Toy-model for quasinormal modes of extremal black holes

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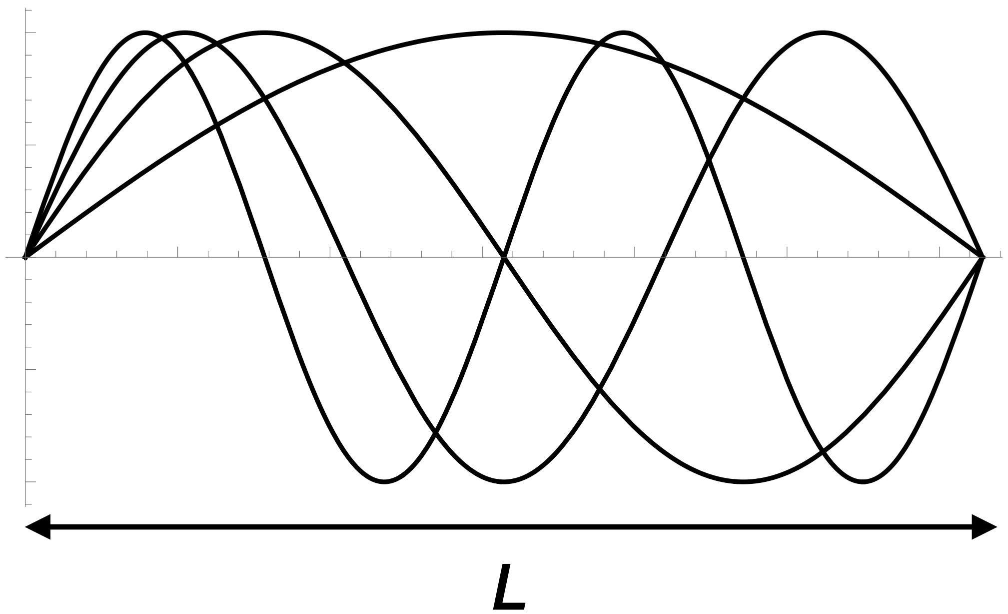
Joint work with Claude Warnick

Plan

- Introduction
- Solvable toy-model
- Nonlinear, conformally invariant wave equation
- Extremal Reissner-Nordström-anti-de Sitter

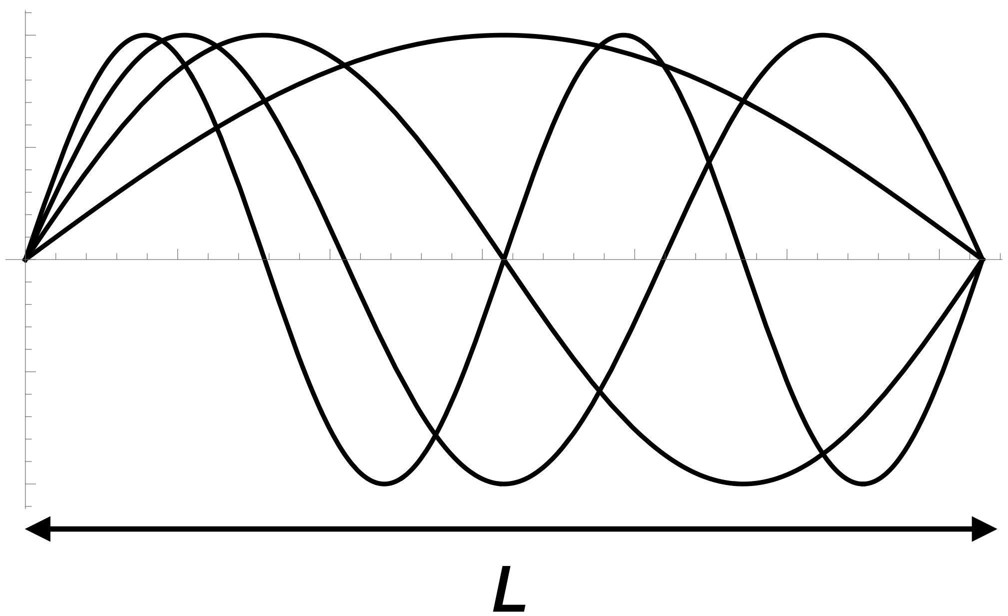
Introduction



$$\nu_n = \frac{(n+1)}{2}\frac{c}{L}$$

$$\psi \sim \sin(\nu_n t)$$

Introduction



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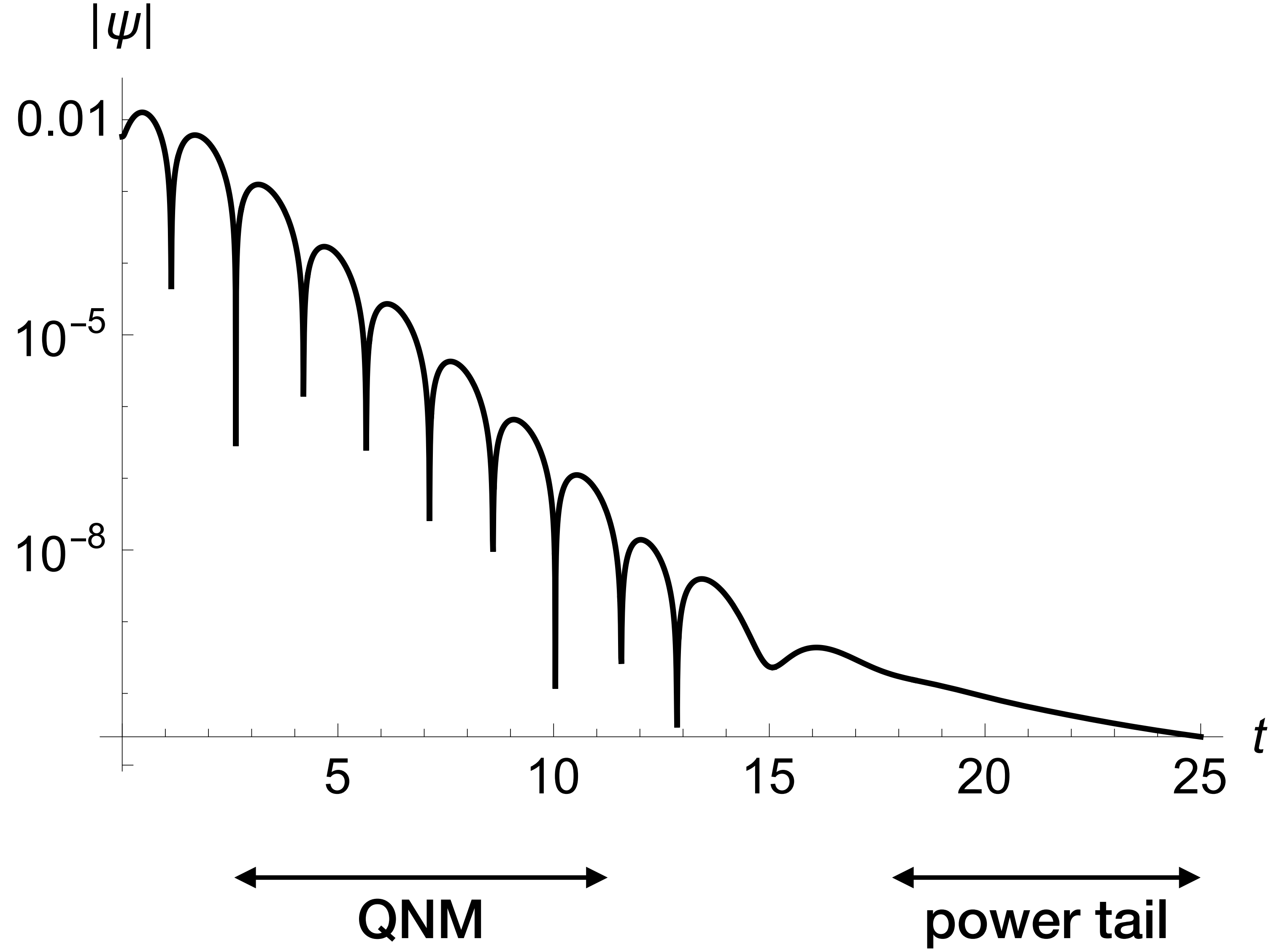
sunn

