Resonant enlargements of the Poincaré/AdS (super)algebras

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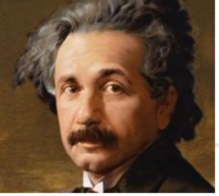


- Einstein theory $\boxed{g_{\mu\nu}}$, $\Gamma^{\lambda}{}_{\mu\nu}(g)$ (torsion $T^{\lambda}{}_{\mu\nu} = 0$, $\nabla_{\rho} g_{\mu\nu} = 0$)

$$S_{Einstein-Hilbert} = \frac{1}{16\pi G} \int \mathrm{d}^4 x \sqrt{-g}\, (R[g] - 2\Lambda)$$

- Palatini theory $\boxed{g_{\mu\nu}}$, $\boxed{\Gamma^{\lambda}{}_{\mu\nu}(g)}$ + Cartan $\boxed{g_{\mu\nu}}$, $\boxed{\Gamma^{\lambda}{}_{\mu\nu}}$

$$S_{Palatini/Cartan} = \frac{1}{16\pi G} \int \mathrm{d}^4 x \sqrt{-g}\, (R[g, \Gamma] - 2\Lambda)$$

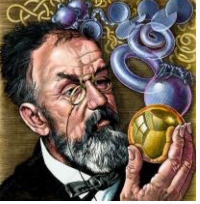
- Einstein-Cartan theory $\boxed{e^a_\mu}$, $\boxed{\omega^{ab}_\mu}$ $g_{\mu\nu} = \eta_{ab}\, e^a_\mu\, e^b_\nu$

$S_{EC}[\omega(e), e] \equiv S_{EH}[g_{\mu\nu}]$

$$S_{EC}(e, \omega) = \frac{1}{32\pi G} \int \epsilon_{abcd} \left(e^c \wedge e^d \wedge R^{ab}(\omega) - \frac{\Lambda}{6} e^a \wedge e^b \wedge e^c \wedge e^d \right)$$

- MacDowell-Mansouri theory $\boxed{A^{IJ}_\mu}$ $A^{ab}_\mu = \omega^{ab}_\mu$, $A^{a4}_\mu = \frac{1}{\ell} e^a_\mu$

$$S_{MM}(A) = \frac{\ell^2}{64\pi G} \int \left(R^{ab} + \frac{1}{\ell^2} e^a \wedge e^b \right) \wedge \left(R^{cd} + \frac{1}{\ell^2} e^c \wedge e^d \right) \epsilon_{abcd}$$



Poincare symmetry of Minkowski space

Poincaré symmetry is a symmetry of Special Relativity described by Minkowski space.

It includes: *rotations* in space, *boosts and translations*

The first two symmetries, ***rotations and boosts*** together make the **Lorentz group**

$$[J_{ab}, J_{cd}] = \eta_{bc}J_{ad} + \eta_{ad}J_{bc} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}$$

Then the ***translations*** group and the **Lorentz group** produce the **Poincaré group**.

$$[J_{ab}, J_{cd}] = \eta_{bc}J_{ad} + \eta_{ad}J_{bc} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}$$
$$[J_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b$$
$$[P_a, P_b] = 0$$

(anti)-De Sitter $\Lambda = \mp \frac{3}{\ell^2}$ (A)dS symmetry

Then the ***translations*** group and the ***Lorentz group*** produce the **AdS group** when

$$J_{ab}$$

$$J_{a4} = \frac{1}{\ell} P_a$$

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} + \eta_{ad} J_{bc} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac}$$
$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b$$
$$[P_a, P_b] = \pm \tfrac{1}{\ell^2} J_{ab}$$

$$J_{AB}$$

$$[J_{AB}, J_{CD}] = \eta_{BC} J_{AD} + \eta_{AD} J_{BC} - \eta_{AC} J_{BD} - \eta_{BD} J_{AC}$$

Gravity as the Gauge Theory with (A)dS group

- Einstein-Cartan theory e^a_μ, ω^{ab}_μ

$$\eta_{ab} = g_{\mu\nu}\, e^\mu_a\, e^\nu_b$$

$$A_\mu = \frac{1}{2} A^{AB}_\mu J_{AB} = \frac{1}{2}\omega^{ab}_\mu J_{ab} + \frac{1}{\ell} e^a_\mu P_a$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

$$S_{MM}(A) = \frac{\ell^2}{64\pi G} \int tr(\hat{F} \wedge \star \hat{F}) = \frac{\ell^2}{64\pi G} \int F^{ab} \wedge F^{bc} \epsilon_{abcd}$$

$$F^{ab} = R^{ab}(\omega) + \frac{1}{\ell^2} e^a \wedge e^b$$

$$F^{a4} = \frac{1}{\ell} T^a = \frac{1}{\ell} D^\omega e^a$$

$$S_{MM}(A) = \frac{\ell^2}{64\pi G} \int \left(R^{ab} + \frac{1}{\ell^2} e^a \wedge e^b \right) \wedge \left(R^{cd} + \frac{1}{\ell^2} e^c \wedge e^d \right) \epsilon_{abcd}$$

Chern-Simons 3D

The most general three-dimensional $\mathcal{N}=1$ Chern-Simons (CS)action is:

$$I_{CS}^{3D} = \frac{k}{4\pi}\int_{\mathcal{M}} \left\langle \mathbb{A}\wedge d\mathbb{A} + \frac{1}{3}\mathbb{A}\wedge[\mathbb{A},\mathbb{A}]\right\rangle .$$

We can transit to the dual 3D fields $\omega^a = \frac{1}{2}\epsilon^{abc}\omega_{bc}$ and generators definition, $J_a = \frac{1}{2}\epsilon_a{}^{bc}J_{ab}$.

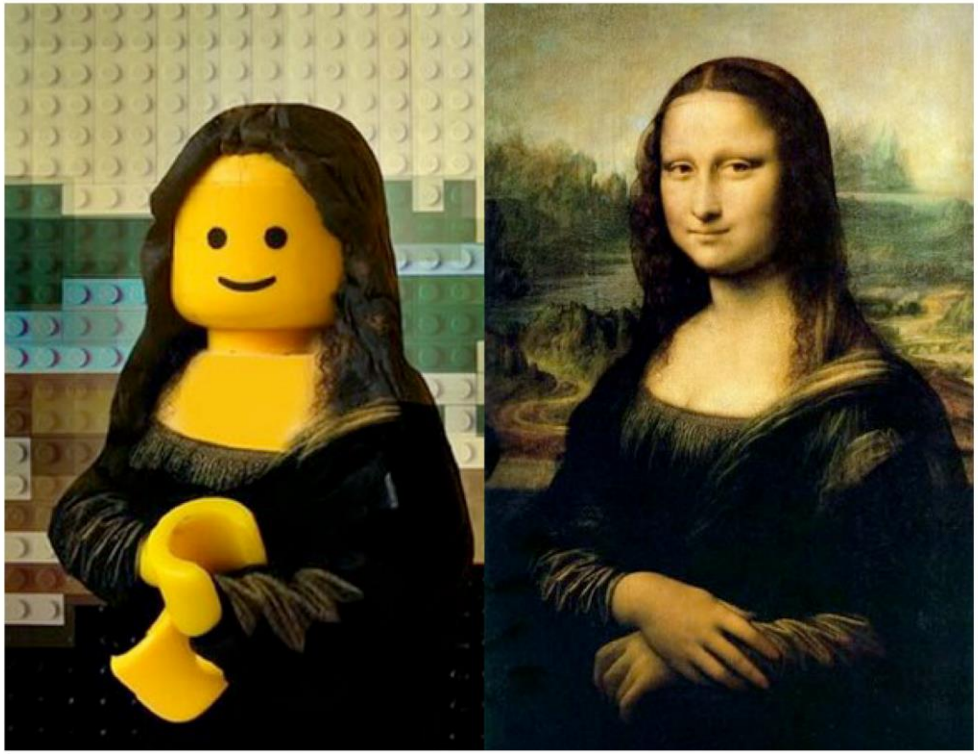
We also introduce: $\mathcal{R}^a = d\omega^a + \frac{1}{2}\epsilon^{abc}\omega_b\omega_c$, $D_\omega e^a = de^a + \epsilon^{abc}\omega_b\, e_c$, the Lorentz covariant derivative acting on a spinor $\mathcal{D}_\omega\psi = d\psi + \frac{1}{2}\omega^a\gamma_a\psi$, as well as define $\mathcal{F} = \mathcal{D}_\omega\psi + \frac{1}{2\ell}e^a\gamma_a\psi$.

