

Scalar curvature operator for loop quantum gravity on a cubical graph

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J. Lewandowski, I.M. Phys. Rev. D 106, 046013 [arXiv:2110.10667]

The 8th Conference of the Polish Society on Relativity
Banach Center IMPAN, Warsaw, Poland

22.09.2022

Scalar curvature in loop quantum gravity

The object of interest: Ricci scalar integrated over the spatial manifold

$$\int_{\Sigma} d^3x \sqrt{q} {}^{(3)}R$$

Relevant to loop quantum gravity

- As a geometrical observable
- As an alternative to the Lorentzian part of the Hamiltonian constraint

$$C = \frac{1}{\beta^2} \frac{\epsilon^{ij} E_i^a E_j^b F_{ab}^k}{\sqrt{|\det E|}} + (1 + \beta^2) \sqrt{|\det E|} {}^{(3)}R$$

Previously: The "Regge" curvature operator (Alesci, Assanioussi, Lewandowski 2014)

$$\int d^3x \sqrt{q} {}^{(3)}R \simeq \sum_{\text{hinges}} (\text{hinge length}) \times (\text{deficit angle})$$

Refers classically to an auxiliary manifold of singular geometry (curvature concentrated on one-dimensional hinges) instead of the actual physical manifold.

Ricci scalar as a function of the Ashtekar variables

The starting point of our construction: Express the Ricci scalar as ${}^{(3)}R(A, E)$

$$q^{ab} = \frac{E_i^a E_i^b}{|\det E|} \quad ({}^{(3)}R)_{ab} = \partial_c \Gamma_{ab}^c - \partial_b \Gamma_{ac}^c + \Gamma_{ab}^c \Gamma_{cd}^d - \Gamma_{ad}^c \Gamma_{bc}^d$$

$$\begin{aligned} |\det E| ({}^{(3)}R) &= -2E_i^a \mathcal{D}_{(a} \mathcal{D}_{b)} E_i^b + 2Q^{ab} E_c^i \mathcal{D}_{(a} \mathcal{D}_{b)} E_i^c \\ &- (\mathcal{D}_a E_i^a)(\mathcal{D}_b E_i^b) - \frac{1}{2}(\mathcal{D}_a E_i^b)(\mathcal{D}_b E_i^a) \\ &+ \frac{5}{2}Q^{ab}(\mathcal{D}_a E_i^c)(\mathcal{D}_b E_i^c) - \frac{1}{2}Q^{ab}Q_{cd}(\mathcal{D}_a E_i^c)(\mathcal{D}_b E_i^d) \\ &+ 2A^{ab}{}_a \mathcal{B}_{cb}{}^c + 2A^{ab}{}_b \mathcal{B}_{ca}{}^c + A^{ab}{}_c \mathcal{B}_{ba}{}^c \\ &+ \frac{1}{2}Q_{ab} \mathcal{A}^{ca}{}_d \mathcal{A}^{db}{}_c - Q^{ab} \mathcal{B}_{ca}{}^c \mathcal{B}_{db}{}^d \\ &+ 2(Q^{ab} \mathcal{B}_{ca}{}^c - \mathcal{A}^{ab}{}_a - \mathcal{A}^{ba}{}_a) \frac{\partial_b |\det E|}{|\det E|} \\ &+ \frac{3}{2}Q^{ab} \frac{\partial_a |\det E|}{|\det E|} \frac{\partial_b |\det E|}{|\det E|} - 2Q^{ab} \frac{\partial_a \partial_b |\det E|}{|\det E|} \end{aligned} \quad \begin{aligned} Q^{ab} &= E_i^a E_i^b \\ Q_{ab} &= E_a^i E_b^i \\ \mathcal{A}^{ab}{}_c &= E_i^a \mathcal{D}_c E_i^b \\ \mathcal{B}_{ab}{}^c &= E_a^i \mathcal{D}_b E_i^c \end{aligned}$$

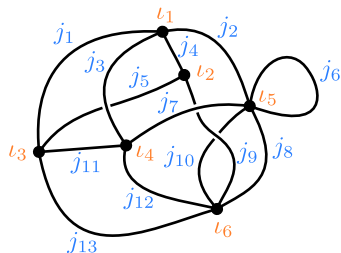
Gauge covariant derivatives: $\mathcal{D}_a E^b = \partial_a E^b + [A_a, E^b]$

$$\mathcal{D}_a \mathcal{D}_b E^c = \partial_a (\mathcal{D}_b E^c) + [A_a, \mathcal{D}_b E^c]$$