$$\psi \sim e^{-\beta t}\sin(\nu_n t)$$



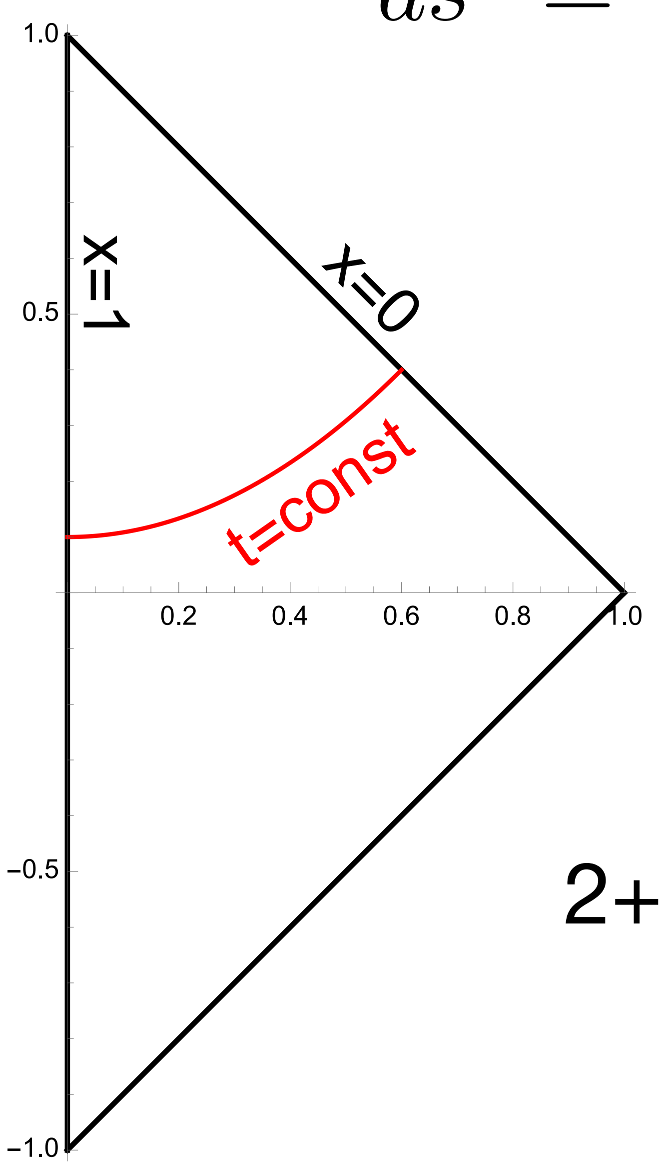
|ψ|
0.01
10^−5
10^−8
t
5
10
15
20
25
QNM
power tail

Toy-model

Setup

$$ds^2 = -\frac{x^2}{1+x^2}dt^2 + \frac{2}{1+x^2}dtdx + \frac{1}{1+x^2}dx^2 + d\theta^2$$

$$(t, x, \theta) \in [0, T) \times [0, 1] \times S^1$$



extremal horizon at x=0

2+1 — dimensional, radially symmetric spacetime

$$\Box\psi = 0 \qquad\qquad \Box\psi - \frac{R}{8}\psi = 0$$

drop most of
1st and 0th order terms

$$-\partial_t^2\psi + \partial_x(x^2\partial_x\psi) + 2\partial_t\partial_x\psi + \partial_\theta^2\psi = 0$$

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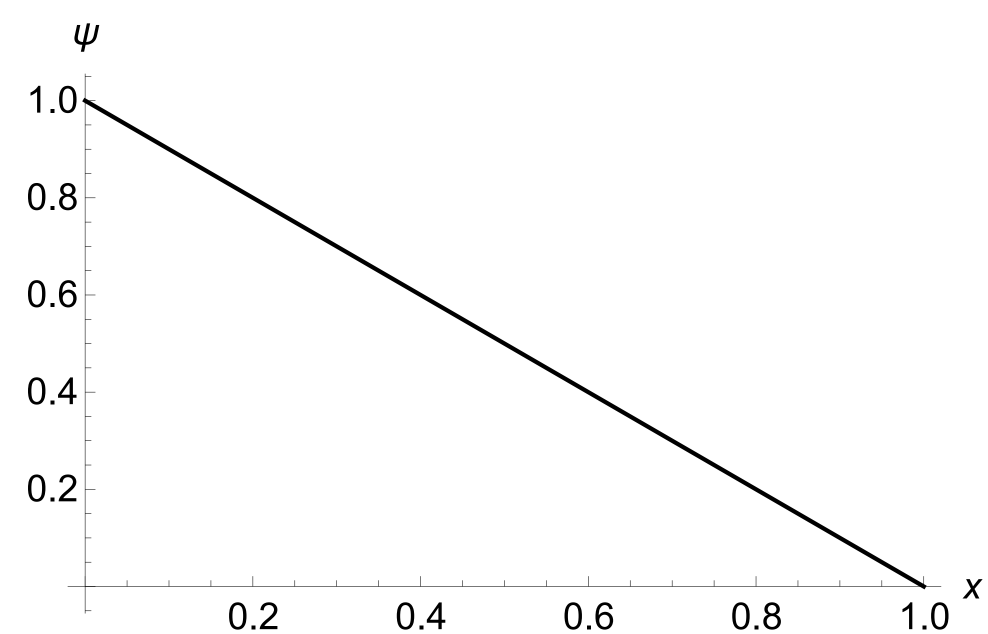
We want solutions such that satisfy

- $\psi|_{t=0} = \psi_0$
- $\partial_t\psi|_{t=0} = \psi_1$
- $\psi|_{x=1} = 0$
- ψ is regular at the horizon



Fourier and Laplace transformations

$$\psi(t,x,\theta) = \frac{1}{2\pi i}\sum_{m\in\mathbb{Z}} e^{im\theta}\int_{\gamma-i\infty}^{\gamma+i\infty} u_{m,s}(x)e^{st}ds,$$

$\Re\gamma > 0$

$$-\frac{d}{dx}\left(x^2\frac{d}{dx}u_{m,s}\right) - 2s\frac{d}{dx}u_{m,s} + (s^2+m^2)u_{m,s} = f_{m,s},$$

initial conditions

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initial conditions

Solutions of homogenous equation — modified Bessel functions

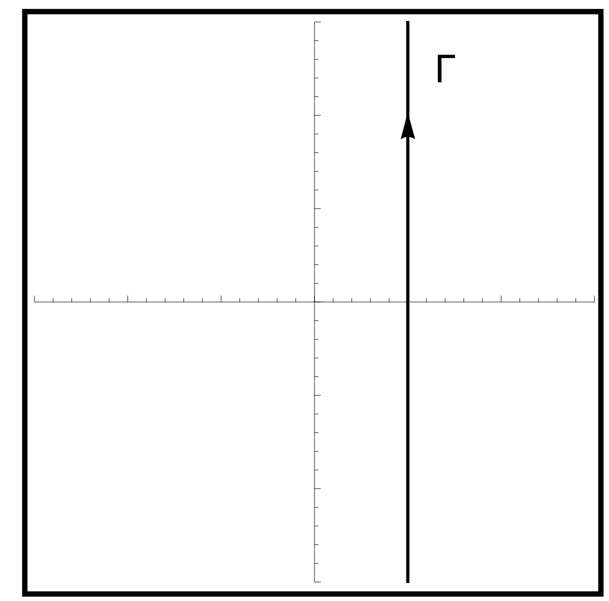
$$u(x) = e^{s/x}\sqrt{\frac{s}{x}}\,\left(a\,K_\lambda\left(\frac{s}{x}\right) + b\,I_\lambda\left(\frac{s}{x}\right)\right)$$