$$\langle [[X_i, X_j]]\, X_k\rangle = \langle X_i\, [[X_j, X_k]]\rangle ,$$

3D CS action being invariant under AdS superalgebra:

$$\begin{aligned} I_{CS} \;&=\; \frac{k}{4\pi}\int \alpha_0\left(\omega^a d\omega_a + \frac{1}{3}\epsilon^{abc}\omega_a\omega_b\omega_c + \frac{1}{\ell^2}e_a D_\omega e^a - \frac{2}{\ell}\bar{\psi}\mathcal{F}\right) \\ &\quad +\alpha_1\left(\frac{2}{\ell}\mathcal{R}^a e_a + \frac{1}{3\ell^3}\epsilon^{abc}e_a e_b e_c + \frac{2}{\ell}\bar{\psi}\mathcal{F}\right) \end{aligned}$$

From gravity to supergravity



Supersymmetry relating bosons with fermions

$$\delta Fermion = Boson$$

$$\delta Boson = Fermion$$

$$\delta Action \to invariant$$

Gauge parameter Θ decomposes into parameters of symmetries:

$$\Theta = \frac{1}{2}\lambda^{ab}\mathcal{M}_{ab} + \xi^a\mathcal{P}_a + \bar{\epsilon}^\alpha Q_\alpha \quad \text{(local Lorentz, translation, supercharge)}$$

Supersymmetry transformations (with $D^\omega\bar{\epsilon} = d\bar{\epsilon} - \frac{1}{4}\omega^{ab}\,\bar{\epsilon}\gamma_{ab}$)

$$\delta_\epsilon e^a = -\ell\kappa\,\bar{\epsilon}\gamma^a\psi\,, \qquad \delta_\epsilon\omega^{ab} = \kappa\,\bar{\epsilon}\gamma^{ab}\psi\,, \qquad \delta_\epsilon\bar{\psi} = \frac{1}{\kappa}(\mathcal{D}^\omega\bar{\epsilon} - \frac{1}{2l}e^a\bar{\epsilon}\gamma_a)\,,$$

Adding grawitino ψ_μ^α spin 3/2 field with constant $\kappa^2 = \frac{4\pi G}{\ell}$ to graviton

$$\mathbb{A}_\mu = \left(\frac{1}{2}\omega_\mu^{ab}\mathcal{J}_{ab} + \frac{1}{\ell}e_\mu^a\mathcal{P}_a\right) + \kappa\bar{\psi}_\mu^\alpha\,Q_\alpha$$

boson
spin-2

fermion
spin-3/2

From gravity to supergravity

Spinor matter $\rightarrow T^a \sim \bar{\psi}\,\gamma^a \psi$

Supergravity by Ferrara, Freedman, Nieuwenhuizen, Townsend

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Cosmological constant in supergravity

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(Received 19 January 1977)

We construct an extension of pure supergravity which contains a cosmological term and a masslike term for the spin-3/2 field. Unlike another recent model which incorporates these features, that presented here is constructed from the usual spin-2, spin-3/2 fields alone.

The action

$$I=\int\left(-\frac{1}{4\kappa^2}\,e e^{a\mu}e^{b\nu}R_{\mu\nu ab}+\frac{3\lambda^2 e}{2}-\tfrac{1}{2}\epsilon^{\lambda\rho\mu\nu}\bar{\psi}_\lambda\gamma_5\gamma_\mu D_\nu\psi_\rho-\lambda\kappa e\bar{\psi}_\lambda\sigma^{\lambda\rho}\psi_\rho\right)d^4x$$

MacDowell-Mansouri: gravity/supergravity as "Yang-Mills" like theory with essential role of cosmological constant and topological Euler term

Adding grawitino ψ^α_μ spin 3/2 field with constant $\kappa^2=\frac{4\pi G}{\ell}$ to graviton

$$\mathbb{A}_\mu=\left(\frac{1}{2}\omega^{ab}_\mu \mathcal{J}_{ab}+\frac{1}{\ell}e^a_\mu\mathcal{P}_a\right)+\kappa\bar{\psi}^\alpha_\mu\,Q_\alpha$$

Gravity $\rightarrow$ Supergravity of MacDowell-Mansouri

$$\mathcal{L}=\mathcal{L}_f+\mathcal{L}_b=\frac{\ell^2}{16\pi G}\left(2\bar{\mathcal{F}}\gamma^5\,\mathcal{F}+\frac{1}{4}F^{(s)ab}\,\epsilon_{abcd}\,F^{(s)cd}\right)$$

$\mathcal{N}=0$ GR

$\mathcal{N}=1$ SUGRA

$\mathcal{N}=2$ SUGRA

$\mathcal{N}$-extended SUGRA with $\mathcal{N}$ gravitinos

Algebras of the generators (Lorentz, translations)

Poincare

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} + \eta_{ad} J_{bc} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac}$$

$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b$$

$$[P_a, P_b] = 0$$

Inönü-Wigner contraction

$$P_a \to \ell \tilde{P}_a$$

$$[\tilde{P}_a, \tilde{P}_b] = \pm \frac{1}{\ell^2} J_{ab}$$

$$\ell \to \infty$$

Inönü-Wigner contraction

AdS

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} + \eta_{ad} J_{bc} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac}$$

$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b$$

$$[P_a, P_b] = J_{ab}$$

dS

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} + \eta_{ad} J_{bc} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac}$$

$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b$$

$$[P_a, P_b] = - J_{ab}$$

What about new bosonic generator ?

$$[P_a, P_b] = Z_{ab}$$

Maxwell algebra [Schrader, Bacry]

Schrader, *The Maxwell group and the quantum theory of particles in classical homogeneous electromagnetic fields*, Fortsch. Phys. **20** (1972) 701

Maxwell algebra, corresponds to the symmetry of fields in the constant electromagnetic background in the flat Minkowski spacetime.

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} + \eta_{ad} J_{bc} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac},$$
$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b; \qquad [P_a, P_b] = Z_{ab},$$
$$[J_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} + \eta_{ad} Z_{bc} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac},$$
$$[Z_{ab}, P_c] = 0; \qquad [Z_{ab}, Z_{cd}] = 0.$$

Semisimple extension of Poincaré [Soroka^2]

D. V. Soroka and V. A. Soroka, "Tensor extension of the Poincaré' algebra," Phys. Lett. B **607**, 302 (2005) [hep-th/0410012]

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} + \eta_{ad} J_{bc} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac},$$
$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b; \qquad [P_a, P_b] = Z_{ab},$$
$$[J_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} + \eta_{ad} Z_{bc} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac},$$
$$[Z_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b,$$
$$[Z_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} + \eta_{ad} Z_{bc} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac}.$$

$$A_\mu = \frac{1}{2}\omega_\mu^{ab} J_{ab} + \frac{1}{\ell} e_\mu^a P_a + \frac{1}{2} k_\mu^{ab} Z_{ab}$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

Cosmological constant term due to the new generator

$$\frac{1}{\ell^2} e_\mu^a e_\nu^b [P_a, P_b] \to Z_{ab}$$

J. A. de Azcarraga, K. Kamimura and J. Lukierski, "Generalized cosmological term from Maxwell symmetries," Phys. Rev. D **83**, 124036 (2011) [arXiv:1012.4402 [hep-th]].

