Evolving black hole with scalar field accretion

talk based on

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Talk outline

- Motivation
- Coordinates adapted to the evolving horizon
- Expanding the field equations near the horizon
- Accretion law
- Instability of the Neumann solution
- Approach to equilibrium and scaling relations
- Outlook and conclusions

Motivation

Can we give an analytic description of a physically realistic evolving BH?
(e.g., in asymptotic flatness, or in a cosmological spacetime)

How does a black hole interact with the cosmic medium?
What is the influence of cosmological evolution on the growth of the horizon?
(previous approaches: McVittie, Bondi, Einstein-Straus, Husain-Martinez-Nuñez, …)

The horizon is part of the unknowns of the problem, but is only determined a posteriori from the solutions.
Is there a way to make it appear explicitly *in the equations*?

Our goal

Develop a general method to study the near-horizon asymptotics of evolving BHs that can be applied for any matter fields and spacetime asymptotics
(in spherical symmetry)

We consider a scalar field as matter, both for simplicity and for its relevance in inflation.

Eddington-Finkelstein coordinates (1)

A general spherically symmetric geometry can be written in EF coordinates

$$ds^2 = -e^{2\beta(v,r)}A(v,r)dv^2 + 2e^{\beta(v,r)}dvdr + r^2 d\Omega^2$$

r is the areal radius, v is an ingoing null coordinate

For a BH, these coordinates are regular at the BH horizon

The apparent horizon is determined by

$$A(v,r) = 0 \Longleftrightarrow \theta_l = 0$$

We must also ensure that $\theta_n < 0$ and $\pounds_n \theta_l < 0$, so that the apparent horizon is a future outer trapping horizon [Hayward '94]

Eddington-Finkelstein coordinates (2)

The equation $A(v, r) = 0$ implicitly defines the horizon as a function of v :

$$r_H = r_H(v)$$

In case there are multiple branches, we may take the outermost one.

The main problem *with these coordinates* is that the apparent horizon can only be determined a posteriori, once we have found a solution to the field equations.

We need to 'extract' the information on the zeroes of A.

New coordinates adapted to the evolving horizon

It is convenient to adopt a new radial coordinate z that is adapted to the evolving horizon

$$\boxed{r = \frac{r_H(v)}{1-z}}$$

In the static case it reduces to [Rezzolla, Zhidenko '14]

In the new coordinates, the horizon is at $z = 0$, while spatial infinity is at $z = 1$.

$$ds^2 = \left(-e^{2\beta(v,z)}A(v,z) + \frac{2e^{\beta(v,z)}\dot{r}_H(v)}{1-z}\right)dv^2 + \frac{2e^{\beta(v,z)}r_H(v)}{(1-z)^2}dvdz + \frac{r_H^2(v)}{(1-z)^2}d\Omega^2$$

The main advantage is that $r_H(v)$ now appears explicitly in the metric
(no longer indirectly through A)

In the new coordinates, both relevant variables z and $\dot{r}_H(v)$ feature explicitly

proximity expansion rate

Einstein-scalar system

$$G_{ab} = \kappa T_{ab} = \kappa \left[\partial_a \phi \partial_b \phi - g_{ab} \left(\frac{1}{2} g^{cd} \partial_c \phi \partial_d \phi + U(\phi) \right) \right]$$

in the coordinates (v, z) the field equations read:

$$2\beta' = \kappa(1-z)\left(\phi'\right)^2 \ ,$$

$$1 - A - (1-z)\left(A' + A\beta'\right) = \frac{\kappa r_H^2}{(1-z)^2} U(\phi) \ ,$$

$$\frac{\dot{A}}{1-z} - \frac{\dot{r}_H}{r_H} A' = \kappa \left(\frac{\dot{r}_H}{r_H} \phi' - \frac{\dot{\phi}}{1-z} \right) \left[(1-z)\phi' A - e^{-\beta} r_H \left(\frac{\dot{r}_H}{r_H} \phi' - \frac{\dot{\phi}}{1-z} \right) \right] \ ,$$