$$\lambda = \sqrt{s^2+m^2+\frac{1}{4}}$$

The Green's function

$$\psi(t,x,\theta) = \frac{1}{2\pi i}\sum_{m\in\mathbb{Z}} e^{im\theta}\int_{\Gamma} e^{st}\int_0^1 G_{m,s}(x,y)f_{m,s}(y)dy\,ds$$

contour in a complex plane



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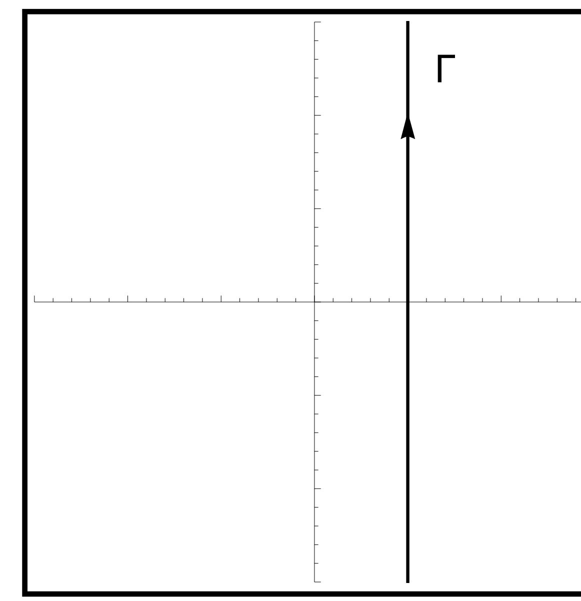

$$G_{m,s}(x,y) = \begin{cases} \dfrac{e^{-2s/y}u_1(x)u_2(y)}{W} & x<y \\ \dfrac{e^{-2s/y}u_2(x)u_1(y)}{W} & x>y \end{cases}$$

$$W = -e^{-s}\sqrt{\frac{2s}{\pi}}K_\lambda(s)$$

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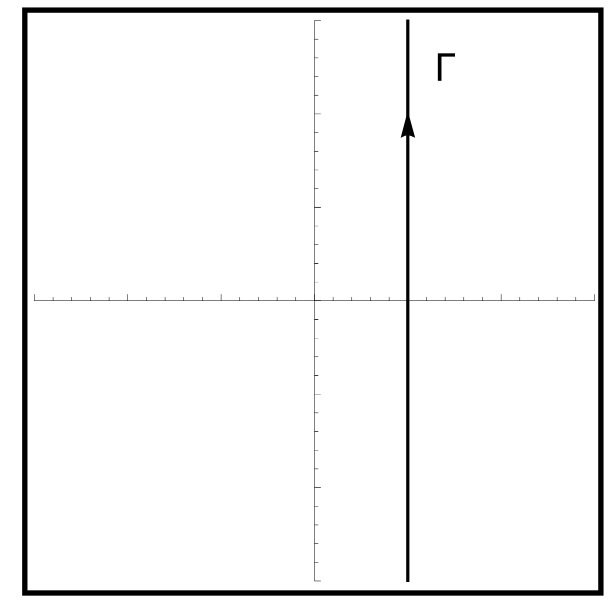

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In principle we can now calculate the solution for any initial data.
(everything is regular in the right half-plane)

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Delicate cancellations → Move contour to the left

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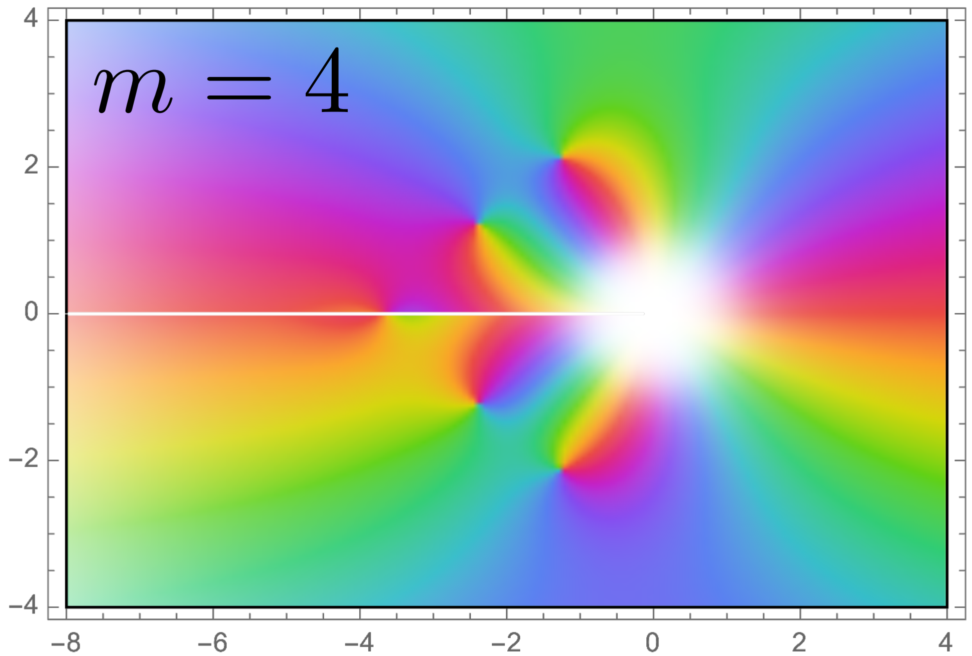