Enlargment of the algebra

Generalized contractions

What if the scaling of the generator is not by the parameter?

$$P_a \to \ell \tilde{P}_a \qquad\qquad P_a \to \lambda_1 \otimes \tilde{P}_a$$

Semigroup expansion

If we define $S = \{\lambda_0, \lambda_1, \ldots\}$ as an abelian semigroup with the multiplication law being

- associative: (a*b)*c=a*(b*c)
- commutative: a*b=b*a

then the Lie algebra $\mathfrak{G} = S \times \mathfrak{g}$

is called S-expanded algebra of $\mathfrak{g}$

F. Izaurieta, E. Rodríguez and P. Salgado, "Expanding Lie (super)algebras through Abelian semigroups," J. Math. Phys. **47**, 123512 (2006) [hep-th/0606215].

P. Salgado and S. Salgado, "$\mathfrak{so}(D-1,1) \otimes \mathfrak{so}(D-1,2)$ algebras and gravity," Phys. Lett. B **728**, 5 (2014).

J. Diaz, O. Fierro, F. Izaurieta, N. Merino, E. Rodríguez, P. Salgado and O. Valdivia, "A generalized action for $(2+1)$-dimensional Chern-Simons gravity," J. Phys. A **45**, 255207 (2012) [arXiv:1311.2215 [gr-qc]].

Semigroup

Semigroup is an algebraic structure consisting of:

- a set
- an associative binary operation (a * b) * c = a * (b * c)

Commutative semigroup is a semigroup where: a * b = b * a

Including identity element e that e * a = a * e = a gives us **Monoid,** which also can be **commutative (abelian) Monoid.**

A001423 Number of semigroups of order n (for n=1,2,3,4,5,6,7,8,9,10)

1, 4, 18, 126, 1160, 15973, 836021, 1843120128, 52989400714478, 12418001077381302684

A058129 Number of monoids (semigroups with identity) of order n.

1, 2, 7, 35, 228, 2237, 31559, 1668997

A001426 Number of **commutative semigroups** of order n.

1, 3, 12, 58, 325, 2143, 17291, 221805, 11545843, 3518930337

A058131 Number of **commutative monoids** (commutative semigroups with identity) of order n.

1, 2, 5, 19, 78, 421, 2637

Resonant decomposition

V_0 is spanned by the Lorentz generator $\tilde{J}_{ab}$

V_1 by the AdS translation generator $\tilde{P}_a$

$$\mathfrak{g} = \mathfrak{so}(D-1,2) = \mathfrak{so}(D-1,1) \oplus \frac{\mathfrak{so}(D-1,2)}{\mathfrak{so}(D-1,1)} = V_0 \oplus V_1$$

$S_0 = \{\lambda_{2i}\}$ and $S_1 = \{\lambda_{2j+1}\}$ for $i, j = 0, 1, 2, \ldots$

resonant subset decomposition $S = S_0 \cup S_1$

This decomposition satisfies

$$S_0 \cdot S_0 \subset S_0 \,, \quad S_0 \cdot S_1 \subset S_1 \,, \quad S_1 \cdot S_1 \subset S_0$$

$$\updownarrow \qquad \updownarrow \qquad \updownarrow$$

$$[V_0, V_0] \subset V_0 \,, \quad [V_0, V_1] \subset V_1 \,, \quad [V_1, V_1] \subset V_0$$

$$\begin{aligned} s_{even} \cdot s_{even} &= s_{even}, \\ s_{even} \cdot s_{odd} &= s_{odd}, \\ s_{odd} \cdot s_{odd} &= s_{even}, \end{aligned}$$

Generators of new algebra

new algebra will be spanned by the $\{J_{ab,(i)}, P_{a,(j)}\}$, where the new generators are related to original $\mathfrak{so}(D-1,2)$ ones through

$$J_{ab,(i)} = \lambda_{2i}\tilde{J}_{ab} \qquad \text{and} \qquad P_{a,(j)} = \lambda_{2j+1}\tilde{P}_a \,.$$

Lie algebra $\mathfrak{G} = S \times \mathfrak{g}$ is called S-expanded algebra of $\mathfrak{g}$

Maxwell-like algebras

Generators

$$\{J_{ab}, P_a, Z_{ab}\}$$

$$
\begin{aligned}
[J_{ab}, J_{cd}] &= \lambda_0\lambda_0(\eta_{bc}\tilde{J}_{ad} - \eta_{ac}\tilde{J}_{bd} + \eta_{ad}\tilde{J}_{bc} - \eta_{bd}\tilde{J}_{ac}) = \eta_{bc}J_{ad} - \eta_{ac}J_{bd} + \eta_{ad}J_{bc} - \eta_{bd}J_{ac}\,,\\
[J_{ab}, Z_{cd}] &= \lambda_0\lambda_2(\eta_{bc}\tilde{J}_{ad} - \eta_{ac}\tilde{J}_{bd} + \eta_{ad}\tilde{J}_{bc} - \eta_{bd}\tilde{J}_{ac}) = \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} + \eta_{ad}Z_{bc} - \eta_{bd}Z_{ac}\,,\\
[Z_{ab}, Z_{cd}] &= \lambda_2\lambda_2(\eta_{bc}\tilde{J}_{ad} - \eta_{ac}\tilde{J}_{bd} + \eta_{ad}\tilde{J}_{bc} - \eta_{bd}\tilde{J}_{ac}) = 0\,,\\
[J_{ab}, P_c] &= \lambda_0\lambda_1(\eta_{bc}\tilde{P}_a - \eta_{ac}\tilde{P}_b) = \eta_{bc}P_a - \eta_{ac}P_b\,,\\
[Z_{ab}, P_c] &= \lambda_2\lambda_1(\eta_{bc}\tilde{P}_a - \eta_{ac}\tilde{P}_b) = 0\,,\\
[P_a, P_b] &= \lambda_1\lambda_1\tilde{J}_{ab} = Z_{ab}\,.
\end{aligned}
$$

$$
\boldsymbol{J}_{ab} = \lambda_0 \otimes \tilde{\boldsymbol{J}}_{ab},\quad
\boldsymbol{Z}_{ab} = \lambda_2 \otimes \tilde{\boldsymbol{J}}_{ab},\quad
\boldsymbol{P}_a = \lambda_1 \otimes \tilde{\boldsymbol{P}}_a,
$$

Lorentz $J_{ab} = J_{ab,(0)} = \lambda_0\tilde{J}_{ab}$

translation $P_a = P_{a,(0)} = \lambda_1\tilde{P}_a$

new generator $Z_{ab} = J_{ab,(1)} = \lambda_2\tilde{J}_{ab}$

The semigroup elements $\{\lambda_0, \lambda_1, \lambda_2, \lambda_3{=}0_S\}$ are *not* real numbers and they are *dimensionless*. In this particular case, they obey the multiplication law

$$
\lambda_\alpha\lambda_\beta = \begin{cases} \lambda_{\alpha+\beta} & \text{when } \alpha+\beta \leq 3,\\ \lambda_3 & \text{when } \alpha+\beta > 3 \end{cases}
$$

$\mathfrak{B}_4$	λ_0	λ_1	λ_2
λ_0	λ_0	λ_1	λ_2
λ_1	λ_1	λ_2	0_S
λ_2	λ_2	0_S	0_S

semigroup $\leftrightarrow$ *algebra*

$[X_{..}, X_{..}]$, $[X_{..}, X_{.}]$ and $[X_{.}, X_{.}]$
$0_S\mathbb{T}_M = 0$

$[\ ,\ ]$	$J_{..}$	$P_{.}$	$Z_{..}$
$J_{..}$	$J_{..}$	$P_{.}$	$Z_{..}$
$P_{.}$	$P_{.}$	$Z_{..}$	0
$Z_{..}$	$Z_{..}$	0	0

Generalized contractions -> semigroup expansion

$[P_a, P_b] = J_{ab}$
AdS

Inönü-Wigner contraction

$P_a \to \ell \tilde{P}_a \qquad \ell \to \infty$

$[\tilde{P}_a, \tilde{P}_b] = 0$
Poincaré

Generators $\{J_{ab}, P_a, Z_{ab}, Z_a\}$

$$\begin{aligned} \boldsymbol{J}_{ab} &= \lambda_0 \otimes \tilde{\boldsymbol{J}}_{ab}, \\ \boldsymbol{Z}_{ab} &= \lambda_2 \otimes \tilde{\boldsymbol{J}}_{ab}, \\ \boldsymbol{P}_a &= \lambda_1 \otimes \tilde{\boldsymbol{P}}_a, \\ \boldsymbol{Z}_a &= \lambda_3 \otimes \tilde{\boldsymbol{P}}_a. \end{aligned}$$

