Hyperheavenly spaces and their application in para-Kähler geometries

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Introduction

Real neutral spaces - 4-dimensional real spaces equipped with a metric of *neutral* signature $(++--)$

- Integrable systems - they admit ASD conformal structures[1]
- Two solids rolling on each other without slipping or twisting[2,3]
- A relation between para-Kähler Einstein spaces and non-integrable twistor distributions[4]

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Research programme: Weak $\mathcal{HH}$-spaces $\longrightarrow$ para-Kähler spaces

Main goal: to find the most general metric of algebraically degenerate para-Kähler Einstein spaces

- Weak nonexpanding $\mathcal{HH}$-spaces[5] (The 7th Conference of the Polish Society on Relativity, Łódź, 2021)
- Weak expanding $\mathcal{HH}$-spaces[6]

Remarks:

- All considerations are local
- All metrics are complex (coordinates are complex, functions are holomorphic) but they have neutral slices

[5] 2022, A. C., *Hyperheavenly spaces and their application in Walker and para-Kähler geometries: Part I*, Journal of Geometry and Physics 179, 104591

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Plan of the presentation:

- Part 1: Criteria of classification (Petrov-Penrose classification, properties of congruences of null strings, properties of intersections of congruences of null strings)
- Part 2: From weak expanding $\mathcal{HH}$-spaces to algebraically degenerate para-Kähler spaces
- Part 3: General metrics of all algebraically degenerate para-Kähler Einstein spaces

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Part 1: Criteria of classification

Petrov-Penrose classification

Penrose theorem says $C_{ABCD} = a_{(A}b_B c_C d_{D)}$, where a_A, b_A, c_A and d_A are *undotted Penrose spinors*.

Complex case		Real neutral case	
Type	$C_{ABCD} =$	Type	$C_{ABCD} =$
[I]	$a_{(A}b_B c_C d_{D)}$	$[\mathrm{I}_r]$	$m_{(A}n_B r_C s_{D)}$
		$[\mathrm{I}_{rc}]$	$m_{(A}n_B a_C \bar{a}_{D)}$
		$[\mathrm{I}_c]$	$a_{(A}\bar{a}_B b_C \bar{b}_{D)}$
[II]	$a_{(A}a_B b_C c_{D)}$	$[\mathrm{II}_r]$	$m_{(A}m_B n_C r_{D)}$
		$[\mathrm{II}_{rc}]$	$m_{(A}m_B a_C \bar{a}_{D)}$
[D]	$a_{(A}a_B b_C b_{D)}$	$[\mathrm{D}_r]$	$m_{(A}m_B n_C n_{D)}$
		$[\mathrm{D}_c]$	$a_{(A}a_B \bar{a}_C \bar{a}_{D)}$
[III]	$a_{(A}a_B a_C b_{D)}$	$[\mathrm{III}_r]$	$m_{(A}m_B m_C n_{D)}$
[N]	$a_A a_B a_C a_D$	$[\mathrm{N}_r]$	$m_A m_B m_C m_D$
[O]	—	$[\mathrm{O}_r]$	—

Spinors a_A, b_A, c_A and d_A are complex, spinors m_A, n_A, r_A and s_A are real; bar stands for the complex conjugation.

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Congruence (foliation) of the null strings

Definition

A congruence (foliation) of null strings is a family of totally null and totally geodesics 2-dimensional holomorphic surfaces, such that for every point $p \in \mathcal{M}$ there exists only one surface of this family such that p belongs to this surface.

If a congruence is self-dual (SD) then null strings are integral manifolds of a 2-dimensional SD distribution $\mathcal{D} = \{m_A a_{\dot{B}}, m_A b_{\dot{B}}\}$, $a_{\dot{A}} b^{\dot{A}} \neq 0$. $\mathcal{D}$ is integrable in the Frobenius sense, if

$$m^B \nabla_{A\dot{M}} m_B = m_A M_{\dot{M}}, \quad A, B = 1, 2, \ \dot{M} = \dot{1}, \dot{2}$$

Spinor field $M_{\dot{M}}$ is called *expansion of the congruence*

- $M_{\dot{M}} \neq 0$ – *expanding congruence* $\mathcal{C}^e$
- $M_{\dot{M}} = 0$ – *nonexpanding congruence* $\mathcal{C}^n$

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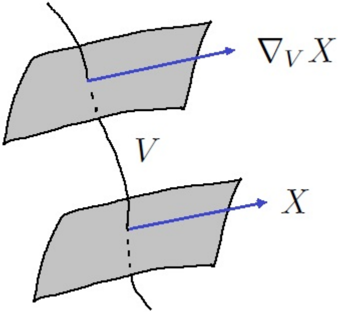
Congruence of the null strings