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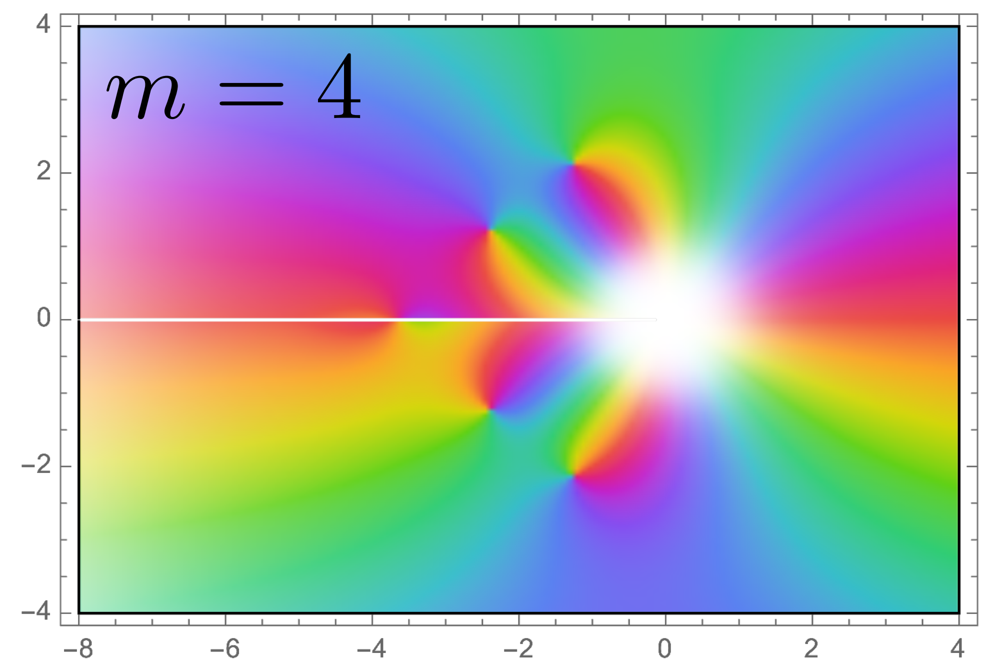
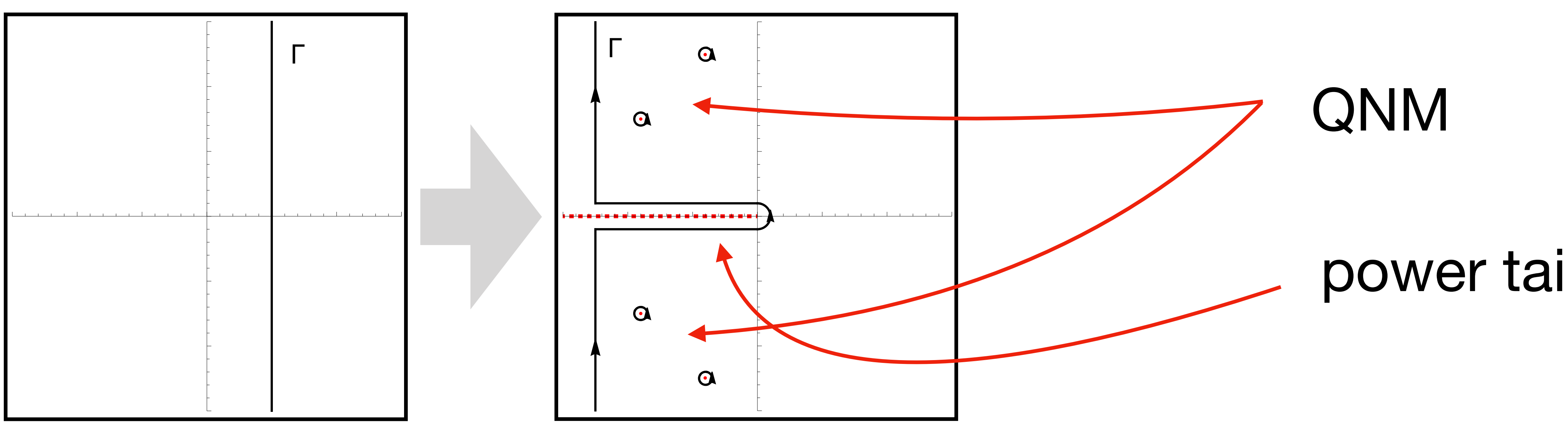
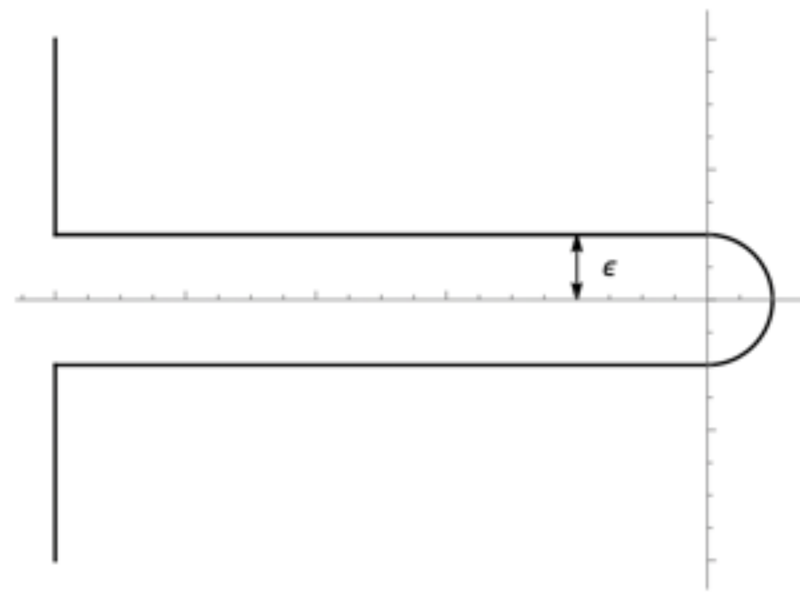




Power tail

Due to the term e^{st} contribution near zero is dominating.



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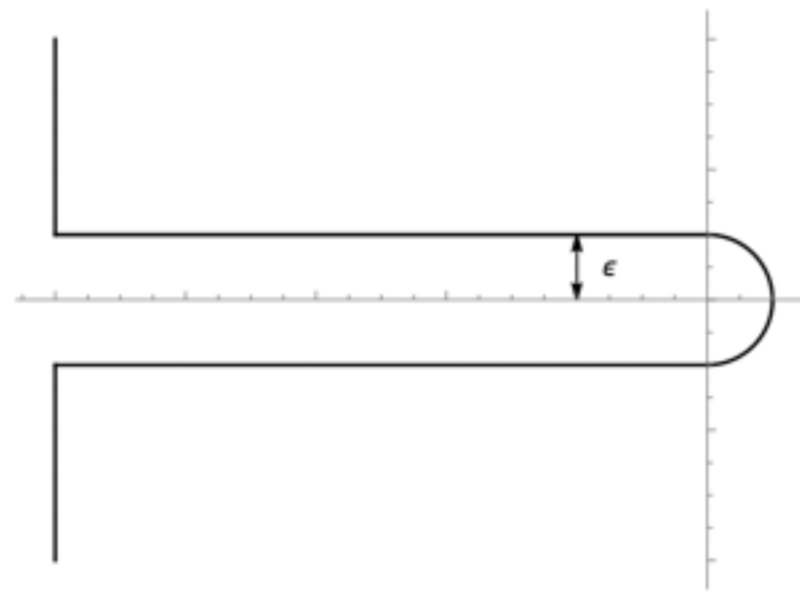
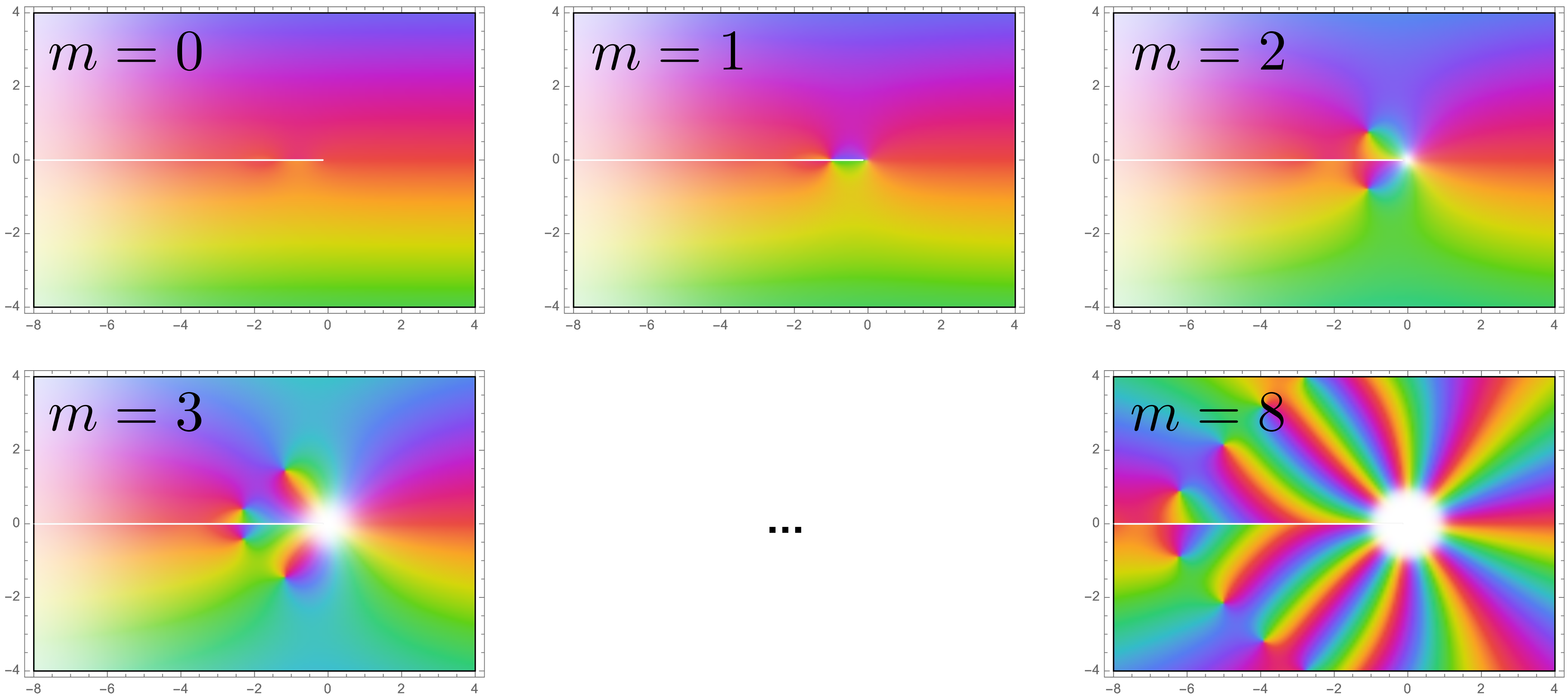