The semigroup elements $\{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ are *not* real numbers and they are *dimensionless*. In this particular case, they obey the multiplication law

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{when } \alpha + \beta \leq 4, \\ \lambda_4, & \text{when } \alpha + \beta > 4. \end{cases}$$

$$\lambda_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \lambda_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$[P_a, P_b] = J_{ab}$
AdS

Semigroup expansion of generators

multiplication tables

new algebras

Semigroup expansion and two algebraic families

Generators	Type $\mathfrak{B}_m$	Type $\mathfrak{C}_m$
J_{ab}, P_a	$\mathfrak{B}_3$ =Poincaré	$\mathfrak{C}_3$ =AdS
J_{ab}, P_a, Z_{ab}	$\mathfrak{B}_4$ =Maxwell	$\mathfrak{C}_4$ =AdS$\oplus$Lorentz
J_{ab}, P_a, Z_{ab}, R_a	$\mathfrak{B}_5$	$\mathfrak{C}_5$
$J_{ab}, P_a, Z_{ab}, R_a, \tilde{Z}_{ab}$	$\mathfrak{B}_6$	$\mathfrak{C}_6$
...	...	...

$$\mathfrak{D}_m = AdS \oplus \mathfrak{B}_{m-2}$$

$$S_E^{(N)} = \{\lambda_\alpha\}_{\alpha=0}^{N+1} \qquad S_{\mathcal{M}}^{(N)} = \{\lambda_\alpha, \alpha = 0, \ldots, N\}$$

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta} & \text{if } \alpha+\beta \leqslant N+1 \\ \lambda_{N+1} & \text{if } \alpha+\beta > N+1 \end{cases} \qquad \lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta} & \text{if } \alpha+\beta \leqslant N, \\ \lambda_{\alpha+\beta-2\left[\frac{N+1}{2}\right]} & \text{if } \alpha+\beta > N, \end{cases}$$

Automatic building blocks for the actions

- Connection   Curvature forms   Invariant tensors

- Born-Infeld theory in even dimensions
- Chern-Simons theory in odd dimensions

$$\mathcal{L}_{CS}^{2n+1}[A] = \kappa(n+1)\int_0^1 \delta t \left\langle A\left(tdA + t^2A^2\right)^n\right\rangle$$

AdS

$$\left\langle J_{a_1 a_2} \cdots J_{a_{2n-1}a_{2n}} P_{a_{2n+1}} \right\rangle = \frac{2^n}{n+1}\epsilon_{a_1 a_2 \cdots a_{2n+1}}$$

Pure Lovelock gravity and Chern-Simons theory

P. K. Concha[1,2]*, R. Durka[3]†, C. Inostroza[4]‡, N. Merino[3]§, E. K. Rodríguez[1,2]¶

New family of Maxwell like algebras

P. K. Concha[1,2]*, R. Durka[3]†, N. Merino[3]‡, E. K. Rodríguez[1,2]§

Question: how many algebras are there?

R. Durka, “Resonant algebras and gravity,” J. Phys. A **50**, no.14, 145202 (2017) doi:10.1088/1751-8121/aa5c0b [arXiv:1605.00059 [hep-th]].

R. Durka and K. Grela, “On the number of possible resonant algebras,” J. Phys. A **53**, no.35, 355202 (2020) doi:10.1088/1751-8121/ab9e8e [arXiv:1911.12814 [hep-th]].

What is the freedom in closing the semigroup multiplication tables?

$$\left[\tilde{J}_{ab}, \tilde{J}_{cd}\right] = \eta_{bc}\tilde{J}_{ad} - \eta_{ac}\tilde{J}_{bd} - \eta_{bd}\tilde{J}_{ac} + \eta_{ad}\tilde{J}_{bc} \quad \rightarrow \quad R^{ab} = d\omega^{ab} + \omega^{ac} \wedge \omega_c{}^b\,,$$

$$\left[\tilde{J}_{ab}, \tilde{P}_c\right] = \eta_{bc}\tilde{P}_a - \eta_{ac}\tilde{P}_b \quad \rightarrow \quad T^a = de^a + \omega^a{}_b \wedge e^b\,.$$

$$\text{Lorentz } J_{ab} = J_{ab,(0)} = \lambda_0 \tilde{J}_{ab}$$

$$\text{translation } P_a = P_{a,(0)} = \lambda_1 \tilde{P}_a$$

$[J_{ab}, J_{cd}]$ and $[J_{ab}, P_c]$

$\lambda_0\lambda_0 = \lambda_0 \qquad \lambda_0\lambda_1 = \lambda_1$

Poincaré-like, AdS-like, Maxwell-like algebras

	λ_0	λ_1	λ_2
λ_0	λ_0	λ_1	λ_2
λ_1	λ_1	0_S	
λ_2	λ_2		

	λ_0	λ_1	λ_2
λ_0	λ_0	λ_1	λ_2
λ_1	λ_1	λ_0	
λ_2	λ_2		

	λ_0	λ_1	λ_2
λ_0	λ_0	λ_1	λ_2
λ_1	λ_1	λ_2	
λ_2	λ_2		

Generators

J_{ab}, P_a, Z_{ab}

- $4\times$ Poincaré-like algebras we could denote as type B_4, BC_4, CB_4, and $C_4 \equiv ISO \oplus Lorentz$:

B_4	λ_0	λ_1	λ_2
λ_0	λ_0	λ_1	λ_2
λ_1	λ_1	0_S	0_S
λ_2	λ_2	0_S	0_S

BC_4	λ_0	λ_1	λ_2
λ_0	λ_0	λ_1	λ_2
λ_1	λ_1	0_S	0_S
λ_2	λ_2	0_S	λ_2

CB_4	λ_0	λ_1	λ_2
λ_0	λ_0	λ_1	λ_2
λ_1	λ_1	0_S	λ_1
λ_2	λ_2	λ_1	λ_0

C_4	λ_0	λ_1	λ_2
λ_0	λ_0	λ_1	λ_2
λ_1	λ_1	0_S	λ_1
λ_2	λ_2	λ_1	λ_2

- none of the AdS-like algebra (associativity is not fulfilled in any configuration)
- $2\times$ Maxwell-like algebras of type $\mathfrak{B}_4$ and $\mathfrak{C}_4$ already introduced in a previous section:

$\mathfrak{B}_4$	λ_0	λ_1	λ_2
λ_0	λ_0	λ_1	λ_2
λ_1	λ_1	λ_2	0_S
λ_2	λ_2	0_S	0_S

$\mathfrak{C}_4$	λ_0	λ_1	λ_2
λ_0	λ_0	λ_1	λ_2
λ_1	λ_1	λ_2	λ_1
λ_2	λ_2	λ_1	λ_2

B_4	J	P	Z
J	J	P	Z
P	P	0	0
Z	Z	0	0

$\tilde{B}_4$	J	P	Z
J	J	P	Z
P	P	0	P
Z	Z	P	J

$\tilde{C}_4$	J	P	Z
J	J	P	Z
P	P	0	0
Z	Z	0	Z

C_4	J	P	Z
J	J	P	Z
P	P	0	P
Z	Z	P	Z

$\mathfrak{B}_4$	J	P	Z
J	J	P	Z
P	P	Z	0
Z	Z	0	0

$\mathfrak{C}_4$	J	P	Z
J	J	P	Z
P	P	Z	P
Z	Z	P	Z

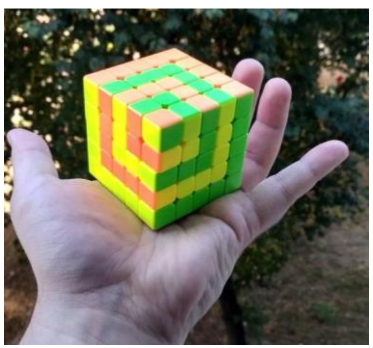
Generators

$\{J_{ab}, P_a, Z_{ab}, R_a\}$

17 x Poincaré-like,
3 x AdS-like,
10 x Maxwell-like

$\mathcal{C}_5$, corresponds to the Klein group

$\mathfrak{C}_5$, correspond to the cyclic group $\mathbb{Z}_4$



Visit website

http://resonantalgebras.wordpress.com

- explicit tables for the algebras labeled by *m* = 3, 4, 5, 6,
- tool checking semigroup associativity