The kinematical framework of loop quantum gravity

The kinematical Hilbert space of LQG is spanned by the (generalized) spin network states

$$|\Gamma, \{j_e\}, \{\iota_v\}\rangle = \left(\prod_{v \in \Gamma} \iota_v \right) \cdot \left(\prod_{e \in \Gamma} D^{(j)}(h_e) \right)$$



The elementary operators of the theory are the holonomy and flux operators

$$h_e = \mathcal{P} \exp \left(- \int_e A \right)$$
$$E(S) = \int_S d^2\sigma n_a(\sigma) E_i^a(\sigma) \tau^i$$

The holonomy acts as a multiplicative operator; the flux essentially as a functional derivative with respect to the Ashtekar connection.

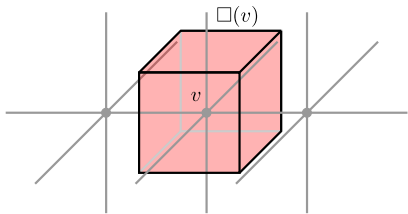
Regularization of the Ricci scalar on a cubical graph

Main assumption/simplification:

Aim to construct the operator on the Hilbert space of a fixed cubical graph.

May seem like a severe restriction. However, several approaches in LQG make extensive use of states defined on cubical graphs:

- Quantum-reduced loop gravity
- Models of effective dynamics
- Algebraic quantum gravity



The integrated Ricci scalar is regularized as a Riemann sum over the cubical partition:

$$\int d^3x \sqrt{q} {}^{(3)}R \simeq \sum_{\square} \epsilon^3 \sqrt{|\det E|}(v) {}^{(3)}R(v)$$

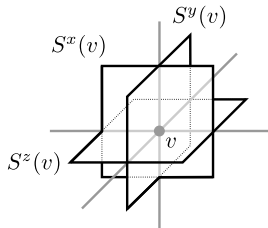
Regularization of gauge covariant derivatives

We use parallel transported flux variables (also known as gauge covariant fluxes) to regularize the gauge covariant derivatives of the triad.

$$\tilde{E}(S, x_0) = \int_S d^2\sigma n_a(\sigma) h_{x(\sigma) \rightarrow x_0} E^a(x(\sigma)) h_{x(\sigma) \rightarrow x_0}^{-1}$$

The holonomies $h_{x(\sigma) \rightarrow x_0}$ perform parallel transport from points on the surface to a fixed point x_0 along a chosen family of paths.

Covariant derivatives of the triad are approximated as finite differences of neighboring parallel transported fluxes.

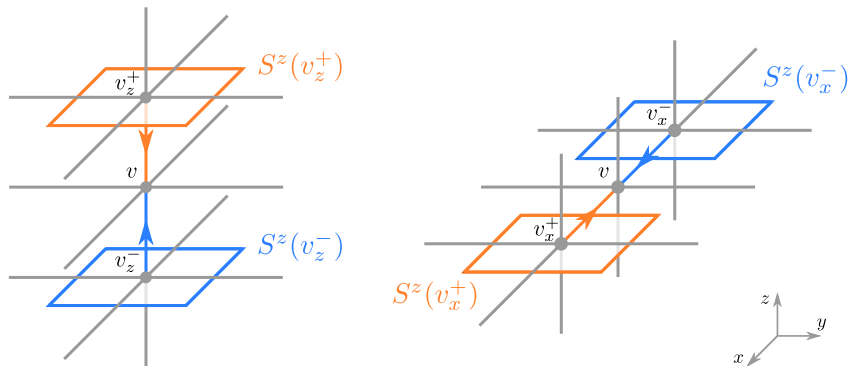


First derivatives:
$$f'(x) \simeq \frac{f(x + \epsilon) - f(x - \epsilon)}{2\epsilon}$$

Second derivatives:
$$f''(x) \simeq \frac{f(x + \epsilon) - 2f(x) + f(x - \epsilon)}{\epsilon^2}$$

$$\frac{\partial^2 f(x, y)}{\partial x \partial y} \simeq \frac{f(x + \epsilon, y + \epsilon) - f(x + \epsilon, y - \epsilon) - f(x - \epsilon, y + \epsilon) + f(x - \epsilon, y - \epsilon)}{4\epsilon^2}$$