For **compactly supported** initial data the asymptotic analysis of the Green's function near $s = 0$ gives:

$$\psi \sim \frac{1}{t}$$

QNMs



$m = 0$					
$m = 1$					
$m = 2$	$-1.067 \pm 0.782i$				
$m = 3$	$-1.171 \pm 1.470i$	$-2.303 \pm 0.409i$			
$m = 4$	$-1.261 \pm 2.143i$	$-2.385 \pm 1.240i$			
$m = 5$	$-1.339 \pm 2.814i$	$-2.497 \pm 1.983i$	$-3.626 \pm 0.865i$		
$m = 6$	$-1.409 \pm 3.487i$	$-2.605 \pm 2.690i$	$-3.711 \pm 1.701i$	$-4.933 \pm 0.403i$	
$m = 7$	$-1.472 \pm 4.161i$	$-2.707 \pm 3.381i$	$-3.821 \pm 2.470i$	$-4.952 \pm 1.325i$	
$m = 8$	$-1.530 \pm 4.836i$	$-2.801 \pm 4.064i$	$-3.933 \pm 3.203i$	$-5.039 \pm 2.164i$	$-6.239 \pm 0.884i$

Numerical results

Plots of $\left|\psi\left(t, 0.8, \frac{\pi}{7}\right)\right|$ for

$$\begin{cases} \psi_0(x,\theta) = (1-x)\cos m\theta \\ \psi_1(x,\theta) = 0 \end{cases}$$

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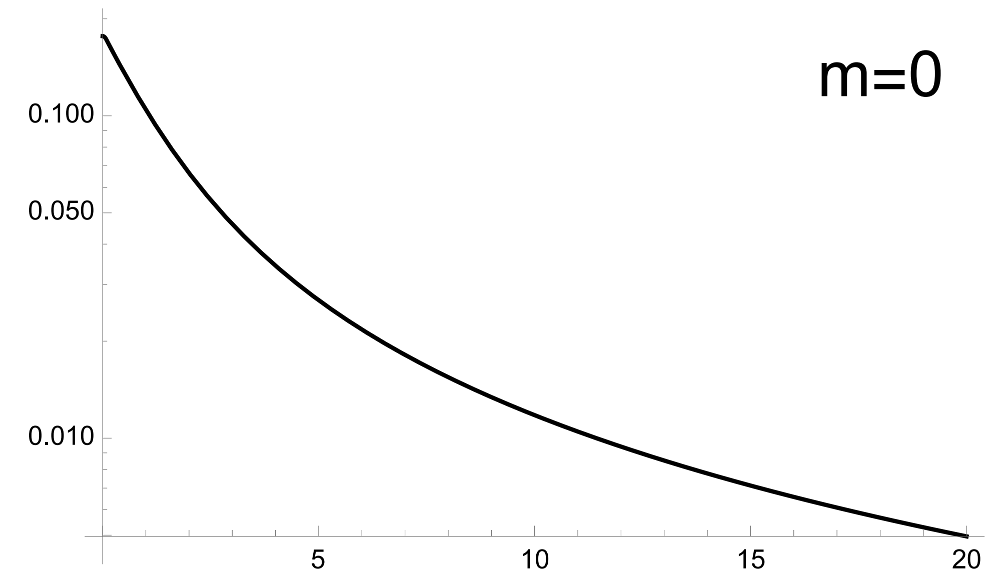