$$2\beta'' A + A'' + \beta' \left(3A' + 2\beta' A\right) - \frac{2e^{-\beta} r_H}{1-z} \left(\frac{\dot{r}_H}{r_H} \beta'' - \frac{\dot{\beta}'}{1-z} \right) = \frac{\kappa e^{-\beta} r_H \phi'}{1-z} \left(\frac{\dot{r}_H}{r_H} \phi' - \frac{2\dot{\phi}}{1-z} \right) - \frac{2\kappa r_H^2 U(\phi)}{(1-z)^4} \ .$$

Klein-Gordon equation:

$$\dot{\phi}' + \frac{\dot{\phi}}{1-z} - \frac{\dot{r}_H}{r_H}(1-z)\phi'' + \frac{e^{\beta}(1-z)^2}{2r_H} \left(A\phi'' + A'\phi' + A\phi'\beta'\right) - \frac{e^{\beta} r_H}{2(1-z)^2} \frac{\partial U}{\partial \phi} = 0$$

z-expansion

We expand the equations in the proximity of the horizon, assuming that both matter and geometry are regular around $z = 0$

Ansätze: analiticity in z

$$A(v,z) = \sum_{n=1}^{\infty} a_n(v) z^n \ , \quad \beta(v,z) = \sum_{n=1}^{\infty} b_n(v) z^n \ , \quad \phi(v,z) = \phi_o(v)\left(1 + \sum_{n=1}^{\infty} c_n(v) z^n\right)$$

NB: $b_0(v) = 0$ is a gauge choice and $a_1 > 0 \Longleftrightarrow \pounds_n \theta_l < 0$

Substitute the ansätze in the field equations and expand.

To first order in z, the solutions are:

$$a_1 = 1 - \kappa\, r_H^2 U(\phi_o) > 0 \ , \quad b_1 = \frac{1}{2}\kappa c_1^2 \phi_o^2 \ , \quad a_1 \dot{r}_H = \kappa \left(c_1 \phi_o \dot{r}_H - r_H \dot{\phi}_o\right)^2$$

Higher order coefficients can similarly be computed, complications are only algebraic.

Physical meaning on the first-order solutions

To gain some insight, let us compute the fluxes in the radial null directions at $z = 0$

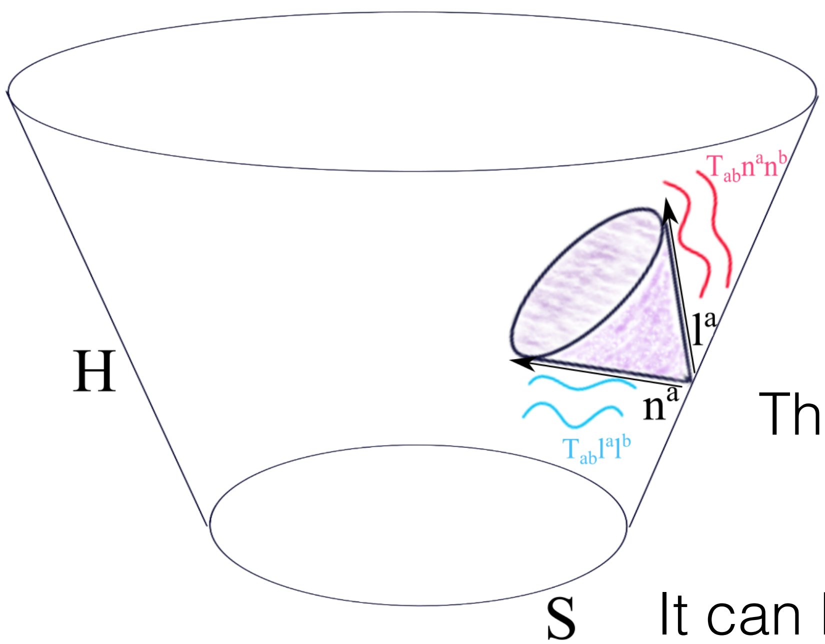
$$T_{ab}n^a n^b = \frac{c_1^2\,\phi_o^2}{r_H^2}$$

