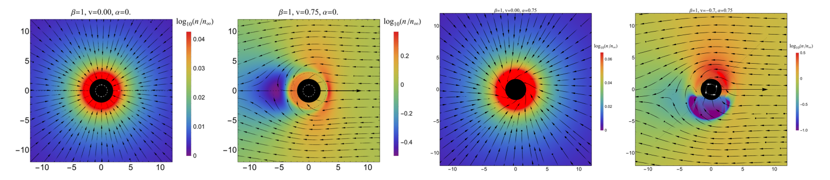
Equatorial accretion on the Kerr black hole

Andrzej Odrzywołek

IFT UJ

PoToR 8, 14:40 Środa, 21.09.2022



With P. Mach & A. Cieślik

Scientific programme: full GR accretion of DM onto black holes

Progress for equatorial plane accretion (2+1 slice)

2+1	$v = 0$	Moving black hole $v \neq 0$
a=0	1D	in progress, 2D
Rotating black hole $a \neq 0$	A. Cieślik, P. Mach, A. Odrzywołek, submitted, arxiv:2208.04218, 1D	in progress, 2D

Progress for standard GR accretion (3+1)

3+1	$v = 0$	Moving black hole $v \neq 0$
$a = 0$	A. Cieślik, P. Mach Phys. Rev. D 102, 024032 (2020), 1D	P. Mach, A. Odrzywołek Phys. Rev. Lett. 126, 101104 (2021), 2D
Rotating black hole $a \neq 0$	in progress, 2D	fire everything! (3D)

Model details

1. black hole metric (equatorial slice of Kerr in Boyer-Lindquist coords)

$$\gamma_{\mu\nu} = \begin{pmatrix} -1 + \frac{2M}{r} & 0 & -\frac{2aM}{r} \\ 0 & \frac{r^2}{a^2 - 2Mr + r^2} & 0 \\ -\frac{2aM}{r} & 0 & \frac{2a^2M}{r} + a^2 + r^2 \end{pmatrix}$$

2. Hamiltonian approach to geodetic motion for ensemble of massive particles *via* Własow equation $\{H, f\} = 0$, where f is the distribution function in phase-space, and $\{\cdot, \cdot\}$ Poisson bracket
3. Hamilton-Jacobi approach using abbrv. action W to find generating function for canonical transformation (old vars, new momenta) making Własow equation trivial, *i.e.*, $\partial f / \partial Q^0 = 0$
4. canonical transformation $\{t, r, \varphi, p_t, p_r, p_\varphi\} \to \{P_0, P_1, P_2, Q^0, Q^1, Q^2\}$ where, e.g:
5. Maxwell - Juttner distribution function at infinity (boosted in moving case)
6. integrate phase-space to obtain currents J_μ, densities, accretion rates (mass, energy, spin) and more (e.g. drag and lift)

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$$H(t, r, \varphi, p_t, p_r, p_\varphi) = \frac{1}{2}\gamma^{\mu\nu} p_\mu p_\nu = \frac{1}{2}\overbrace{\left(-\frac{p_t^2 \left(a^2(2M + r) + r^3 \right)}{r\Delta} - \frac{4aMp_t p_\varphi}{r\Delta} + \frac{p_\varphi^2 (1 - 2M/r)}{\Delta} + \frac{p_r^2 \Delta}{r^2} \right)}^{m^2}$$

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$$W = -P_1 t + P_2 \varphi + \epsilon_r \int \frac{r\sqrt{(a^2 + r^2)\left({P_1}^2 - {P_0}^2\right) + \frac{2M(P_2 - aP_1)^2}{r} + 2M{P_0}^2 r - {P_2}^2}}{\Delta} dr$$