Nonexpanding congruence = distribution $\mathcal{D}$ is parallely propagated:

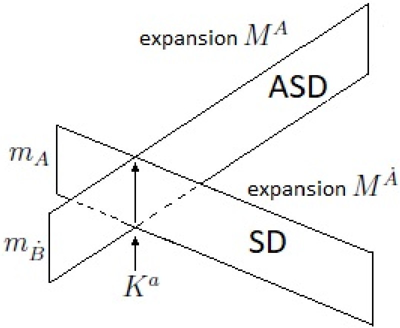
$$\nabla_V X \in \mathcal{D} \text{ for any vector field } V \text{ and any vector field } X \in \mathcal{D}$$



Intersection of SD and ASD congruences of the null strings

Consider two congruences of null strings

- a SD congruence generated by a spinor m_A with an expansion given by a spinor $M_{\dot{A}}$
- an ASD congruence generated by a spinor $m_{\dot{A}}$ with an expansion given by a spinor M_A



Intersection of SD and ASD congruences of the null strings

Intersection of these congruences constitutes a congruence of null geodesics. It is given by the vector field $K^a \sim m^A m^{\dot{B}}$. Define *complex expansion* θ and *complex twist* ϱ by the formulas

$$\theta := \frac{1}{2}\nabla^a K_a \quad \sim \quad m_A M^A + m_{\dot{A}} M^{\dot{A}}$$

$$\varrho^2 := \frac{1}{2}\nabla_{[a} K_{b]} \, \nabla^a K^b \quad \sim \quad m_A M^A - m_{\dot{A}} M^{\dot{A}}$$

- $[++]$: $\theta \neq 0, \varrho \neq 0$; expanding, twisting
- $[+-]$: $\theta \neq 0, \varrho = 0$; expanding, nontwisting
- $[-+]$: $\theta = 0, \varrho \neq 0$; nonexpanding, twisting
- $[--]$: $\theta = 0, \varrho = 0$; nonexpanding, nontwisting

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Symbol

In complex and real neutral geometries the following symbol is used

$$[\mathrm{SD}_{\mathrm{Weyl}}] \otimes [\mathrm{ASD}_{\mathrm{Weyl}}]$$

where

$$\begin{aligned} \mathrm{SD}_{\mathrm{Weyl}}, \mathrm{ASD}_{\mathrm{Weyl}} &= \{\mathrm{I}, \mathrm{II}, \mathrm{D}, \mathrm{III}, \mathrm{N}, \mathrm{O}\} \text{ in complex spaces} \\ \mathrm{SD}_{\mathrm{Weyl}}, \mathrm{ASD}_{\mathrm{Weyl}} &= \{\mathrm{I}_r, \mathrm{I}_{rc}, \mathrm{I}_c, \mathrm{II}_r, \mathrm{II}_{rc}, \mathrm{D}_r, \mathrm{D}_c, \mathrm{III}_r, \mathrm{N}_r, \mathrm{O}_r\} \\ & \quad \text{in neutral spaces} \end{aligned}$$

for example

$$[\mathrm{D}] \otimes [\mathrm{N}]$$

Symbol

An extension of this symbol reads

$$\{[\mathrm{SD}_{\mathrm{Weyl}}]^{i_1 i_2 \cdots} \otimes [\mathrm{ASD}_{\mathrm{Weyl}}]^{j_1 j_2 \cdots}, [k_{i_1 j_1}, k_{i_1 j_2}, ... k_{i_2 j_1}, k_{i_2 j_2}, ...]\}$$

where

$$i_1, i_2, ..., j_1, j_2, ... = \{n, e\}$$

n stands for nonexpanding congruence, e stands for expanding congruence and

$$k_{i_1 j_1}, k_{i_1 j_2}, ... k_{i_2 j_1}, k_{i_2 j_2} ... = \{++, +-, -+, --\}$$

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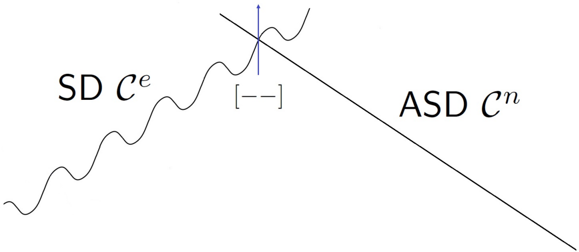
Example 1