First derivatives

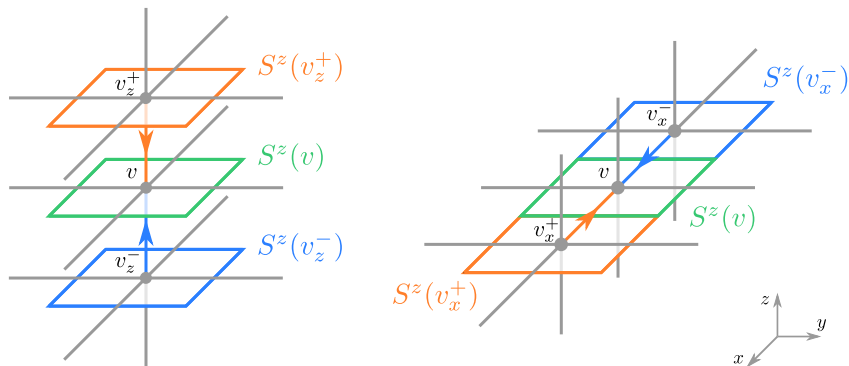


The covariant derivative $\mathcal{D}_a E^b(v)$ is approximated by the discretized variable

$$\Delta_a E(S^b, v) \equiv \frac{\tilde{E}(S^b(v_a^+), v) - \tilde{E}(S^b(v_a^-), v)}{2}$$

For small regularization parameter: $\Delta_a E(S^b, v) = \epsilon^3 \mathcal{D}_a E^b(v) + \mathcal{O}(\epsilon^4)$

Second derivatives



The diagonal components of the second covariant derivative are discretized as

$$\Delta_{aa}E(S^b, v) \equiv \tilde{E}(S^b(v_a^+), v) - 2\tilde{E}(S^b(v), v) + \tilde{E}(S^b(v_a^-), v)$$

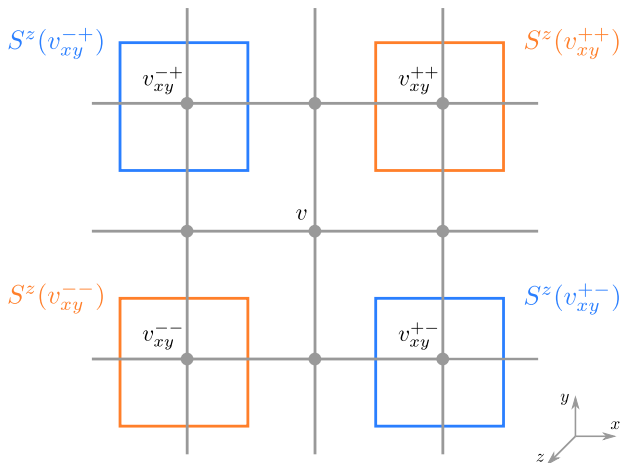
For small regularization parameter: $\Delta_{aa}E(S^b, v) = \epsilon^4 \mathcal{D}_a^2 E^b(v) + \mathcal{O}(\epsilon^5)$

Mixed second derivatives

The regularization of the mixed second derivative

$$\mathcal{D}_a \mathcal{D}_b E^c(v)$$

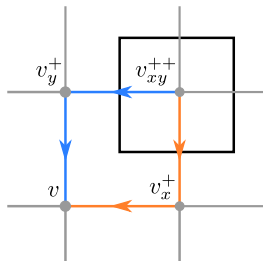
involves the four nodes diagonally adjacent to v in the (a, b) coordinate plane.



Symmetric regularization of mixed second derivatives

Two equally good paths are now available for parallel transport to the central node. Hence introduce

$$\begin{aligned}\tilde{E}(S^z(v_{xy}^{++}), v)_{\text{sym.}} &\equiv \frac{1}{2} \left(\tilde{E}(S^z(v_{xy}^{++}), v)_{v_{xy}^{++} \rightarrow v_x^+ \rightarrow v} \right. \\ &\quad \left. + \tilde{E}(S^z(v_{xy}^{++}), v)_{v_{xy}^{++} \rightarrow v_y^+ \rightarrow v} \right)\end{aligned}$$



Then we define the discretized variable

$$\begin{aligned}\Delta_{ab}E(S^c, v) &\equiv \frac{1}{4} \left(\tilde{E}(S^c(v_{ab}^{++}, v)_{\text{sym.}}) - \tilde{E}(S^c(v_{ab}^{+-}, v)_{\text{sym.}}) \right. \\ &\quad \left. - \tilde{E}(S^c(v_{ab}^{-+}, v)_{\text{sym.}}) + \tilde{E}(S^c(v_{ab}^{--}, v)_{\text{sym.}}) \right)\end{aligned}$$