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$$P_0 = \sqrt{-2H} \equiv m, P_1 = -p_t, P_2 = p_\varphi, Q^2 = \varphi - \epsilon_r X\left(\frac{r}{M}, \frac{-p_t}{m}, \epsilon_\sigma \frac{p_\varphi + a p_t}{Mm}, \epsilon_\sigma \frac{a}{M}\right)$$

where signs $\epsilon_{r,\sigma} = \pm 1$ and *elliptic „X" function* [A. Cieślik talk]

$$X(\xi, \varepsilon, \lambda, \alpha) = \int_\xi^\infty \frac{\lambda + \frac{\alpha\varepsilon}{1-\frac{2}{\xi}}}{\left(\xi^2 + \frac{\alpha^2}{1-\frac{2}{\xi}}\right)\sqrt{\varepsilon^2 - \left(1 - \frac{2}{\xi}\right)\left(1 + \frac{\lambda^2}{\xi^2}\right) - \frac{\alpha^2 + 2\varepsilon\lambda\alpha}{\xi^2}}}\, d\xi$$

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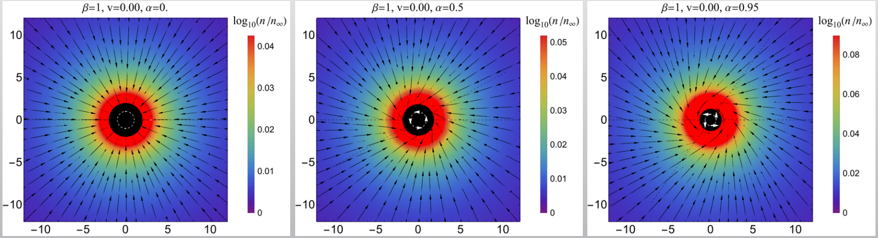
$$J_\mu = \iiint p_\mu f(P_0, P_1, P_2, Q^2) \delta \left(m(r, p_t, p_r, p_\varphi) - m_0 \right) \sqrt{-|\gamma^{\mu\nu}|} dp_t dp_r dp_\varphi$$

Accretion flow



Properties of the flow

- Kerr black hole rotates counterclockwise,
- streamlines show spatial part of particle current J^μ,
- color show surface density $n_s = \sqrt{-J^\mu J_\mu}$,
- streamlines twist near photon circles,
- shrinking horizon (black disk) and resized prograde/retrograde photon circles.

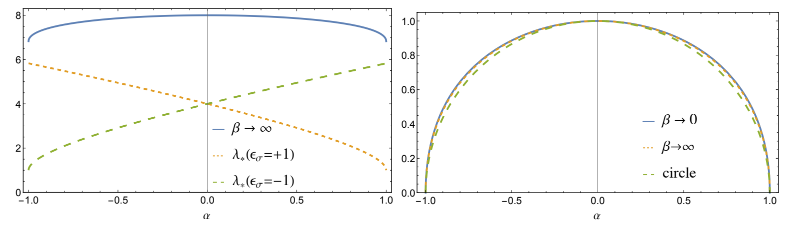
Equatorial Kerr mass/energy accretion rates

Simple formulas can be derived for cold ($\beta \equiv \frac{mc^2}{k_B T} \to \infty$) and hot ($\beta \to 0$) limits.

$$\lim_{\beta\to\infty} \dot{M} = 8M\rho_\infty \cos^2\left(\frac{\arcsin\alpha}{4}\right), \qquad \rho_\infty \text{ – asymptotic } \textbf{surface} \text{ density}$$

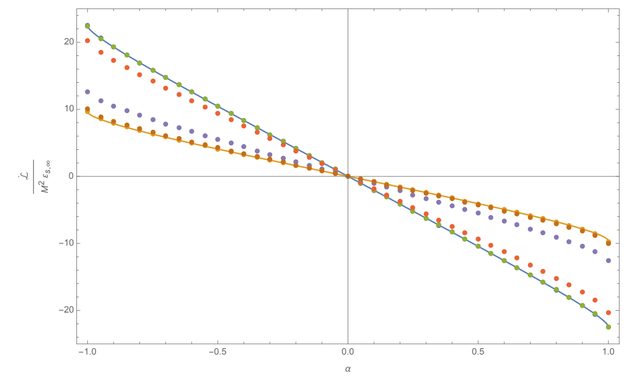
$$\lim_{\beta\to 0} \dot{M} = 6\sqrt{3}M\rho_\infty \cos\left(\frac{\arcsin\alpha}{3}\right), \qquad \alpha \text{ – dimensionless Kerr parameter}$$

Above formulas lead to remarkable, circular curves (after rescalling).