outgoing flux

$$T_{ab}l^a l^b = \left(\dot\phi_o - c_1\phi_o\frac{\dot r_H}{r_H}\right)^2$$

ingoing flux

Combining this with the first-order solution, we get the accretion law



$$\dot r_H = \frac{\kappa\, r_H^2}{1-\mathscr{E}}T_{ab}l^a l^b = \frac{\kappa\, r_H^2}{1-\mathscr{E}}(\pounds_l\phi)^2 \geq 0$$

$$\mathscr{E} \equiv \kappa\, r_H^2 U(\phi_o) = r_H^2\Lambda_{\rm eff}$$

Other geometric derivations:
[Ashtekar, Krishnan ‘04; Booth *et al.* ’06]

This result is consistent with the area theorem

It can be rewritten as a “Bondi-like” accretion formula

$$\boxed{\dot M = \frac{16\pi G^2}{1-\mathscr{E}}M^2(\pounds_l\phi)^2}$$

Neumann boundary conditions

The scalar field obeys the boundary condition $(\partial_z \phi)|_{z=0} = 0 \Longleftrightarrow c_1 = 0$

(equivalent to the condition $T_{ab} n^a n^b = 0$ at $z = 0$)

$\boxed{\text{This solution \underline{does not admit a static limit}}}$

$$b_3 = \frac{\left((1 - \mathscr{E}) r_H \dot{\mathscr{E}} - 2\mathscr{K}\right)^2}{24 \mathscr{K}^3}$$

$\mathscr{E} \equiv \kappa\, r_H^2 U(\phi_o)\ , \quad \mathscr{K} \equiv \kappa\, r_H^2 (\dot{\phi}_o)^2$

$$R = \frac{1}{r_H^2} \left\{ 4\mathscr{E} + \left[2 - r_H (1 - \mathscr{E}) \frac{\dot{\mathscr{E}}}{\mathscr{K}} \right] z + \mathcal{O}(z^2) \right\}$$

In the $\dot{\phi}_o \to 0$ limit, these quantities diverge
(they are not the only ones)

This shows that the <u>outgoing flux cannot be zero at all times</u>, but only relaxes to zero in the static limit.

Approaching equilibrium (1)

We expand the equations around a static solution

(similar in spirit to a non-linear generalization of quasi-normal modes)

horizon $$r_H(v) = r_H^{(0)} + r_H^{(1)}(v) + r_H^{(2)}(v) + \dots$$

dynamical fields
$$A(v,z) = A^{(0)}(z) + A^{(1)}(v,z) + A^{(2)}(v,z) + \dots \ ,$$
$$\beta(v,z) = \beta^{(0)}(v) + \beta^{(1)}(v,z) + \beta^{(2)}(v,z) + \dots \ ,$$
$$\phi(v,z) = \phi^{(0)} + \phi^{(1)}(v,z) + \phi^{(2)}(v,z) + \dots \ .$$

scalar field at the horizon $$\phi_o(v) = \phi^{(0)} + \phi_o^{(1)}(v) + \phi_o^{(2)}(v) + \dots \qquad [\phi_o^{(n)}(v) = \phi^{(n)}(v,0)]$$

Schwarzschild-de Sitter as a static background:

$$A^{(0)}(z) = z + \frac{\kappa}{3}(r_H^{(0)})^2 U(\phi^{(0)})\left[(1-z) - \frac{1}{(1-z)^2}\right] \ , \quad \beta^{(0)}(v) = 0$$

the scalar field must be in equilibrium: $$\left.\frac{\partial U}{\partial \phi}\right|_{\phi^{(0)}} = 0$$

Approaching equilibrium (2)

