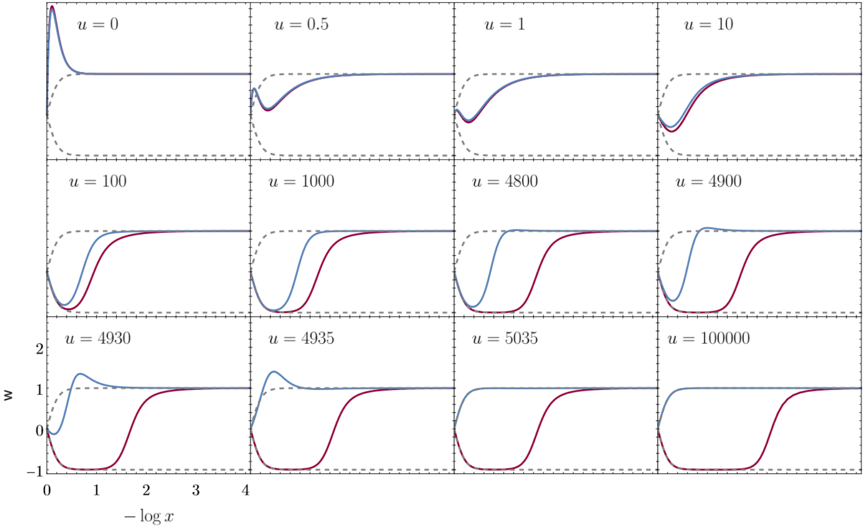
- Define null coordinate $u = t - r$ and compactify $x = r^{-1/2}$

Soliton Resolution Conjecture

Any f with $E(f) \neq 0$ tends to either the vacuum $f = \pm 1$ or an alternating chain of rescaled kinks/antikinks plus radiation [Jendrej & Lawrie (2021)]

$$1 + \sum_{j=1}^{N} (-1)^{N+j} \left(Q(r/\lambda_j) - 1\right)$$

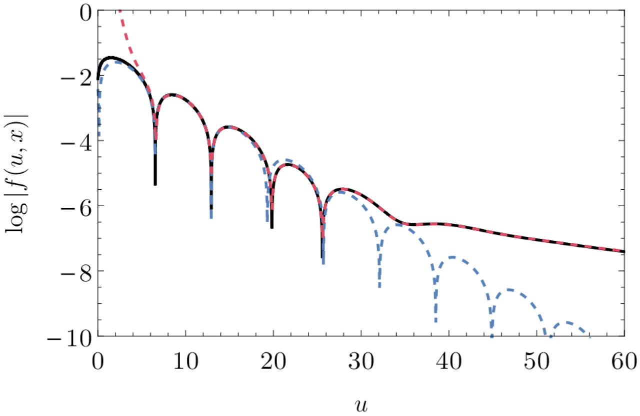
Solitons III



[Bizoń, BC, & Maliborski (2021)]

Solitons III

- Perturbations of kink solution have a single QNM $\omega = -0.364271 \pm 0.476856i$



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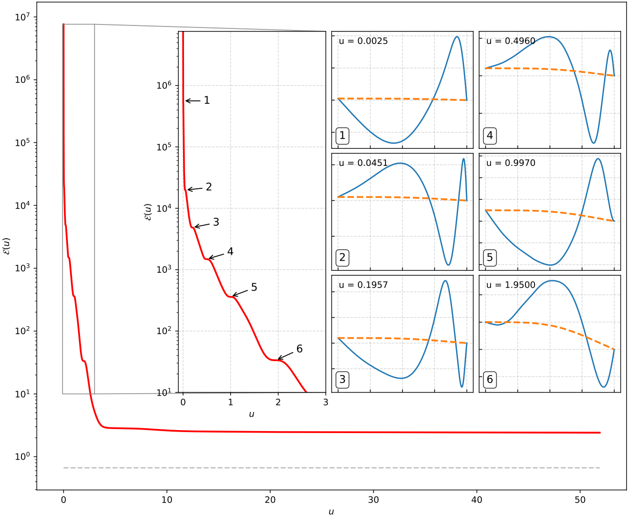
Solitons III

- Large (initial) energy solitons exhibit energy plateaus



Solitons IV

- Linearize around the half-kink: $f(u,x) = Q(x) + \phi(u,x)$

$$(g'^2 - h'^2)\partial_u^2\phi - 2h'\partial_{ux}^2\phi - A(x)\partial_u\phi = \partial_x^2\phi + B(x)\partial_x\phi + C(x)\phi$$

- First-order reduction in time $\psi \equiv \partial_u\phi$ and separate full operator in non/self-adjoint parts

$$\Phi = \begin{pmatrix}\phi \\ \psi\end{pmatrix} \qquad \partial_\tau\Phi = iL\Phi \qquad L = \frac{1}{i}\begin{pmatrix}0 & 1 \\ L_1 & L_2\end{pmatrix}$$

$$L_1 = \frac{1}{g'^2 - h'^2}\Big[\partial_x(p(x)\partial_x) - q(x)\Big]$$

$$L_2 = \frac{1}{g'^2 - h'^2}\Big[2h'\partial_x + A(x)\Big]$$

In Progress

- Yang-Mills soliton QNMs, analyze the stability characteristics using the pseudospectrum
- Connection to resonance windows? Staccato radiation?
- Extend to other solitons (e.g. ϕ^4, sine-Gordon), types of perturbations, choices of compactification