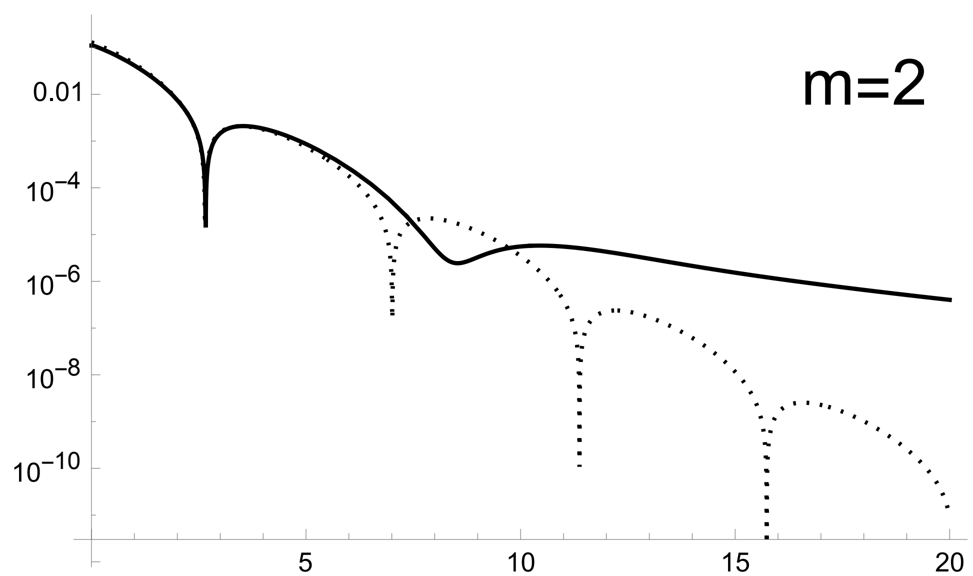

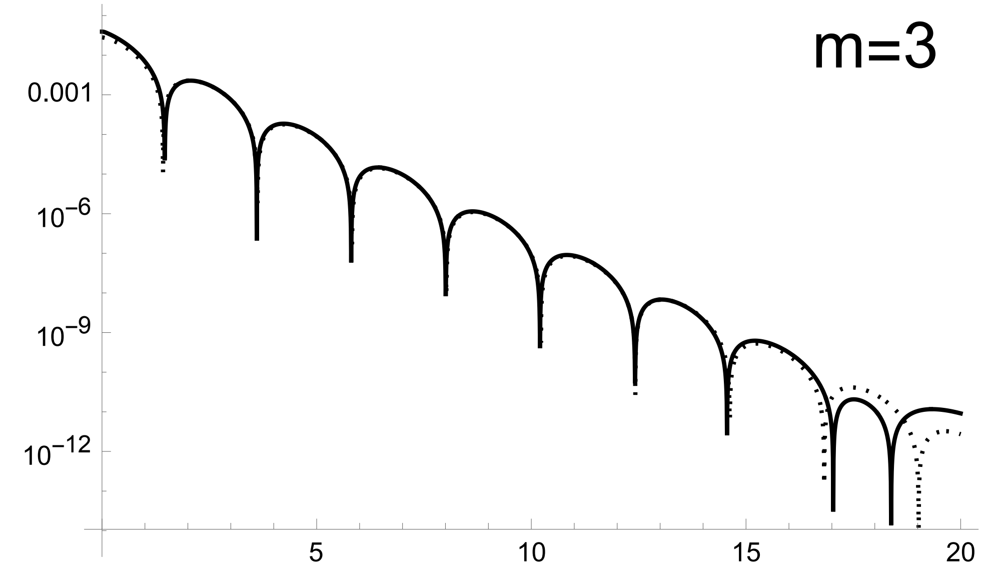

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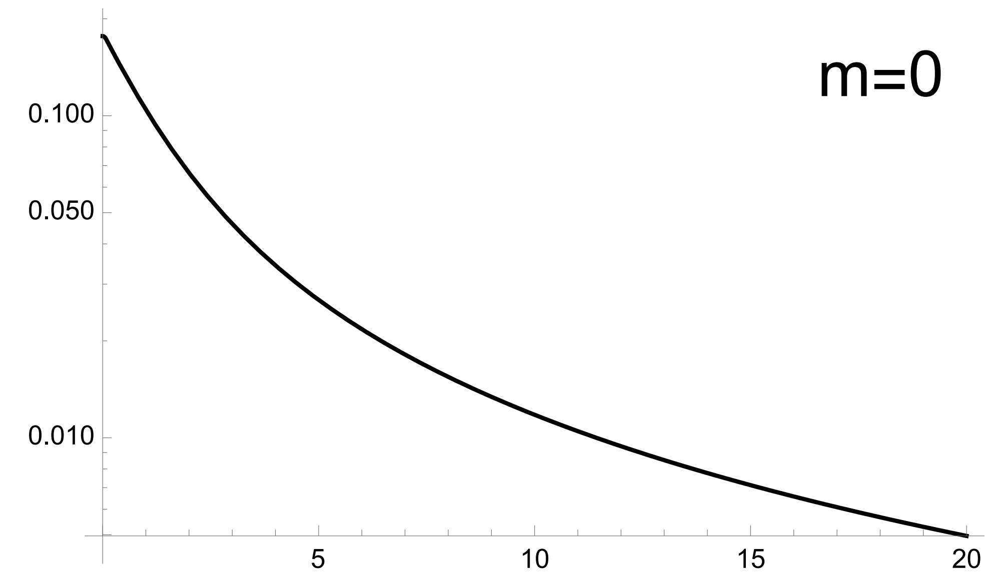

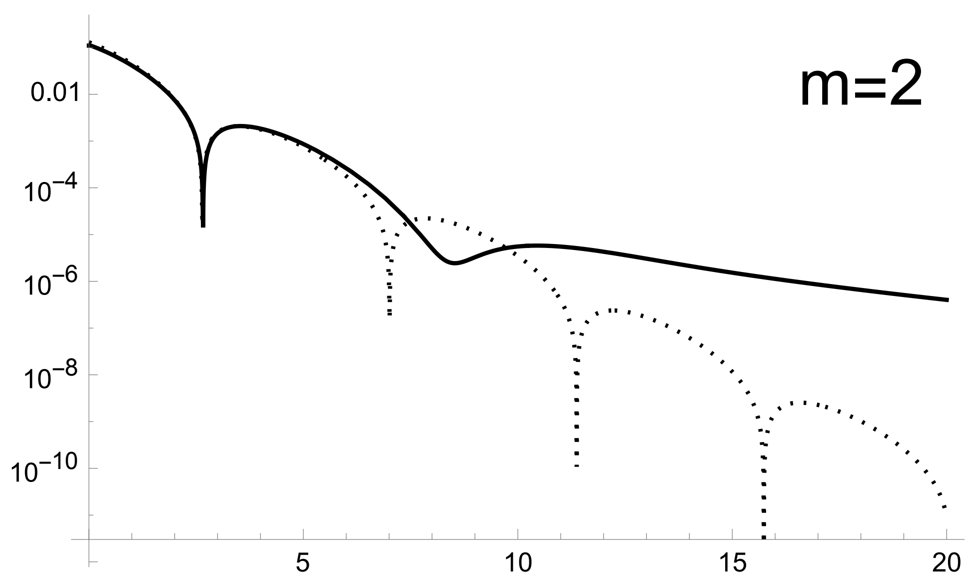

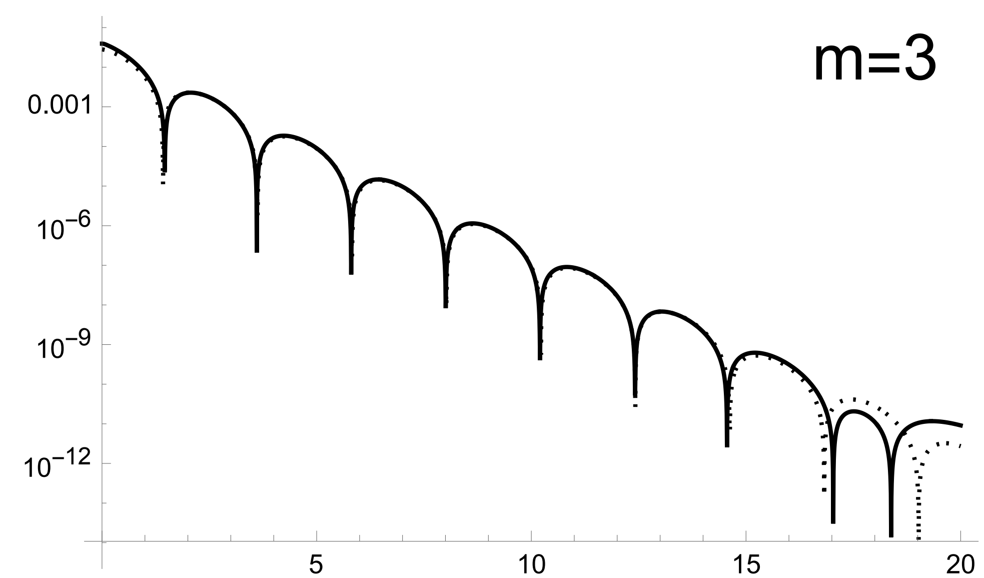

The power tails differ from t^{-1}
since the initial data are not compactly supported

Nonlinear problem

Conformally invariant equation

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$$-\frac{x^3}{1+x^2}\partial_x\psi - \frac{x}{1+x^2}\partial_t\psi - \frac{-1+2x^2}{4(1+x^2)^2}\psi - \lambda|\psi|^4\psi = 0$$

Conformal factor depending only on x does not change QNFs

$$(g, \psi) \to (\Omega^2 g, \Omega^{-1/2} \psi)$$

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$$\Omega(x) = \sqrt{1 + x^2}$$

$$- \partial_t^2 \psi + \partial_x (x^2 \partial_x \psi) + 2 \partial_t \partial_x \psi + \partial_\theta^2 \psi$$

$$+ \frac{1}{4} \left(1 + \frac{2x^2}{(1 + x^2)^2} \right) \psi - \lambda |\psi|^4 \psi = 0$$

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We start with the linear problem

Leaver method

$$\psi(t,x,\theta) = e^{st}e^{im\theta}u(x)$$

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The Taylor expansion:

$$u(x) = \sum_{k=0}^{\infty} H_k (1-x)^k$$

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Then we want: $H_0 = 0$

$$\lim_{k \to \infty} H_k = 0$$ — quantisation condition

Leaver method

$$H_k = \frac{1}{(k-1)k}\left[2(k-1)(k-1+s)H_{k-1} - \left((k-2)(k-1) - (s^2+m^2) + \frac{1}{4}\right)H_{k-2} + \sum_{j=0}^{k-2} a_{k-j-2}H_j\right]$$