$$\{[\cdot]^e \otimes [\cdot]^n, [--]\}$$



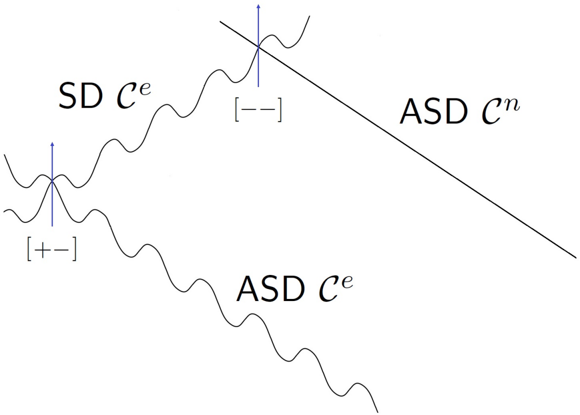
One SD, one ASD congruence

If there is one SD and one ASD congruence of null strings, there are 7 subtypes:

Type / Subtype	Conditions
Type $[\cdot]^n \otimes [\cdot]^n$	$M^{\dot{A}} = M^A = 0$
$[--]$	-
Type $[\cdot]^n \otimes [\cdot]^e$	$M^{\dot{A}} = 0,\ M^A \neq 0$
$[--]$	$m_A M^A = 0$
$[++]$	$m_A M^A \neq 0$
Type $[\cdot]^e \otimes [\cdot]^e$	$M^{\dot{A}} \neq 0,\ M^A \neq 0$
$[--]$	$m_A M^A = m_{\dot{A}} M^{\dot{A}} = 0$
$[+-]$	$m_A M^A = m_{\dot{A}} M^{\dot{A}} \neq 0$
$[-+]$	$m_A M^A = -m_{\dot{A}} M^{\dot{A}} \neq 0$
$[++]$	$m_A M^A \pm m_{\dot{A}} M^{\dot{A}} \neq 0$

Example 2

$$\{[\,\cdot\,]^e \otimes [\,\cdot\,]^{ne}, [--, +-]\}$$



One SD, two ASD congruences

If there is one SD and two ASD congruence of null strings, there are 24 subtypes:

Type / Subtype	Conditions
Type $[\cdot]^n \otimes [\cdot]^{nn}$	$M^{\dot{A}} = N^A = M^A = 0$
$[--,--]$	-
Type $[\cdot]^n \otimes [\cdot]^{ne}$	$M^{\dot{A}} = M^A = 0,\ N^A \neq 0$
$[--,--]$	$m_A N^A = 0$
$[--,++]$	$m_A N^A \neq 0$
Type $[\cdot]^n \otimes [\cdot]^{ee}$	$M^{\dot{A}} = 0,\ M^A \neq 0,\ N^A \neq 0$
$[--,--]$	$m_A M^A = m_A N^A = 0$
$[--,++]$	$m_A M^A = 0,\ m_A N^A \neq 0$
$[++,++]$	$m_A M^A \neq 0,\ m_A N^A \neq 0$
Type $[\cdot]^e \otimes [\cdot]^{nn}$	$M^{\dot{A}} \neq 0,\ M^A = N^A = 0$
$[--,++]$	$m_{\dot{A}} M^{\dot{A}} = 0,\ n_{\dot{A}} M^{\dot{A}} \neq 0$
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One SD, two ASD congruences