NOTE: $\dot{M}(\alpha)$ varies proportionally to shrinking/expanding of both (clockwise & anti-clockwise) marginally bound circles.

Equatorial Kerr momentum accretion rates



The coefficient (slope)

$$-12\sqrt{3} \leqslant \quad \delta \equiv \frac{d}{d\alpha}\left(\frac{\dot{\mathcal{L}}}{M^2\varepsilon_\infty}\right)\Bigg|_{\alpha=0} \quad \leqslant -8$$

varies from $\delta = -8$ in the limit $\beta \to \infty$ to $\delta = -12\sqrt{3}$ for $\beta \to 0$.
NOTE: slope δ is negative, therefore accretion **slow down** black hole rotation.

Stopping time

Kerr parameter $\alpha = a/M$ change due to accretion of both mass-energy and angular momentum

$$\frac{d\alpha}{dt} = \frac{d}{dt}\frac{J}{M^2} = \frac{J}{M^2}\left(\frac{\dot{\mathcal{L}}}{J} - 2\frac{\dot{\mathcal{E}}}{M}\right) = \alpha\left(\frac{\dot{\mathcal{L}}}{J} - 2\frac{\dot{\mathcal{E}}}{M}\right).$$

In the limit of $\beta \to \infty$ (cold matter) we get, for small values of α,

$$\frac{1}{\alpha}\frac{d\alpha}{dt} \simeq -24\rho_\infty.$$

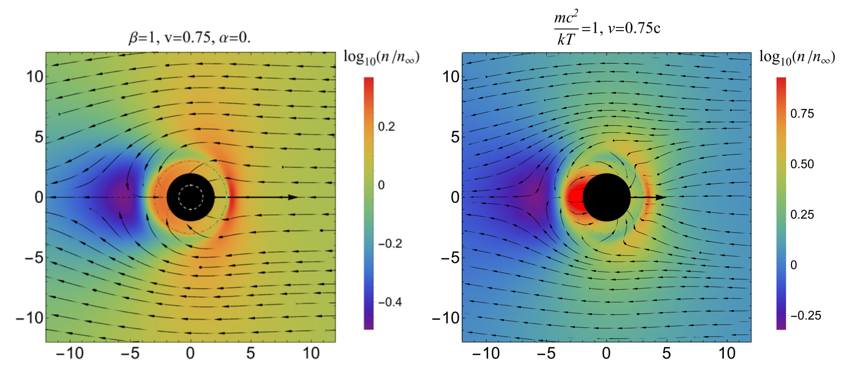
Therefore, in this limit the rotation of the black hole **slows down exponentially**, with same timescale for **any mass** M (e.g. primordial, stellar, IMBH or SMBH).

The stopping time scale

$$\tau = \frac{c}{24G\rho_\infty} \qquad \text{(in SI units)} \quad \tau \sim 10^9 \text{ years for Milky Way DM.}$$

NOTE large 1/24 coefficient.

2+1 (left) vs 3+1 (right)



Essential results

Summarry

1. "circular" shape of $\dot{M}(\alpha)$ relation due to shrinking/expanding photon circles
2. nearly linear dependence of angular momentum accretion $\dot{\mathcal{L}}(\alpha)$, except$(\alpha \pm 1)$
3. accretion slow down black hole rotation with timescale $\tau = c/(24G\rho_\infty)$
4. using present-day density of Dark Matter timesale is of the order of 10^9 years for **ALL** black hole masses *(extrapolating 2+1 → 3+1)*
5. surprising differences between 2+1 and 3+1 cases
6. crucial step forward towards 3+1 Kerr accretion and Magnus effect