where $$\frac{x^2}{2(1+x^2)^2} = \sum_{n=0}^{\infty} a_n(1-x)^n$$

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$$H_0 = 0$$

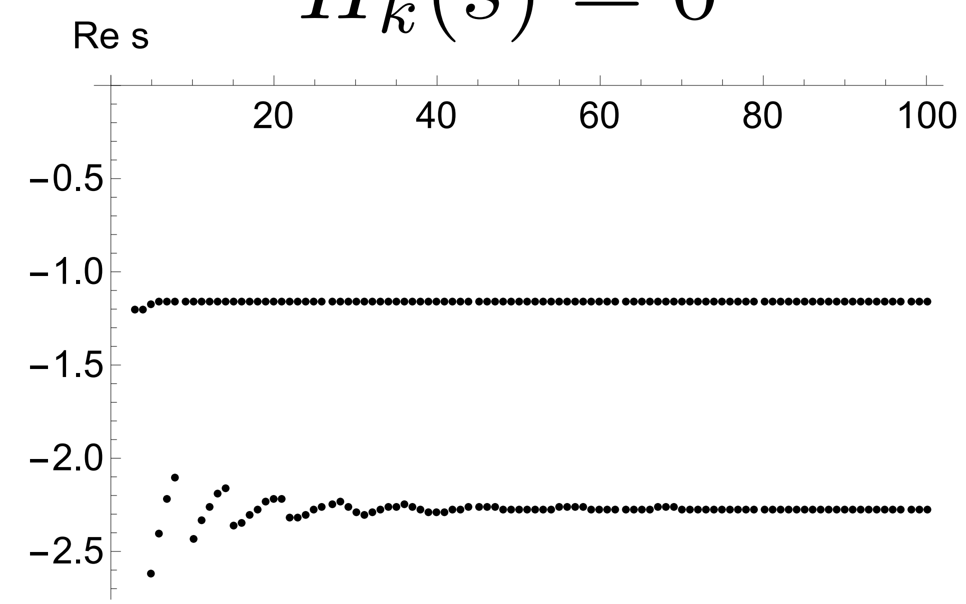
$$H_1 = 1$$

$$H_2 = 1 + s$$

$$H_3 = \frac{39}{16} + 2s + \frac{5s^2}{6}$$

$$H_4 = \frac{31}{8} + \frac{71s}{16} + \frac{7s^2}{3} + \frac{s^3}{2}$$

$$H_k(s) = 0$$



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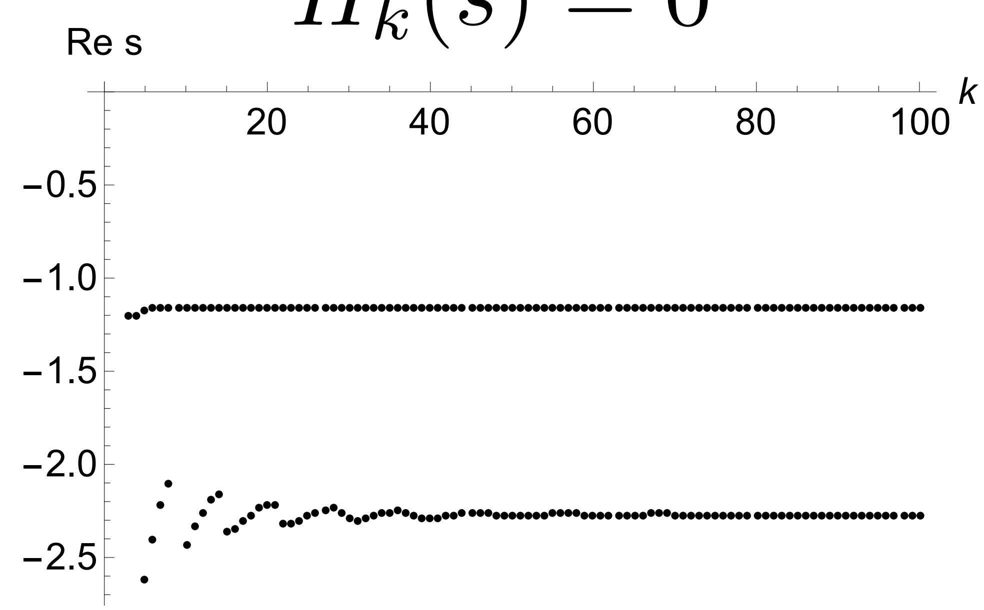

More advanced methods are available for more complicated cases

Nonlinear equation

For larger fields nonlinearities introduce many effects.

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but QNM are usually small…

$$-\partial_t^2\phi_m + \partial_x(x^2\partial_x\phi_m) + 2\partial_t\partial_x\phi_m - m^2\phi_m + \frac{1}{4}\left(1 + \frac{2x^2}{(1+x^2)^2}\right)\phi_m$$

$$-\lambda(1+x^2)\sum_{m_1,m_2,m_3,m_4}\phi_{m_1}\bar{\phi}_{m_2}\phi_{m_3}\bar{\phi}_{m_4}\phi_{m-m_1+m_2-m_3+m_4} = 0.$$

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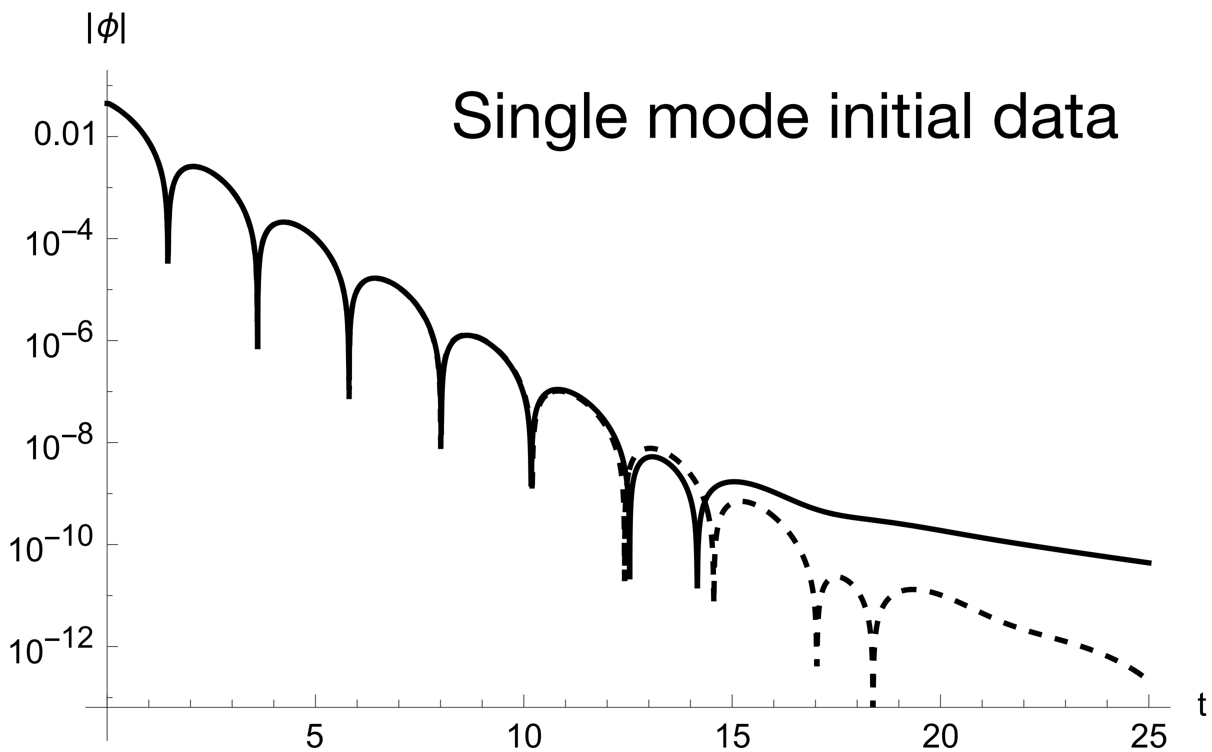
Mode mixing — but there are no QNM for m=0 and m=1.