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Type $[\cdot]^e \otimes [\cdot]^{ne}$	$M^{\dot{A}} \neq 0,\ M^A = 0,\ N^A \neq 0$
$[--,+-]$	$m_{\dot{A}} M^{\dot{A}} = 0,\ m_A N^A = n_{\dot{A}} M^{\dot{A}} \neq 0$
$[--,-+]$	$m_{\dot{A}} M^{\dot{A}} = 0,\ m_A N^A = -n_{\dot{A}} M^{\dot{A}} \neq 0$
$[--,++]$	$m_{\dot{A}} M^{\dot{A}} = 0,\ m_A N^A \pm n_{\dot{A}} M^{\dot{A}} \neq 0$
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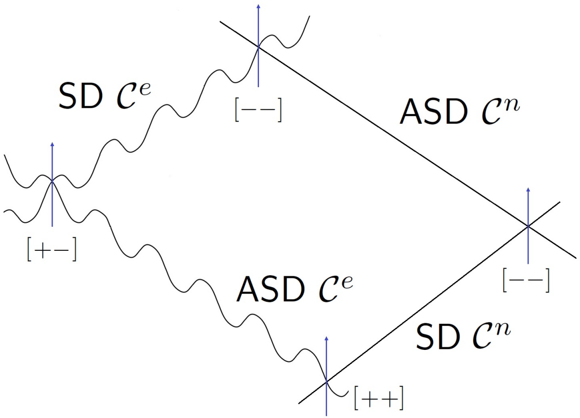
$[++,++]$	$m_{\dot{A}} M^{\dot{A}} \neq 0,\ m_A N^A \pm n_{\dot{A}} M^{\dot{A}} \neq 0$
Type $[\cdot]^e \otimes [\cdot]^{ee}$	$M^{\dot{A}} \neq 0,\ M^A \neq 0,\ N^A \neq 0$
$[--,+-]$	$m_A M^A = m_{\dot{A}} M^{\dot{A}} = 0,\ m_A N^A = n_{\dot{A}} M^{\dot{A}} \neq 0$
$[--,-+]$	$m_A M^A = m_{\dot{A}} M^{\dot{A}} = 0,\ m_A N^A = -n_{\dot{A}} M^{\dot{A}} \neq 0$
$[--,++]$	$m_A M^A = m_{\dot{A}} M^{\dot{A}} = 0,\ m_A N^A \pm n_{\dot{A}} M^{\dot{A}} \neq 0$
$[+-,+-]$	$m_A M^A = m_{\dot{A}} M^{\dot{A}} \neq 0,\ m_A N^A = n_{\dot{A}} M^{\dot{A}} \neq 0$
$[+-,-+]$	$m_A M^A = m_{\dot{A}} M^{\dot{A}} \neq 0,\ m_A N^A = -n_{\dot{A}} M^{\dot{A}} \neq 0$
$[-+,-+]$	$m_A M^A = -m_{\dot{A}} M^{\dot{A}} \neq 0,\ m_A N^A = -n_{\dot{A}} M^{\dot{A}} \neq 0$
$[++,+-]$	$m_A M^A \pm m_{\dot{A}} M^{\dot{A}} \neq 0,\ m_A N^A = n_{\dot{A}} M^{\dot{A}} \neq 0$
$[++,-+]$	$m_A M^A \pm m_{\dot{A}} M^{\dot{A}} \neq 0,\ m_A N^A = -n_{\dot{A}} M^{\dot{A}} \neq 0$
$[++,++]$	$m_A M^A \pm m_{\dot{A}} M^{\dot{A}} \neq 0,\ m_A N^A \pm n_{\dot{A}} M^{\dot{A}} \neq 0$

Example 3

$$\{[\cdot]^{en} \otimes [\cdot]^{ne}, [--, +-, --, ++]\}$$



Two SD, two ASD congruences

If there are two SD and two ASD congruences of null strings, there are 89 subtypes.

Part 2: From weak $\mathcal{HH}$-spaces to algebraically degenerate para-Kähler spaces

Para-Kähler spaces (types $[\text{any}] \otimes [\text{D}]^{nn}$)

Definition

Para-Kähler space (pK space) is a pair $(\mathcal{M}, ds^2)$ where $\mathcal{M}$ is a 4-dimensional real differential manifold and ds^2 is a smooth metric of the signature $(++--)$ which satisfies the following condition:

- *there exist two different (complementary) nonexpanding congruences of null strings*

The metric of para-Kähler spaces

The metric of any para-Kähler space can be brought to the form

$$\frac{1}{2}ds^2 = M_{qx}dqdx + M_{qy}dqdy + M_{px}dpdx + M_{py}dpdy$$

where $M = M(q,p,x,y)$.

For para-Kähler Einstein spaces function M satisfies

$$M_{qx}M_{py} - M_{qy}M_{px} = e^{-\Lambda M}$$

Weak $\mathcal{HH}$-spaces (types $[\mathrm{deg}]^e \otimes [\mathrm{any}]$ or $[\mathrm{deg}]^n \otimes [\mathrm{any}]$)

Definition

Weak hyperheavenly space (weak $\mathcal{HH}$-space) is a pair $(\mathcal{M}, ds^2)$ where $\mathcal{M}$ is a 4-dimensional complex analytic differential manifold and ds^2 is a holomorphic metric which satisfies the following conditions:

- *there exists a congruence of SD null strings generated by a spinor m_A*

$$m^A m^B \nabla_{A\dot{M}} m_B = 0$$

- *the self-dual Weyl spinor C_{ABCD} is algebraically degenerate and m_A is a multiple Penrose spinor i.e.*

$$C_{ABCD}\, m^A m^B m^C = 0$$

The metric of weak $\mathcal{HH}$-spaces

The metric of a weak $\mathcal{HH}$-space can be brought to the form

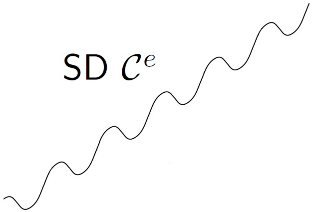
$$\frac{1}{2}ds^2 = \phi^{-2}\left(dqdy - dpdx + \mathcal{A}\,dp^2 - 2\mathcal{Q}\,dpdq + \mathcal{B}\,dq^2\right)$$