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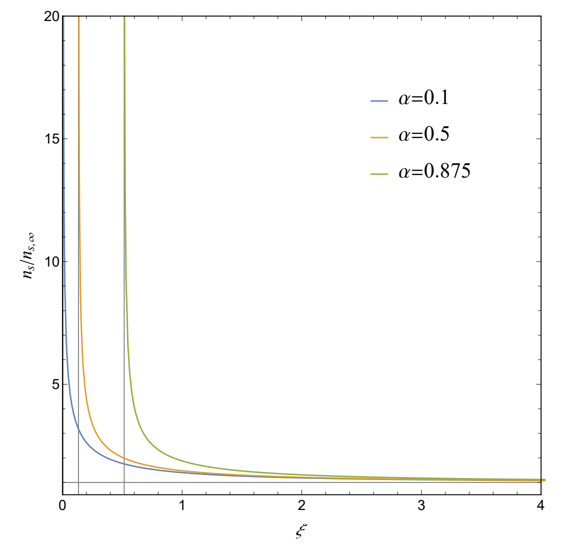
EXTRA SLIDES

Surface density



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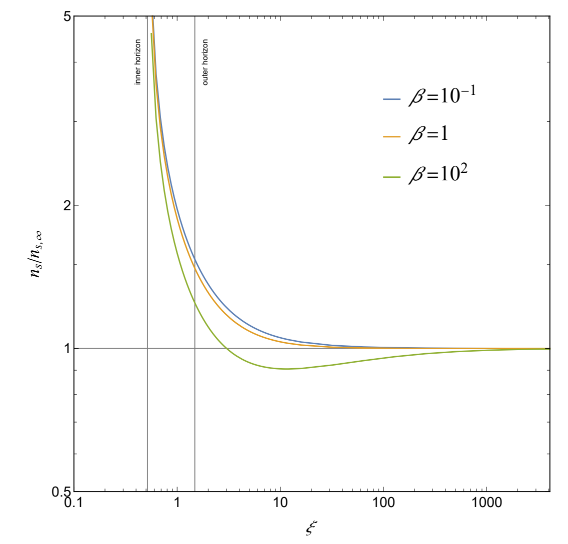
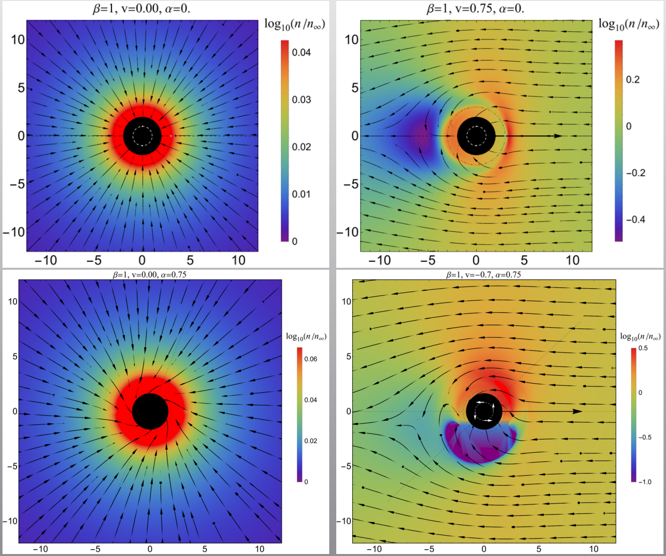


Tentative prerliminary results: particle current J^{μ}



Tentative prerliminary results: lift via Żukowski theorem

According to *Kutta-Żukowski Theorem* lift force coefficient c_l can be computed as

$$c_l = \rho_\infty v_\infty \Gamma,$$

where circulation Γ can be computed by line integral over closed contour encircling black hole

$$\Gamma = \oint J^\mu ds$$

Using circle of constant radius ξ as a contour, we get

$$\Gamma = \int_{-\pi}^{\pi} J^\varphi(\xi, \phi)\xi \, d\phi$$