Existence of potential for linearized Weyl tensor as a generalized Poincare lemma

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Decomposition of Riemann tensor

$$R_{\mu\nu\alpha\beta} = W_{\mu\nu\alpha\beta} + \frac{1}{2}(R_{\alpha[\mu}g_{\nu]\beta} - R_{\beta[\mu}g_{\nu]\alpha}) + \frac{1}{6}Rg_{\alpha[\mu}g_{\nu]\beta}$$

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Important identity for Weyl tensor

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Vacuum Einstein equations can be treated as a spin-2 field equations

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$$dF = 0\,, \qquad d*F = 0$$

First equation quarantee existing of four-potential A_{μ}. The second equation is obtained from lagrangian

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and there are obtained using third order potential $A_{\mu\nu\alpha}$ from lagrangian

$$L = \frac{1}{16}\big\langle \mathrm{TL}(A_{\mu\nu[\alpha|\beta]} + A_{\alpha\beta[\mu|\nu]}), \mathrm{TL}(A_{\mu\nu[\alpha|\beta]} + A_{\alpha\beta[\mu|\nu]})\big\rangle = \frac{1}{16}\langle W, W\rangle$$

Spin-2 field vs electrodynamics

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What is quarantee, that such potential A exists?

Existence of potential $A_{\mu\nu\alpha}$ for Weyl-candidate tensor was suggested by Lanczos in 1962 and fully proved by F. Bampi and G. Caviglia in 1983:

THEOREM

Let (M, g) be a Riemmanian manifold and let $W_{\mu\nu\alpha\beta}$ be a tensor having symmetries of Weyl tensor. Then there exists a tensor $A_{\mu\nu\alpha}$ such that

$$W_{\mu\nu\alpha\beta} = \mathrm{TL}(A_{\mu\nu[\alpha|\beta]} + A_{\alpha\beta[\mu|\nu]}) \,,$$

where by TL *we denoted a projection to traceless part of a given tensor.*

Lanczos potential

Existence of potential $A_{\mu\nu\alpha}$ for Weyl-candidate tensor was suggested by Lanczos in 1962 and fully proved by F. Bampi and G. Caviglia in 1983:

Theorem

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My idea is to show that on a flat Riemmanian manifold, existing of potential for every spin-n field is a consequence of identity type

$$\mathrm{TL}(W_{\mu\nu[\alpha\beta|\gamma]}) = 0$$

and generalized Poincaré lemma.

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Let (Ω, d) *be an N-complex. Let denote by* d_p *restriction of* d *to* Ω_p *(subspace of degree p). p-th cohomology space of degree k of N-complex* (Ω, d) *is a space*

$$H_p^k(V) := \ker \mathrm{d}_p^k \,/\, \mathrm{Im}\, \mathrm{d}_{p-(N-k)}^{N-k}$$

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$H_p^k = 0$ means just that if $\mathrm{d}_p^k v = 0$, then there exists some $w \in \mathrm{d}_{p-(N-k)}$, że $v = \mathrm{d}^{(N-k)} v$.

Definition

Let M be a riemmanian manifold, and Y be an Young diagram of length $|Y|$. Traceless Young symmetrizer is an operator $\mathbf{Y} : T^*M^{\otimes^{|Y|}} \to T^*M^{\otimes^{|Y|}}$ *defined as*

$$\mathbf{Y} = \lambda_Y \cdot \mathrm{TL} \circ A_{\mathrm{col}} \circ S_{\mathrm{row}}$$

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Let (Y_p) be a sequence of Young diagrams such that $|Y| = p$ and Y_p consists of the k rows of length $(N-1)$ and one row of length r, where $p = (N-1)k + r$, $r < p$. For a given riemannian manifold M we define

$$\Omega_N(M) := \bigoplus_p \mathbf{Y}_p T^*M^{\otimes^p}, \qquad \mathrm{d}(T_p) := \mathbf{Y}_{p+1} \circ \nabla T_p \ \text{ dla } \ T_p \in \mathbf{Y}_p T^*M^{\otimes^p}$$

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Proposition

Let M be a flat riemannian manifold. Then $(\Omega_N(M), \mathrm{d})$ is an N-complex.

It just means, that $\mathrm{d}^N = 0$.

Examples for several N

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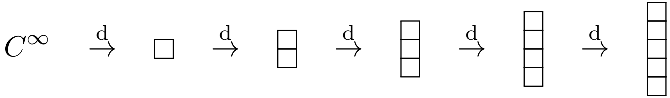

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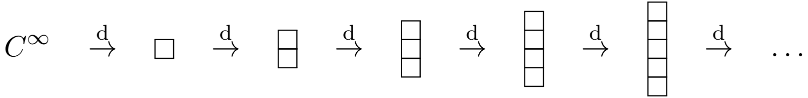

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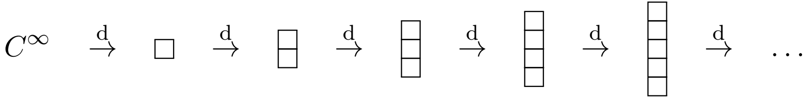

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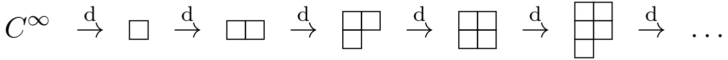

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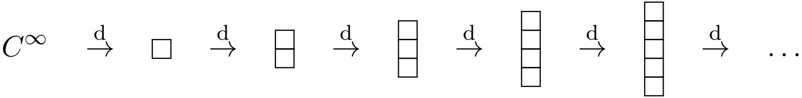