where (q,p,x,y) are local coordinates; $\mathcal{A} = \mathcal{A}(q,p,x,y)$, $\mathcal{Q} = \mathcal{Q}(q,p,x,y)$ and $\mathcal{B} = \mathcal{B}(q,p,x,y)$ are arbitrary holomorphic functions and

$$\begin{array}{rl} \text{for weak expanding } \mathcal{HH}\text{-space} & \phi = \phi(q,p,x,y),\ |\phi_x| + |\phi_y| \neq 0 \\ \text{for weak nonexpanding } \mathcal{HH}\text{-space} & \phi = 1 \end{array}$$

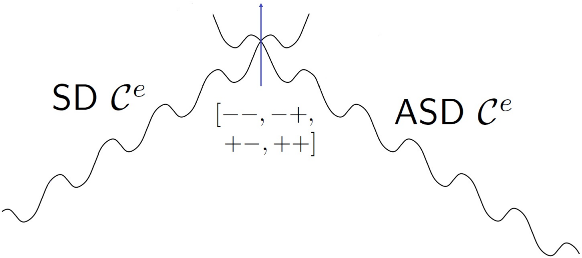
Schemes

$[\text{deg}]^e \otimes [\text{any}]$ (weak expanding $\mathcal{HH}$-space)



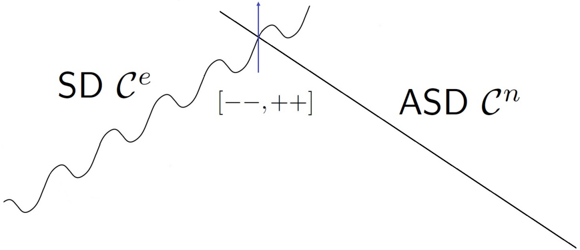
Schemes

$$[\text{deg}]^e \otimes [\text{any}] \longrightarrow [\text{deg}]^e \otimes [\text{any}]^e$$



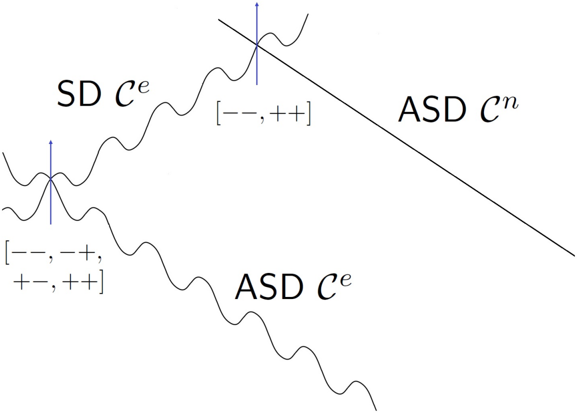
Schemes

$$[\text{deg}]^e \otimes [\text{any}]^e \longrightarrow [\text{deg}]^e \otimes [\text{deg}]^n$$



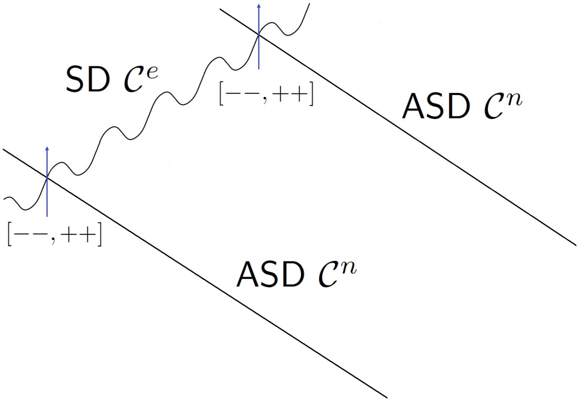
Schemes

$$[\deg]^e \otimes [\deg]^n \longrightarrow [\deg]^e \otimes [\deg]^{ne}$$



Schemes

$[\deg]^e \otimes [\deg]^{ne} \longrightarrow [\deg]^e \otimes [\mathrm{D}]^{nn}$ (para-Kähler spaces)



Schemes

Nonexpanding case:

$$[\mathrm{deg}]^n \otimes [\mathrm{any}]$$

$\downarrow$ $\longleftarrow$ let $\mathcal{C}^e_{m\dot{A}}$ exists $\longrightarrow$

$$\{[\mathrm{deg}]^n \otimes [\mathrm{any}]^e, [--]\}$$
$$\{[\mathrm{deg}]^n \otimes [\mathrm{any}]^e, [++]\}$$