Further enlargement, coming with the new translational $R_a = P_{a,(1)} = \lambda_3 \tilde{P}_a$ generator, brings much richer collection of the algebras:

- 17× Poincaré-like
- 3× AdS-like of type $\mathcal{B}_5$, $\mathcal{C}_5$, and $\mathcal{D}_5 \equiv AdS \oplus AdS$

$\mathcal{B}_5$	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_0	λ_3	λ_2
λ_2	λ_2	λ_3	0_S	0_S
λ_3	λ_3	λ_2	0_S	0_S

$\mathcal{C}_5$	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_0	λ_3	λ_2
λ_2	λ_2	λ_3	λ_0	λ_1
λ_3	λ_3	λ_2	λ_1	λ_0

$\mathcal{D}_5$	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_0	λ_3	λ_2
λ_2	λ_2	λ_3	λ_2	λ_3
λ_3	λ_3	λ_2	λ_3	λ_2

- 10× Maxwell-like: three of them $\mathfrak{B}_5, \mathfrak{C}_5, \mathfrak{D}_5$ were already derived in a previous section

$\mathfrak{B}_5$	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_2	λ_3	0_S
λ_2	λ_2	λ_3	0_S	0_S
λ_3	λ_3	0_S	0_S	0_S

$\mathfrak{C}_5$	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_2	λ_3	λ_0
λ_2	λ_2	λ_3	λ_0	λ_1
λ_3	λ_3	λ_0	λ_1	λ_2

$\mathfrak{D}_5$	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_2	λ_3	λ_2
λ_2	λ_2	λ_3	λ_2	λ_3
λ_3	λ_3	λ_2	λ_3	λ_2

but surprisingly there are five others with the 0_S (from them only one is being presented below) and additional two without the zero elements

	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_2	λ_1	0_S
λ_2	λ_2	λ_1	λ_2	0_S
λ_3	λ_3	0_S	0_S	0_S

	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_2	λ_1	λ_2
λ_2	λ_2	λ_1	λ_2	λ_1
λ_3	λ_3	λ_2	λ_1	λ_2

	λ_0	λ_1	λ_2	λ_3
λ_0	λ_0	λ_1	λ_2	λ_3
λ_1	λ_1	λ_2	λ_1	λ_2
λ_2	λ_2	λ_1	λ_2	λ_1
λ_3	λ_3	λ_2	λ_1	λ_0

Resonant (super)algebras

$$
\begin{cases}
\left[\square_{ab},\, \square_{cd}\right] = \eta_{bc}\square_{ad} - \eta_{ac}\square_{bd} - \eta_{bd}\square_{ac} + \eta_{ad}\square_{bc}\,, \\
\left[\square_{ab},\, \boxdot_{c}\right] = \eta_{bc}\boxdot_{a} - \eta_{ac}\boxdot_{b}\,, \\
\left[\boxdot_{a},\, \boxdot_{b}\right] = \square_{ab}\,, \\
\left[\square_{ab},\, \bigcirc_{\alpha}\right] = \frac{1}{2}\left(\Gamma_{ab}\right)_{\alpha}^{\ \beta}\bigcirc_{\beta}\,, \\
\left[\boxdot_{a},\, \bigcirc_{\alpha}\right] = \frac{1}{2}\left(\Gamma_{a}\right)_{\alpha}^{\ \beta}\bigcirc_{\beta}\,, \\
\left\{\bigcirc_{\alpha},\, \bigcirc_{\beta}\right\} = -\left(C\Gamma^{a}\right)_{\alpha\beta}\boxdot_{a} + \frac{1}{2}\left(C\Gamma^{ab}\right)_{\alpha\beta}\square_{ab}\,.
\end{cases}
$$

Available content of generators concerns

$\square_{ab} \rightarrow J_{ab}, Z_{ab}, \ldots$ (Lorentz – like)

$\boxdot_{a} \rightarrow P_{a}, U_{a}, \ldots$ (translation – like)

$\bigcirc_{\alpha} \rightarrow Q_{\alpha}, Y_{\alpha}, \ldots$ (supercharge – like)

Algebraic and physical requirements to be satisfied:

- holding the same structure constants as original super AdS;
- preservation by the Lorentz generator, i.e. for all generators $[J, X] \sim X$;
- anticommutator $\{Q, Q\}$ being non-zero;
- fulfilling graded super-Jacobi identities.

Jacobi identities

$$[[P_a,P_b],J_c]+[[P_b,J_c],P_a]+[[J_c,P_a],P_b]=0$$
$$[[P_a,P_b],Q_\alpha]+[[P_b,Q_\alpha],P_a]+[[Q_\alpha,P_a],P_b]=0$$
$$\{[P_a,Q_\alpha],Q_\beta\}+[\{Q_\alpha,Q_\beta\},P_a]-\{[Q_\beta,P_a],Q_\alpha\}=0$$
$$[\{Q_\lambda,Q_\alpha\},Q_\beta]+[\{Q_\alpha,Q_\beta\},Q_\lambda]+[\{Q_\beta,Q_\lambda\},Q_\alpha]=0$$

If one explicitly use structure constants, then the super-Jacobi identity reads

$$\begin{aligned}0 &= \left(\left(X_i,X_j\right),X_k\right)\oplus\left(\left(X_j,X_k\right),X_i\right)\oplus\left(\left(X_k,X_i\right),X_j\right)\\ &= {f_{i,j}}^m\left(X_m,X_k\right)\oplus {f_{j,k}}^m\left(X_m,X_i\right)\oplus {f_{k,i}}^m\left(X_m,X_j\right)\\ &= {f_{i,j}}^m\,{f_{m,k}}^n\underbrace{X_n}_{A}\oplus {f_{j,k}}^m\,{f_{m,i}}^n\underbrace{X_n}_{B}\oplus {f_{k,i}}^m\,{f_{m,j}}^n\underbrace{X_n}_{C}\,.\end{aligned}$$

Equivalent to check if A,B,C are of the same "type".

J	J	J
J	J	P
J	J	Q
J	P	J
J	P	P
J	P	Q
J	Q	J
J	Q	P
J	Q	Q
P	J	J
P	J	P
P	J	Q
P	P	J
P	P	P
P	P	Q
P	Q	J
P	Q	P
P	Q	Q
Q	J	J
Q	J	P
Q	J	Q
Q	P	J
Q	P	P
Q	P	Q
Q	Q	J
Q	Q	P
Q	Q	Q

Commutation tables

Poincare

$[.,.]$	J	P
J	J	P
P	P	0

$[.,.]$	Q
J	Q
P	0

$\{.,.\}$	Q
Q	P

AdS

$[.,.]$	J	P
J	J	P
P	P	J

$[.,.]$	Q
J	Q
P	Q

$\{.,.\}$	Q
Q	$P+J$

Algebra candidates and Jacobi Identities

Assuming that algebra contains: $p+1$, n and f of even indexed bosonic generators, odd indexed, and fermionic ones, respectively, then the generated total number of candidates configurations reads

$$\overline{Alg} = (p+2)^{\frac{p(p+1)}{2}} (n+1)^{pn} (p+2)^{\frac{n(n+1)}{2}} (f+1)^{(p+n)f} ((p+2)(n+1)-1)((p+2)(n+1))^{\frac{(f+1)f}{2}-1}$$

$$\overline{\text{Jacobi}} = \binom{p+n+f+2}{3}$$

Having given candidate Alg_i with a unique set of (anti)commutation rules, the upper number of checks to execute is equal to:

$$6 \cdot \overline{\text{Alg}} \cdot \overline{\text{Jacobi}}$$

Factor six accounts for two rounds of super-commutator evaluations in a single super-Jacobi identity. $(-1)^{|x||z|}[x,[y,z]] + (-1)^{|y||x|}[y,[z,x]] + (-1)^{|z||y|}[z,[x,y]] = 0$

Case of $JPZU + QY$ requires performing:

- 35 unique super-Jacobi identities
- multiplied by six super-commutator substitutions
- for each of 344 373 768 possible algebra candidates.