We perform a double expansion:
analytic ansatz in z at each order in the perturbative series

$$A^{(n)}(v,z)=\sum_{k=1}^{\infty}a_k^{(n)}(v)z^k\ ,\quad \beta^{(n)}(v,z)=\sum_{k=1}^{\infty}b_k^{(n)}(v)z^k\quad \phi^{(n)}(v,z)=\phi_o^{(n)}(v)+\sum_{k=1}^{\infty}l_k^{(n)}(v)z^k$$

This amounts to mapping the Einstein equations to a (infinite-dimensional) dynamical system for the variables $\{r_H^{(n)},\,\phi_o^{(n)},\,a_k^{(n)},\,b_k^{(n)},\,l_k^{(n)}\}$

To second order in perturbation theory and in the z-expansion, the solution is:

$$A(v,z)=\left\{1-\kappa\left[(r_H^{(0)})^2\left(U(\phi^{(0)})+\frac{1}{2}\frac{\partial^2 U}{\partial\phi^2}\bigg|_{\phi^{(0)}}(\phi_o^{(1)})^2\right)+2r_H^{(0)}r_H^{(2)}U(\phi^{(0)})\right]\right\}z$$

$$-\kappa\left\{(r_H^{(0)})^2\left[\left(1-\frac{\kappa}{4}\left(l_1^{(1)}\right)^2\right)U(\phi^{(0)})+\frac{1}{2}\left(\phi_o^{(1)}+l_1^{(1)}\right)\phi_o^{(1)}\frac{\partial^2 U}{\partial\phi^2}\bigg|_{\phi^{(0)}}\right]+2r_H^{(0)}r_H^{(2)}U(\phi^{(0)})+\frac{1}{4}\left(l_1^{(1)}\right)^2\right\}z^2+\dots$$

$$\beta(v,z)=\frac{\kappa}{2}\left(l_1^{(1)}\right)^2 z-\frac{\kappa}{4}l_1^{(1)}\left(l_1^{(1)}-4l_2^{(1)}\right)z^2+\dots\ ,$$

$$\phi(v,z)=\left(\phi^{(0)}+\phi_o^{(1)}+\phi_o^{(2)}\right)+\left(l_1^{(1)}+l_1^{(2)}\right)z+\left(l_2^{(1)}+l_2^{(2)}\right)z^2+\dots\ .$$

Note that the second order solution for the geometry depends on $\phi_o^{(1)},\,l_1^{(1)},\,l_2^{(1)}$,
although *not* on higher order corrections to the scalar field.

Approaching equilibrium (3)

The remaining coefficients satisfy the following dynamical equations

$$\dot{l}_1^{(1)} = -\dot{\phi}_o^{(1)} + \frac{r_H^{(0)}}{2}\phi_o^{(1)}\frac{\partial^2 U}{\partial \phi^2}\bigg|_{\phi^{0)}} - \frac{l_1^{(1)}}{2r_H^{(0)}}\left(1 - \kappa\,(r_H^{(0)})^2 U(\phi^{(0)})\right),$$

$$\dot{l}_2^{(1)} = \frac{r_H^{(0)}}{4}\phi_o^{(1)}\frac{\partial^2 U}{\partial \phi^2}\bigg|_{\phi^{(0)}} + \frac{l_1^{(1)}}{4r_H^{(0)}}\left[3 - (r_H^{(0)})^2\left(\kappa\, U(\phi^{(0)}) - \frac{\partial^2 U}{\partial \phi^2}\bigg|_{\phi^{(0)}}\right)\right] - \frac{l_2^{(1)}}{r_H^{(0)}}\left(1 - \kappa(r_H^{(0)})^2 U(\phi^{(0)})\right)$$

$$\dot{r}_H^{(2)} = \frac{\kappa(r_H^{(0)})^2}{1 - \mathscr{E}^{(0)}}(\dot{\phi}_o^{(1)})^2$$

<u>NB</u>: $\dot{\phi}_o^{(1)}$ is still undetermined at this stage (boundary data), while $\dot{r}_H^{(1)} = 0$

$$T_{ab}n^a n^b\Big|_{z=0} \simeq \frac{(l_1^{(1)})^2}{(r_H^{(0)})^2}\,, \quad T_{ab}l^a l^b\Big|_{z=0} \simeq (\dot{\phi}_o^{(1)})^2$$

From the $\dot{l}_1^{(1)}$ equation, we see that setting $l_1^{(1)} = 0$ implies that the scalar field must *climb up* the potential! This explains the absence of a static limit for the Neumann solution.