$\downarrow$ $\longleftarrow$ let $\mathcal{C}_{m\dot{A}} = \mathcal{C}^n$ $\longrightarrow$

$$\{[\mathrm{deg}]^n \otimes [\mathrm{deg}]^n, [--]\}$$

Expanding case:

$$[\mathrm{deg}]^e \otimes [\mathrm{any}]$$

$\downarrow$

$$\{[\mathrm{deg}]^e \otimes [\mathrm{any}]^e, [--]\}$$
$$\{[\mathrm{deg}]^e \otimes [\mathrm{any}]^e, [+-]\}$$
$$\{[\mathrm{deg}]^e \otimes [\mathrm{any}]^e, [-+]\}$$
$$\{[\mathrm{deg}]^e \otimes [\mathrm{any}]^e, [++]\}$$

$\downarrow$

$$\{[\mathrm{deg}]^e \otimes [\mathrm{deg}]^n, [--]$$
$$\{[\mathrm{deg}]^e \otimes [\mathrm{deg}]^n, [++]$$

Schemes

Nonexpanding case:

$$\{[\text{deg}]^n \otimes [\text{deg}]^{ne}, [--,--]\}$$
$$\{[\text{deg}]^n \otimes [\text{deg}]^{ne}, [--,++]\}$$

Expanding case:

$$\{[\text{deg}]^e \otimes [\text{deg}]^{ne}, [--,+-]\}$$
$$\{[\text{deg}]^e \otimes [\text{deg}]^{ne}, [--,-+]\}$$
$$\{[\text{deg}]^e \otimes [\text{deg}]^{ne}, [--,++]\}$$
$$\{[\text{deg}]^e \otimes [\text{deg}]^{ne}, [++,--]\}$$
$$\{[\text{deg}]^e \otimes [\text{deg}]^{ne}, [++,+-]\}$$
$$\{[\text{deg}]^e \otimes [\text{deg}]^{ne}, [++,++]\}$$

$$\longleftarrow \text{let } \mathcal{C}_{n^{\dot{A}}} = \mathcal{C}^n \longrightarrow$$

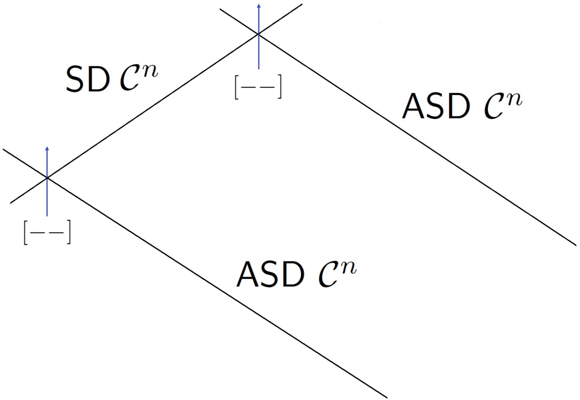
$$\{[\text{deg}]^n \otimes [\text{D}]^{nn}, [--,--]\}$$

$$\{[\text{deg}]^e \otimes [\text{D}]^{nn}, [--,++]\}$$
$$\{[\text{deg}]^e \otimes [\text{D}]^{nn}, [++,++]\}$$

Part 3: Para-Kähler Einstein spaces

Para-Kähler Einstein spaces: Class 1

$\{[\deg]^n \otimes [\mathrm{D}]^{nn}, [--, --]\}$; SD Weyl $= \{\mathrm{II}, \mathrm{D}\}$



Para-Kähler Einstein spaces: Class 1

The metric of an Einstein space of the type $[\mathrm{II}]^n \otimes [\mathrm{D}]^{nn}$ or $[\mathrm{D}]^{nn} \otimes [\mathrm{D}]^{nn}$ can be brought to the form

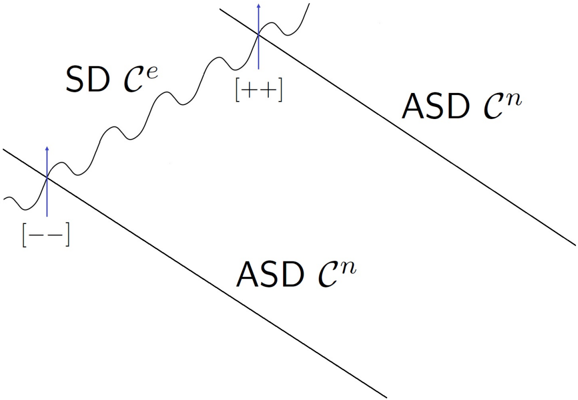
$$\frac{1}{2}ds^2 = dqdy - dpdx + \left(\frac{\Lambda}{2}x^2 + \Omega\right) dp^2 + \left(\frac{\Lambda}{2}y^2 + \Sigma\right) dq^2$$

where (q, p, x, y) are local coordinates; $\Omega = \Omega(q, p)$ and $\Sigma = \Sigma(q, p)$ are functions such that

$$\text{for the type } [\mathrm{II}]^n \otimes [\mathrm{D}]^{nn} : |\Sigma_p| + |\Omega_q| \neq 0;$$
$$\text{for the type } [\mathrm{D}]^{nn} \otimes [\mathrm{D}]^{nn} : \Sigma = \Omega = 0.$$