Nonlinear equation



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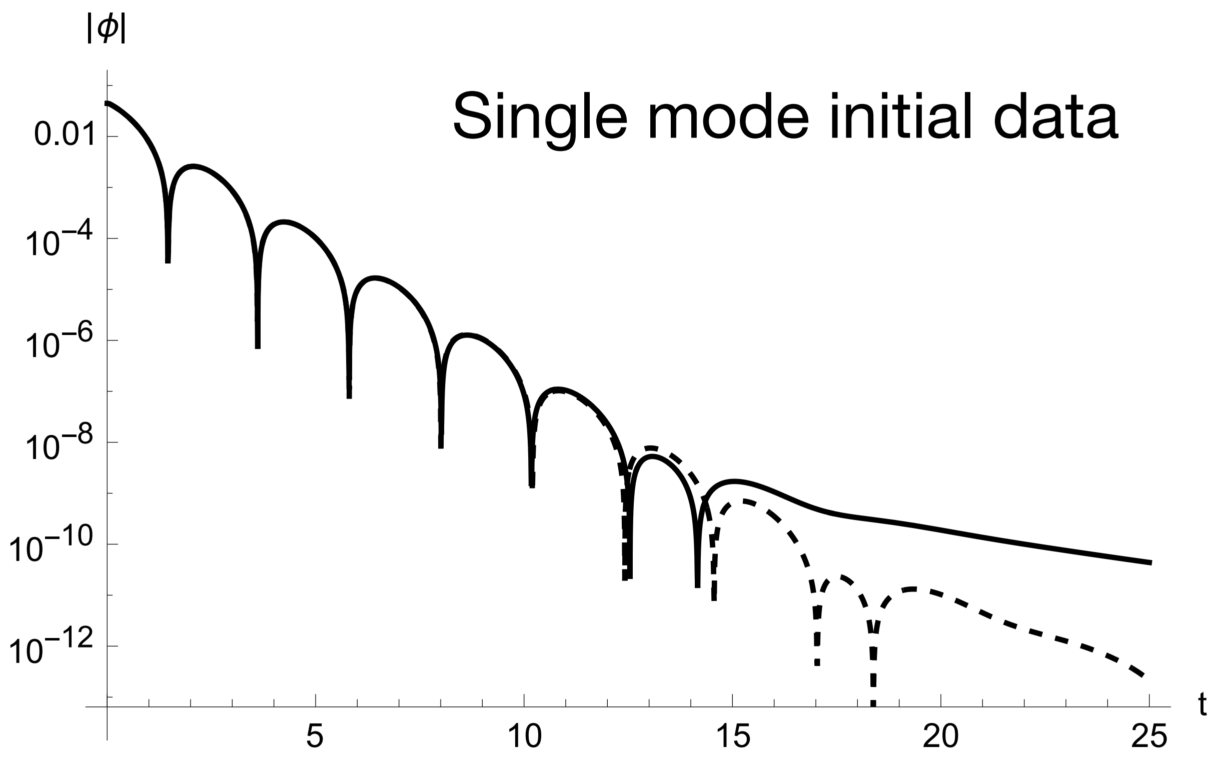

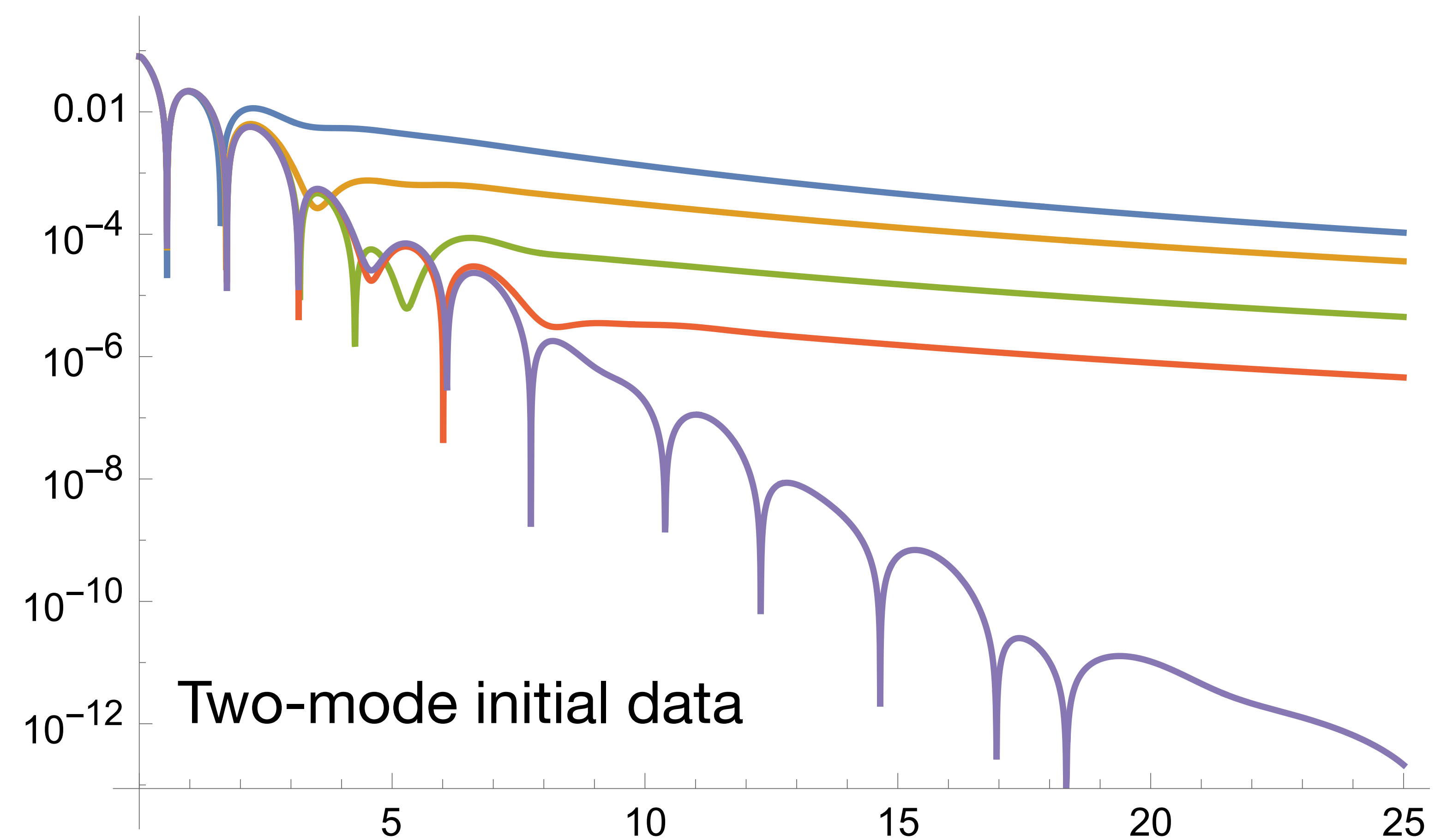

Extremal RNAdS

Single horizon at r_* $\qquad \rho = r/r_* \qquad \lambda = r_*^2/l^2$

$$ds^2 = -f(\rho)dt^2 + f(\rho)^{-1}dr^2 + d\Omega^2$$

$$f(\rho) = 1 - \frac{2(1+2\lambda)}{\rho} + \frac{1+3\lambda}{\rho^2} + \lambda\rho^2$$

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$$-a_{tt}(x)\partial_t^2\psi + \frac{2\alpha(x)^2}{(1+6\lambda)^2}\partial_t\partial_x\psi +$$

$$+\alpha(x)^2\partial_x\left(\frac{x^2}{1+6\lambda+2(1+4\lambda)x}\partial_x\psi\right) + (a_0(x) - l(l+1)\alpha(x))\psi = 0$$

Work in progress…

Summary

- The completely solvable toy-model gives us some insights regarding QNM.
- Using conformal invariance one can swiftly change between various equivalent problems.
- Nonlinearities may impact the behaviour of the system also in the weak field regime.

Homogenous problem

$$-\frac{d}{dx}\left(x^2\frac{d}{dx}u\right) - 2s\frac{d}{dx}u + (s^2+m^2)u = 0$$

$$z = \frac{s}{x}$$

$$u(x) = e^z\sqrt{z}\,v(z)$$

$$z^2\frac{d^2v}{dz^2} + z\frac{dv}{dz} - \left(z^2+\lambda^2\right)v = 0$$

$$\lambda = \sqrt{s^2+m^2+\frac{1}{4}}$$

Solutions: the modified Bessel functions $K_\lambda(z),\ I_\lambda(z)$

$$u(x) = e^{s/x}\sqrt{\frac{s}{x}}\left(a\,K_\lambda\left(\frac{s}{x}\right) + b\,I_\lambda\left(\frac{s}{x}\right)\right)$$