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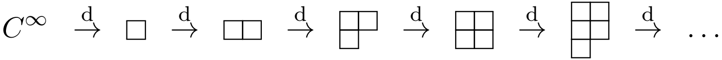

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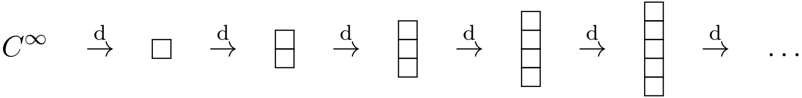

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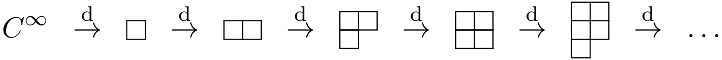

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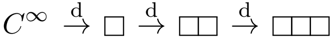

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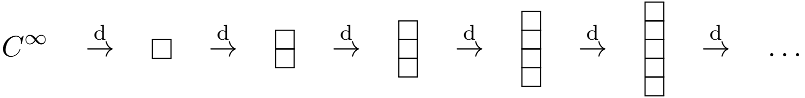

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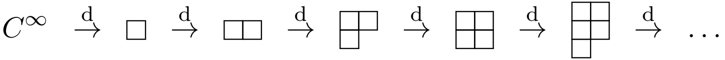

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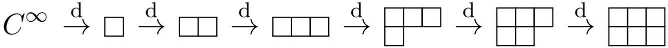

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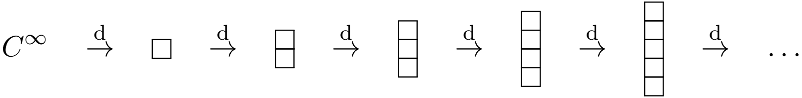

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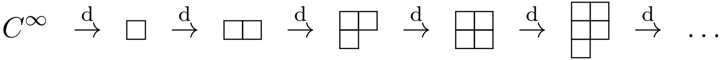

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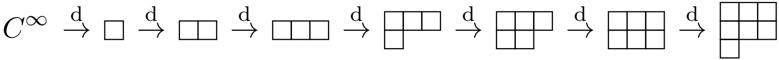

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$$C^\infty \xrightarrow{\mathrm{d}} \ydiagram{1} \xrightarrow{\mathrm{d}} \ydiagram{1,1} \xrightarrow{\mathrm{d}} \ydiagram{1,1,1} \xrightarrow{\mathrm{d}} \ydiagram{1,1,1,1} \xrightarrow{\mathrm{d}} \ydiagram{1,1,1,1,1} \xrightarrow{\mathrm{d}} \ldots$$

$N = 3 \quad \longrightarrow \quad$ spin-2 field (or traceless part of linearized gravity:)

$$C^\infty \xrightarrow{\mathrm{d}} \ydiagram{1} \xrightarrow{\mathrm{d}} \ydiagram{2} \xrightarrow{\mathrm{d}} \ydiagram{2,1} \xrightarrow{\mathrm{d}} \ydiagram{2,2} \xrightarrow{\mathrm{d}} \ydiagram{2,2,1} \xrightarrow{\mathrm{d}} \ldots$$

$N = 4 \quad \longrightarrow \quad$ spin-3 field:

$$C^\infty \xrightarrow{\mathrm{d}} \ydiagram{1} \xrightarrow{\mathrm{d}} \ydiagram{2} \xrightarrow{\mathrm{d}} \ydiagram{3} \xrightarrow{\mathrm{d}} \ydiagram{3,1} \xrightarrow{\mathrm{d}} \ydiagram{3,2} \xrightarrow{\mathrm{d}} \ydiagram{3,3} \xrightarrow{\mathrm{d}} \ydiagram{3,3,1} \xrightarrow{\mathrm{d}} \ldots$$

Cohomology of $(\Omega_N(M), \mathrm{d})$ (generalized Poincaré lemma)

Theorem

Let M be simply connected, flat, riemannian manifold. Wówczas

$$H_{k(N-1)}\left(\Omega_N(M)\right) = 0 \quad \forall_{k\in\mathbb{N}}$$

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Corollary

Let D be a simply connected domain in a Minkowski space, and $p = (N-1)k$. If tensor $T \in T^*D^{\otimes^p}$ (tensor of rectangular Young diagram) fulfills

$$\mathrm{d}^k T = 0\,,$$

then there exists a tensor $S \in T^*D^{\otimes^{p-(N-k)}}$, such that

$$\mathrm{d}^{N-k} S = T\,.$$

Spin n field can be represented as traceless tensor

$F_{\mu_1\nu_1\mu_2\nu_2\ldots\mu_n\nu_n} \in \Omega_{n+1}(M)$ with Young symmetry $\underbrace{\begin{array}{|c|c|c|c|}\hline \mu_1 & \mu_2 & \ldots & \mu_n \\ \hline \nu_1 & \nu_2 & \ldots & \nu_n \\ \hline \end{array}}_{n}$.

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Because of dimension equal 4, it fulfills automaticaly

$$\mathrm{d}\, F = 0 \qquad \left(\Longleftrightarrow \mathrm{TL}(\nabla_{[\alpha} F_{\beta\gamma]\mu_2\nu_2\ldots\mu_n\nu_n}) = 0\right) ,$$

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Field equations

$$\nabla_{[\alpha}F_{\beta\gamma]\mu_2\nu_2\ldots\mu_n\nu_n} = 0$$

are obtained from lagrangian

$$L = \int_M \langle \mathrm{d}^n\, A, \mathrm{d}^n\, A\rangle\,.$$

THANK YOU FOR YOUR ATTENTION