Para-Kähler Einstein spaces: Class 2

$\{[\mathrm{deg}]^e \otimes [\mathrm{D}]^{nn}, [++, --]\}$; SD Weyl $= \{\mathrm{II}, \mathrm{D}, \mathrm{III}, \mathrm{N}\}$



Para-Kähler Einstein spaces: Class 2

The metric of an Einstein space of the type $\{[\text{deg}]^e \otimes [\text{D}]^{nn}, [++,--]\}$ can be brought to the form

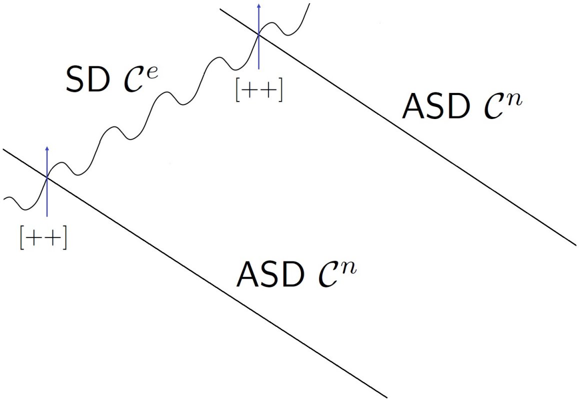
$$\begin{aligned}\frac{1}{2}ds^2 \quad = \quad & x^{-2}\Big\{dqdy - dpdx + \left(Ax^3 + \frac{\Lambda}{3}\right)dp^2 \\ & -2\left(Ayx^2 + \frac{\Lambda}{3}\frac{y}{x} + \frac{M_p}{2\Lambda}x^2\right)dqdp \\ & +\left(Ay^2x + \frac{\Lambda}{3}\frac{y^2}{x^2} + \frac{M_p}{\Lambda}xy + Nx - My\right)dq^2\Big\}\end{aligned}$$

where $\Lambda \neq 0$ is a cosmological constant, $A = A(q,p)$, $M = M(q,p)$ and $N = N(q,p)$ are arbitrary holomorphic (real smooth) functions.

$$\begin{aligned}\{[\text{II}]^e \otimes [\text{D}]^{nn}, [++,--]\} \quad &: \quad A \neq 0, 2C^{(2)}C^{(2)} - 3C^{(3)}C^{(1)} \neq 0 \\ \{[\text{III}]^e \otimes [\text{D}]^{nn}, [++,--]\} \quad &: \quad A = 0, M_{pp} \neq 0 \\ \{[\text{N}]^e \otimes [\text{D}]^{nn}, [++,--]\} \quad &: \quad A = 0, M = M_0 = \{-1,0,1\}, \\ & \quad\ 3N_{pp} - 2M_0^2 p \neq 0\end{aligned}$$

Para-Kähler Einstein spaces: Class 3

$\{[\text{deg}]^e \otimes [\text{D}]^{nn}, [++, ++]\}$; SD Weyl $= \{\text{II}, \text{D}, \text{III}, \text{N}\}$



Para-Kähler Einstein spaces: Class 3

The metric of an Einstein space of the type $\{[\mathrm{deg}]^e \otimes [\mathrm{D}]^{nn}, [++,++]\}$ can be brought to the form

$$\frac{1}{2}ds^2 = x^{-2}\Big\{dqdy - dpdx + \Big(Ax^3 - Cx^2 - Bx(1-2y) + \frac{\Lambda}{3}(1-3y+3y^2)\Big)\,dp^2$$
$$-2\left(Ax^2y - By(1-y) - Mx^2 + \frac{\Lambda}{3}\frac{y(1-y)(1-2y)}{x}\right)dqdp$$
$$+\left(Axy^2 - Cy(1-y) + Mx(1-2y) + \frac{\Lambda}{3}\frac{y^2(1-y)^2}{x^2} + \frac{2B_q + C_p}{2\Lambda}x\right)dq^2\Big\}$$

where $\Lambda \neq 0$ is a cosmological constant, $A = A(q,p)$, $B = B(q,p)$, $C = C(q,p)$ and $M = M(q,p)$ are arbitrary holomorphic (real smooth) functions.