This approximates the symmetric part of the mixed second derivative at v :

$$\Delta_{ab}E(S^c, v) = \epsilon^4 \mathcal{D}_{(a} \mathcal{D}_{b)} E^c(v) + \mathcal{O}(\epsilon^5)$$

Quantization

After regularization on the cubical lattice, we have

$$\sqrt{q}^{(3)}R = \mathcal{R}(E^a, \mathcal{D}_a E^b, \mathcal{D}_{(a} \mathcal{D}_{b)} E^c, \sqrt{|\det E|}, \partial_a \sqrt{|\det E|})$$
$$\int d^3x \sqrt{q}^{(3)}R \simeq \sum_{\square} \mathcal{R}(\tilde{E}(S^a), \Delta_a E(S^b), \Delta_{ab} E(S^c), V(\square), \Delta_a V(\square))$$

Every factor appearing here can now be promoted into an operator in LQG.

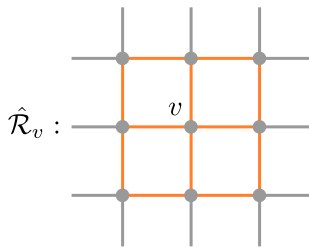
The result is an operator of the form

$$\int d^3x \widehat{\sqrt{q}^{(3)}R} = \sum_{v \in \Gamma_{\text{cubical}}} \hat{\mathcal{R}}_v$$

on the Hilbert space of the chosen cubical graph.

Properties:

- $SU(2)$ gauge invariant
- Densely defined adjoint operator
- Level of complexity roughly similar to the Euclidean part of Thiemann's Hamiltonian



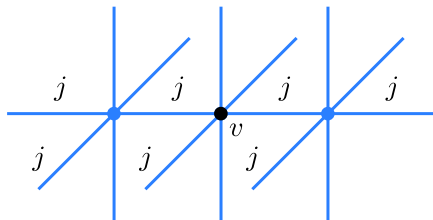
Example: Curvature of reduced spin network states

To study the curvature operator in the setting of a simple example, we compute its action on the "reduced spin network states" of quantum-reduced loop gravity. (Alesci, Cianfrani 2013; Alesci, Bahrami, Pranzetti 2018; I.M. 2020; ...)

The reduced spin network states are defined on a cubical graph. Each edge of the state carries a "reduced holonomy" of the form

$$D_{jj}^{(j)}(h_e)$$

An example calculation: Expectation value of the curvature.

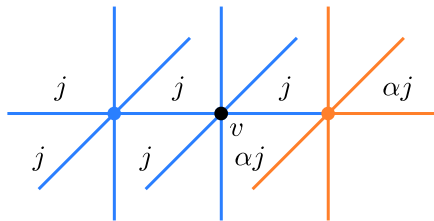


$$\langle \mathcal{R}_v \rangle = -21\sqrt{j}$$

$$\Delta \mathcal{R}_v = \sqrt{\langle \mathcal{R}_v^2 \rangle - \langle \mathcal{R}_v \rangle^2} \simeq 7.3\sqrt{j}$$

Example: Curvature of reduced spin network states

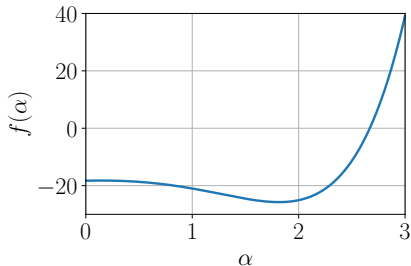
Example of a state with a positive expectation value:



$$\langle \mathcal{R}_v \rangle = f(\alpha) \sqrt{j}$$

For $\alpha \gtrsim 2.7$ the expectation value of curvature at v becomes positive.

Is the operator nevertheless "too negative"? A detailed semiclassical analysis would help to clarify the situation.



Summary and outlook

We have defined a new operator representing the three-dimensional scalar curvature in loop quantum gravity.

- The operator is restricted to the Hilbert space of a fixed cubical graph
- Classical starting point: Ricci scalar expressed directly as a function of the densitized triad and its gauge covariant derivatives
- The covariant derivatives are quantized by using parallel transported flux variables to discretize them on the cubical lattice provided by the graph

Open questions/topics for future work:

- Generalize the construction to other kinds of graphs (e.g. four-valent nodes)
- Extension to the entire kinematical Hilbert space of LQG? (Modified regularization of derivatives needed to obtain a symmetric operator)
- Semiclassical properties?
- Physical applications?

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Thank you for your attention!