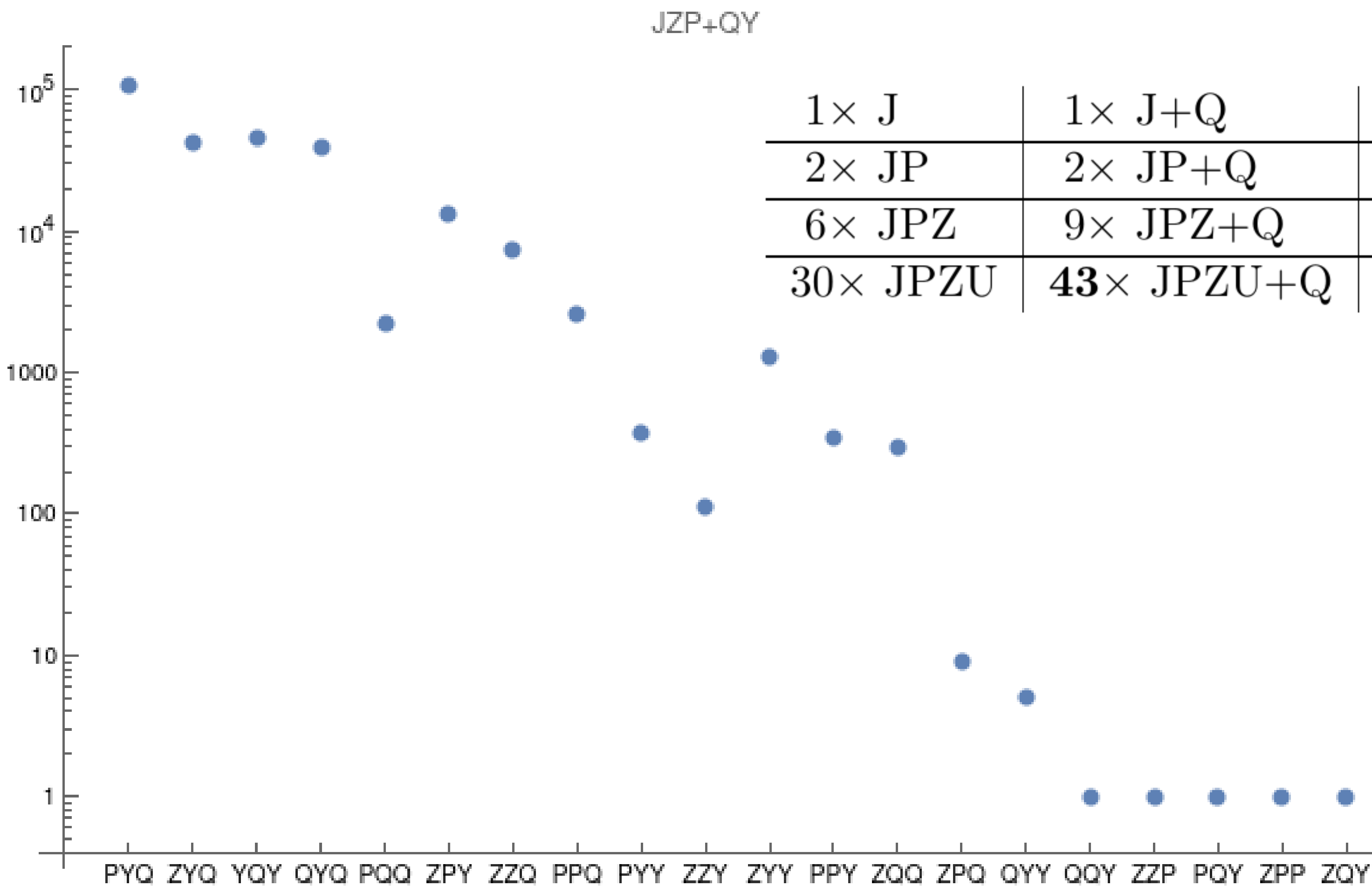
Mathematica

Resonant superalgebras for supergravity

Remigiusz Durka (Wroclaw U.), Krzysztof M. Graczyk (Wroclaw U.) (Aug 23, 2021)

Published in: *Eur.Phys.J.C* 82 (2022) 3, 254 • e-Print: 2108.10304 [hep-th]

Dynamical searching algorithm that "learns" the most problematic Jacobi identities and uses them in the search process.



1× J	1× J+Q	0× J+QY
2× JP	2× JP+Q	**10**× JP+QY
6× JPZ	9× JPZ+Q	**102**× JPZ+QY
30× JPZU	**43**× JPZU+Q	**667**× JPZU+QY

Resonant algebras JP+QY

	J	P
J	J	P
P	P	0

	Q	Y
J	Q	Y
P	Y	0

	Q	Y
Q	$P+J$	P
Y	P	0

	Q	Y
Q	P	0
Y	0	0

	Q	Y
Q	J	P
Y	P	0

	Q	Y
J	Q	Y
P	0	0

	Q	Y
Q	P	P
Y	P	P

	Q	Y
Q	P	P
Y	P	0

	Q	Y
Q	P	0
Y	0	P

	Q	Y
Q	P	0
Y	0	0

	J	P
J	J	P
P	P	J

	Q	Y
J	Q	Y
P	Y	Q

	Q	Y
Q	$P+J$	$P+J$
Y	$P+J$	$P+J$

	Q	Y
Q	P	J
Y	J	P

	Q	Y
Q	J	P
Y	P	J

Q^I_α

$Q^1_\alpha \equiv Q_\alpha$ and $Q^2_\alpha \equiv Y_\alpha$

N=2 resonant algebra JPZ+QY+T

$$[J_{ab},J_{cd}]=\eta_{bc}J_{ad}-\eta_{ac}J_{bd}+\eta_{ad}J_{bc}-\eta_{bd}J_{ac}\,,$$
$$[J_{ab},Z_{cd}]=\eta_{bc}Z_{ad}-\eta_{ac}Z_{bd}+\eta_{ad}Z_{bc}-\eta_{bd}Z_{ac}\,,$$
$$[Z_{ab},Z_{cd}]=\eta_{bc}Z_{ad}-\eta_{ac}Z_{bd}+\eta_{ad}Z_{bc}-\eta_{bd}Z_{ac}\,,$$
$$[J_{ab},P_b]=\eta_{bc}P_a-\eta_{ac}P_b\,,$$
$$[Z_{ab},P_b]=\eta_{bc}P_a-\eta_{ac}P_b\,,$$
$$[P_a,P_b]=Z_{ab}\,,$$
$$[J_{ab},Q^I_\alpha]=\frac{1}{2}\,(\Gamma_{ab})_\alpha^{\ \beta}\,Q^I_\beta\,,\qquad [P_a,Q^I_\alpha]=\frac{1}{2}\,(\Gamma_a)_\alpha^{\ \beta}\,Q^I_\beta\,,$$
$$[Z_{ab},Q^I_\alpha]=\frac{1}{2}\,(\Gamma_{ab})_\alpha^{\ \beta}\,Q^I_\beta\,,\qquad [T,Q^I_\alpha]=\epsilon^{IJ}Q^J$$
$$\{Q^I_\alpha,Q^J_\beta\}=-\delta^{IJ}\left((\Gamma^aC)_{\alpha\beta}\,P_a-\frac{1}{2}(\Gamma^{ab}C)_{\alpha\beta}Z_{ab}\right)+\epsilon^{IJ}C_{\alpha\beta}T\,.$$