Approaching equilibrium (4)

Following the structure of the $\dot{l}^{(1)}$ equation, we propose: $\boxed{r_H^{(0)}\dot{\phi}_o^{(1)} = -\gamma\,\phi_o^{(1)} + \xi\, l_1^{(1)}}$

With this assumption, the dynamics boils down to the following autonomous dynamical system

$$\boxed{\begin{aligned} r_H^{(0)}\dot{\phi}_o^{(1)} &= -\gamma\,\phi_o^{(1)} + \xi\, l_1^{(1)}\,, \\ r_H^{(0)}\dot{l}_1^{(1)} &= \left(\gamma + \frac{(r_H^{(0)})^2}{2}\frac{\partial^2 U}{\partial\phi^2}\bigg|_{\phi^{(0)}}\right)\phi_o^{(1)} - \frac{1}{2}\Big(1 - \mathscr{E}^{(0)} + 2\xi\Big) l_1^{(1)} \end{aligned}}$$

The solution $\phi_o^{(1)} = l_1^{(1)} = 0$ is an *attractive fixed point* provided that:

$$2(\gamma+\xi) + 1 - \mathscr{E}^{(0)} > 0\,, \quad 0 < \gamma\left(1 - \mathscr{E}^{(0)}\right) - \xi (r_H^{(0)})^2 \frac{\partial^2 U}{\partial \phi^2}\bigg|_{\phi^{(0)}} \leq \frac{1}{8}\Big[2(\gamma+\xi) + 1 - \mathscr{E}^{(0)}\Big]^2$$

In the large time limit, we get (λ_1 is the least negative eigenvalue)

$$\phi_o^{(1)}(v) \sim p\, e^{\lambda_1 v/r_H^{(0)}}\,, \quad l_1^{(1)}(v) \sim \xi^{-1} p(\gamma+\lambda_1)\, e^{\lambda_1 v/r_H^{(0)}}\,, \quad r_H^{(2)}(v) \sim \Delta r_H + \frac{\kappa p^2 \lambda_1}{2(1-\mathscr{E}^{(0)})} r_H^{(0)}\, e^{2\lambda_1 v/r_H^{(0)}}$$

Approaching equilibrium (5)

We obtain the following scaling relations

$$\boxed{\left|\frac{r_H(v) - r_H^f}{r_H^{(0)}}\right| \sim \kappa(\phi_o^{(1)}(v))^2 \,, \quad l_1^{(1)}(v) \sim \phi_o^{(1)}(v)}$$

These are "universal" since they *do not* depend (except for prefactors) on the shape of the potential, boundary data, etc

(However, they do depend on our modelling of $\dot{\phi}_o^{(1)}$ as a linear combination of $\phi_o^{(1)}, l_1^{(1)}$)

<u>Potentially testable with numerical simulations</u>

Summary

- We introduced a new radial coordinate $z = 1 - r_H(v)/r$ adapted to the evolving horizon
- The equations of motion can be solved order by order in z. The first order solution gives an exact Bondi-like accretion law.
- The solution with Neumann boundary conditions at the horizon does not admit a static limit.
- Near-equilibrium black-holes can be studied introducing a double expansion (perturbative and in z). The Einstein equations are mapped to an infinite-dimensional dynamical system.
- The approach to equilibrium is characterized by universal scaling relations.

Future work

- Matching the near-horizon solutions to the region far from the BH
- Similar analysis for different matter fields (e.g., hydrodynamic matter, gauge fields) and for alternatives to general relativity
- Going beyond spherical symmetry