[,]	J	P	Z	T
J	J	P	Z	0
P	P	Z	P	0
Z	Z	P	Z	0
T	0	0	0	0

[,]	Q	Y
J	Q	Y
P	Q	Y
Z	Q	Y
T	Y	Q

{ , }	Q	Y
Q	P + Z	T
Y	T	P + Z

[,]	J	P	T
J	J	P	0
P	P	J	0
T	0	0	0

[,]	Q	Y
J	Q	Y
P	Q	Y
T	Y	Q

{ , }	Q	Y
Q	P + J	T
Y	T	P + J

N=2 resonant superalgebra for supergravity

Remigiusz Durka (Wroclaw U.), Krzysztof M. Graczyk (Wroclaw U.) (May 12, 2022)

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$$I=\frac{k}{4\pi}\int\Big[\alpha_0\left(\omega^a d\omega_a+\frac{1}{3}\epsilon^{abc}\omega_a\omega_b\omega_c\right)$$
$$+\alpha_1\left(\frac{2}{\ell}\mathcal{R}^a e_a+\frac{1}{3\ell^3}\epsilon^{abc}e_ae_be_c+\frac{2}{\ell}e_aD_\omega h^a+\frac{1}{\ell}\epsilon^{abc}e_ah_bh_c+\frac{2}{\ell}\bar{\psi}\mathcal{F}+\frac{2}{\ell}\bar{\chi}\mathcal{G}+\frac{2}{\ell^2}a\,f(a)\right)$$
$$+\alpha_2\left(\frac{1}{\ell^2}e_aD_\omega e^a+2h_a\mathcal{R}^a+\frac{1}{\ell^2}\epsilon^{abc}e_ae_bh_c+h_aD_\omega h^a+\frac{1}{3}\epsilon^{abc}h_ah_bh_c+\frac{2}{\ell}\bar{\psi}\mathcal{F}+\frac{2}{\ell}\bar{\chi}\mathcal{G}+\frac{2}{\ell^2}a\,f(a)\right)\Big]\,.$$

$$I^{AdS}_{CS}=\frac{k}{4\pi}\int\Big[\alpha_0\left(\omega^a d\omega_a+\frac{1}{3}\epsilon^{abc}\omega_a\omega_b\omega_c+\frac{1}{\ell^2}e_aD_\omega e^a+\frac{2}{\ell}\bar{\psi}\mathcal{F}+\frac{2}{\ell}\bar{\chi}\mathcal{G}+\frac{2}{\ell^2}a\,f(a)\right)$$
$$+\alpha_1\left(\frac{2}{\ell}\mathcal{R}^a e_a+\frac{1}{3\ell^3}\epsilon^{abc}e_ae_be_c+\frac{2}{\ell}\bar{\psi}\mathcal{F}+\frac{2}{\ell}\bar{\chi}\mathcal{G}+\frac{2}{\ell^2}a\,f(a)\right)\Big]\,.$$

• • •

Applications

Bi-metric theories

$$J_{ab}, P_a \quad \rightarrow \quad \omega^{ab}, e^a \quad \rightarrow g_{\mu\nu}$$

$$Z_{ab}, R_a \quad \rightarrow \quad h^{ab}, k^a \quad \rightarrow h_{\mu\nu}$$

$$S = -\frac{M_g^2}{2}\int d^4x\sqrt{-g}R(g) + \int d^4x V(g,h) - \frac{M_h^2}{2}\int d^4x\sqrt{-h}R(h)$$

$$\langle J_{ab}\, Z_{cd}\rangle \neq 0$$

30 different JPZR algebras → 30 different theories

Einstein-Hilbert action from 5D Chern-Simons

Chern-Simons theory is characterized by the action

Algebra: AdS

$$S_{CS}^{(5D)} = \kappa \int d^5x \left(\frac{1}{\ell} R^{ab} R^{cd} e^e + \frac{2}{3}\frac{1}{\ell^3} R^{ab} e^c e^d e^e + \frac{1}{5}\frac{1}{\ell^5} e^a e^b e^c e^d e^e \right) \epsilon_{abcde}$$

We DON'T HAVE the Einstein action limit

Algebra: $\mathfrak{B}_5$ with the generators: *Jab, Pa, Zab, Ra,*

$$L_{\text{CS}}^{\mathfrak{B}_5} = \alpha_1 \ell^2 \varepsilon_{abcde} R^{ab} R^{cd} e^e + \alpha_3 \varepsilon_{abcde} \left(\frac{2}{3} R^{ab} e^c e^d e^e + 2\ell^2 k^{ab} R^{cd} T^e + \ell^2 R^{ab} R^{cd} h^e \right)$$

when α_1 vanishes we almost there

No matter content $\delta k^{ab} = 0$ and $\delta h^a = 0$ leads to the right action!
But what about variations? (let's asume that $T^a = 0$)

$$\delta L_{\text{CS}}^{(5)} = 2\alpha_3 \varepsilon_{abcde} R^{ab} e^c e^d \delta e^e + \alpha_3 \ell^2 \varepsilon_{abcde} R^{ab} R^{cd} \delta h^e$$

$$\begin{cases} \varepsilon_{abcde} R^{ab} e^c e^d = 0 \\ \ell^2 \varepsilon_{abcde} R^{ab} R^{cd} = 0 \end{cases}$$

Action in the critic limit

$\ell = 0$ leads to GR.

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Geometric Model of Topological Insulators from the Maxwell Algebra

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(Dated: October 18, 2016)

We propose a novel geometric model of three-dimensional topological insulators in presence of an external electromagnetic field. The gapped boundary of these systems supports relativistic quantum Hall states and is described by a Chern-Simons theory with a gauge connection that takes values in the Maxwell algebra. This represents a non-central extension of the Poincaré algebra and takes into account both the Lorentz and magnetic-translation symmetries of the surface states. In this way, we derive a relativistic version of the Wen-Zee term, and we show that the non-minimal coupling between the background geometry and the electromagnetic field in the model is in agreement with the main properties of the relativistic quantum Hall states in the flat space.

Maxwell algebra

$$S_{\rm CS}[\mathcal{A}] = \int \mathrm{tr}\ \mathrm{CS}(\mathcal{A})$$

→

$$\mathcal{A}_\mu = \mathcal{A}_\mu^A X_A = \frac{1}{\beta}\, e_\mu^a P_a + \omega_\mu^a J_a + \widehat{A}_\mu^a Z_a,$$

Model of topological insulator

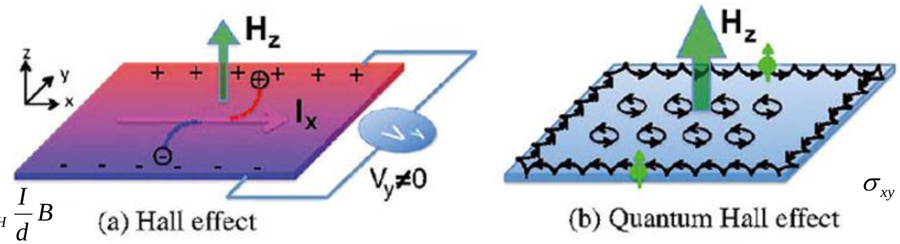
$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} + \eta_{ad} J_{bc} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac},$$
$$[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b; \qquad [P_a, P_b] = Z_{ab},$$
$$[J_{ab}, Z_{cd}] = \eta_{bc} Z_{ad} + \eta_{ad} Z_{bc} - \eta_{ac} Z_{bd} - \eta_{bd} Z_{ac},$$
$$[Z_{ab}, P_c] = 0; \qquad [Z_{ab}, Z_{cd}] = 0.$$

$$S_{\rm CS}[e, \omega, \widehat{A}] = \int \mathrm{tr}[\varrho_1 \mathrm{CS}(\omega) + \varrho_2\, \mathrm{e} \wedge \mathrm{D}_\omega \mathrm{e} + \varrho_3 \widehat{A} \wedge (R + \varrho_4\, e \wedge e) + \varrho_5 \widehat{A} \wedge D_\omega \widehat{A}],$$

Quantum Hall Effect

Topological Insulator is a material that behaves as an insulator (internal electric charges do not flow freely in its Interior) while permitting the movement of charges on its boundary.

Hall effects

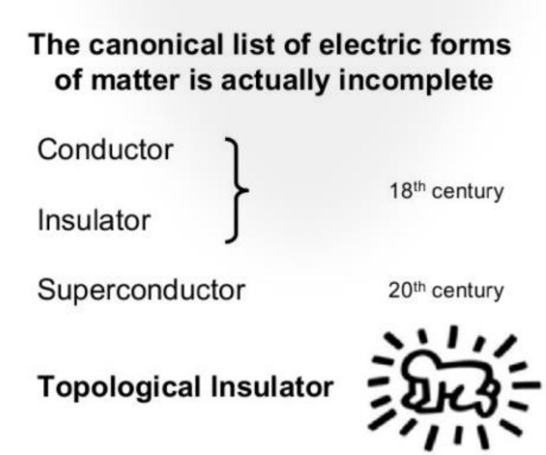


$V_H = \frac{-I}{ned} B = R_H \frac{I}{d} B$ (a) Hall effect

(b) Quantum Hall effect

$\sigma_{xy} = \frac{I}{V_H} = \nu \frac{e^2}{h}$

$\sigma_{xy} = \frac{1}{\text{Resistance}}$

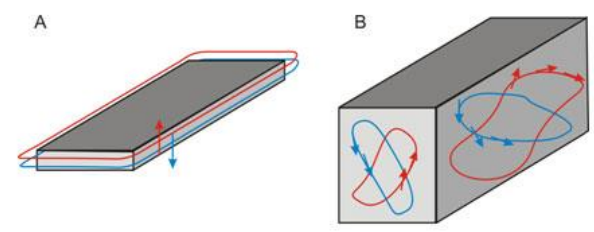


Terms of the lowest order in derivatives

$$S_{ind} = \frac{\nu}{4\pi} \int \left[AdA + 2\bar{s}\, \omega dA + \beta' \, \omega d\omega \right] .$$

Geometric terms:

- AdA – Chern-Simons term (ν: Hall conductance, filling factor)
- ωdA – Wen-Zee term ($\bar{s}$: orbital spin, Hall viscosity, Wen-Zee shift)
- $\omega d\omega$ – "gravitational CS term" (β': Hall viscosity - curvature, thermal Hall effect, orbital spin variance)



2D topological insulator   3D topological insulator

Immersed EM field in Maxwellian field!

$$S_{\mathfrak{C}_4}[e,\omega,\hat{A}] = \frac{k}{4\pi}\int \alpha_0\, CS(\omega) + \frac{k}{4\pi}\int \alpha_2 \left(\frac{1}{\ell^2}\, e_a \wedge D_\omega e^a + 2\hat{A}_a \wedge (R^a + \frac{1}{2\ell^2}\epsilon^{abc}\, e_b \wedge e_c) + \hat{A}_a \wedge D_\omega \hat{A}^a\right) + \frac{k}{4\pi}\int \alpha_2\, \frac{1}{3}\, \epsilon^{abc}\, \hat{A}_a \wedge \hat{A}_b \wedge \hat{A}_c\,.$$

$$S_{\mathfrak{C}_4}[e,\omega,A] = \frac{\nu}{4\pi}\int AdA + \frac{\nu}{2\pi}\int A\wedge(d\omega^0 + \omega_1\wedge\omega_2 + \frac{1}{\ell^2}e_1\wedge e_2) + \frac{\nu}{4\pi}\frac{\alpha_0}{\alpha_2}\int CS(\omega) + \frac{\nu}{4\pi}\frac{1}{\ell^2}\int e_a \wedge D_\omega e^a\,.$$

Table 1: Lagrangian content for the $\{J_a, P_a, Z_a\}$ resonant algebras

	B_4	$\tilde{B}_4$	$\tilde{C}_4$	C_4	$\mathfrak{B}_4$	$\mathfrak{C}_4$
$C(\omega)$	α_0	α_0	α_0	α_0	α_0	α_0
$2\,\hat{A}_a R^a$	α_2	α_2	α_2	α_2	α_2	α_2
$\hat{A}_a D_\omega \hat{A}^a$	-	α_0	α_2	α_2	-	α_2
$\frac{1}{3}\epsilon^{abc}\hat{A}_a\hat{A}_b\hat{A}_c$	-	α_2	α_2	α_2	-	α_2
$\frac{1}{\ell^2}\epsilon^{abc}\hat{A}_a e_b e_c$	-	-	-	-	-	α_2
$\frac{1}{\ell^2}e_a T^a$	-	-	-	-	α_2	α_2

Table 2: Lagrangian counter-content for the $\{J_a, P_a, Z_a\}$ resonant algebras

	B_4	$\tilde{B}_4$	$\tilde{C}_4$	C_4	$\mathfrak{B}_4$	$\mathfrak{C}_4$
$\frac{2}{\ell}e_a R^a$	α_1	α_1	α_1	α_1	α_1	α_1
$\frac{2}{\ell}e_a D_\omega \hat{A}^a$	-	α_1	-	α_1	-	α_1
$\frac{1}{\ell}e_a\hat{A}_b\hat{A}_c\,\epsilon^{abc}$	-	α_1	-	α_1	-	α_1
$\frac{1}{3\ell^3}e_a e_b e_c\,\epsilon^{abc}$	-	-	-	-	-	α_1

gauge breaking ansatz
gauge breaking ansatz

$$\hat{A}^0_\mu = A_\mu, \qquad \hat{A}^1_\mu = 0, \qquad \text{and} \qquad \hat{A}^2_\mu = 0$$

$$S_{ind} = \frac{\nu}{4\pi}\int \left[AdA + 2\bar{s}\,\omega dA + \beta'\,\omega d\omega\right]$$

B_4	J	P	Z
J	J	P	Z
P	P	0	0
Z	Z	0	0

$\tilde{B}_4$	J	P	Z
J	J	P	Z
P	P	0	P
Z	Z	P	J

$\tilde{C}_4$	J	P	Z
J	J	P	Z
P	P	0	0
Z	Z	0	Z

C_4	J	P	Z
J	J	P	Z
P	P	0	P
Z	Z	P	Z

$\mathfrak{B}_4$	J	P	Z
J	J	P	Z
P	P	Z	0
Z	Z	0	0

$\mathfrak{C}_4$	J	P	Z
J	J	P	Z
P	P	Z	P
Z	Z	P	Z

$S_{CS}[A] = \frac{k}{4\pi}\int d^3x\epsilon^{\mu\rho\sigma}A_\mu\partial_\rho A_\sigma$ one computes the current that arises from the Chern-Simons term $J_i = \frac{\delta S_{CS}[A]}{\delta A_i} = \frac{k}{2\pi}\epsilon_{ji}E_j$ and the charge density $J_0 = \frac{\delta S_{CS}[A]}{\delta A_0} = \frac{k}{2\pi}B$ (see [1]). This means that the Hall conductivity takes the value of $\sigma_{xy} = \frac{k}{2\pi}$. Then the Chern-Simons level corresponds to the filling factor ν of the Landau levels, accordingly to $k = \frac{e^2\nu}{\hbar}$.

The Hall viscosity coefficient, standing in front of the torsional term and describing the response of the quantum Hall fluid, is defined as $\eta_H = \frac{\nu\bar{s}B_0}{4\pi}$ [7, 8]. This is in agreement with our framework after relating ℓ to the magnetic length $\ell = \sqrt{\hbar c/|e|B_0}$ with B_0 being an external magnetic field [9]. Then we have $\frac{\nu}{4\pi}\frac{1}{\ell^2} = \frac{\nu B_0}{4\pi} = \eta_H$ with once again $\bar{s} = 1$.

Thank you for your attention!