$$\{[\mathrm{II}]^e \otimes [\mathrm{D}]^{nn}, [++,++]\} \quad : \quad A \neq 0, 2C^{(2)}C^{(2)} - 3C^{(3)}C^{(1)} \neq 0$$
$$\{[\mathrm{III}]^e \otimes [\mathrm{D}]^{nn}, [++,++]\} \quad : \quad A = 0, 2M_p + C_q + \frac{B}{\Lambda}(2B_q + C_p) \neq 0$$
$$\{[\mathrm{N}]^e \otimes [\mathrm{D}]^{nn}, [++,++]\} \quad : \quad A = 0, 2M_p + C_q + \frac{B}{\Lambda}(2B_q + C_p) = 0$$

Para-Kähler Einstein spaces: Class 3

The metric of an Einstein space of the type $\{[\mathrm{N}]^e \otimes [\mathrm{D}]^{nn}, [++,++]\}$ can be brought to the form

$$\begin{aligned}
\frac{1}{2}ds^2 \quad = \quad & x^{-2}\Big\{dqdy - (P_t dt + P_q dq)dx \\
& + \left(-\left(t - \frac{B^2}{\Lambda}\right)x^2 - Bx(1-2y) + \frac{\Lambda}{3}(1-3y+3y^2)\right)(P_t dt + P_q dq)^2 \\
& -2\left(-By(1-y) - \left(\frac{N}{2} + \frac{B^3}{3\Lambda^2}\right)x^2 + \frac{\Lambda}{3}\frac{y(1-y)(1-2y)}{x}\right)dq(P_t dt + P_q dq) \\
& + \Bigg[-\left(t - \frac{B^2}{\Lambda}\right)y(1-y) + \left(\frac{N}{2} + \frac{B^3}{3\Lambda^2}\right)x(1-2y) + \frac{\Lambda}{3}\frac{y^2(1-y)^2}{x^2} \\
& + \frac{1}{2\Lambda}\left(2B_q + \frac{1}{P_t} - \frac{2B_t}{P_t}(2P_q - N_t)\right)x\Bigg]dq^2\Big\}
\end{aligned}$$

where $\Lambda \neq 0$ is a cosmological constant, $B := \Lambda(P_q - N_t)$; $N = N(q,t)$ and $P = P(q,t)$ are arbitrary holomorphic (real smooth) functions such that $P_t \neq 0$ and

$$\begin{aligned}
& -\frac{1}{2\Lambda}\frac{1}{P_t}\frac{\partial}{\partial t}\left(\frac{1}{P_t}\frac{\partial}{\partial t}\left(2B_q - 2\frac{P_q}{P_t}B_t + \frac{1}{P_t} - \frac{2}{\Lambda}\frac{1}{P_t}BB_t\right)\right) - \frac{1}{2}\frac{1}{P_t}\partial_t(NB) \\
& + \frac{\Lambda}{6}\left(N_q - \frac{P_q}{P_t}N_t\right) - \frac{4t}{3\Lambda}\frac{1}{P_t}BB_t + \frac{2t}{3}\frac{1}{P_t} + \frac{t}{3}\left(B_q - \frac{P_q}{P_t}B_t\right) - \frac{2}{3\Lambda}\frac{1}{P_t}B^2 \neq 0
\end{aligned}$$

Para-Kähler Einstein spaces: Class 4

Type $[\mathrm{I}] \otimes [\mathrm{D}]^{nn}$. No examples.

Para-Kähler Einstein spaces: Summary

- Algebraically special pKE-spaces

$$\left.\begin{array}{l}\text{Class 1: } \{[\text{deg}]^n \otimes [\text{D}]^{nn}, [--,--]\} \\ \text{Class 2: } \{[\text{deg}]^e \otimes [\text{D}]^{nn}, [++,--]\} \\ \text{Class 3: } \{[\text{deg}]^e \otimes [\text{D}]^{nn}, [++,++]\}\end{array}\right\} \text{all known in all the generality}$$

- Algebraically general pKE-spaces

$$\text{Class 4: } [\text{I}] \otimes [\text{D}]^{nn} \quad \text{no examples are known}$$

Para-Kähler Einstein spaces: Summary

- Algebraically special pKE-spaces

$$\left.\begin{array}{l} \text{Class 1: } \{[\text{deg}]^n \otimes [\text{D}]^{nn}, [--,--]\} \\ \text{Class 2: } \{[\text{deg}]^e \otimes [\text{D}]^{nn}, [++,--]\} \\ \text{Class 3: } \{[\text{deg}]^e \otimes [\text{D}]^{nn}, [++,++]\} \end{array}\right\} \text{all known in all the generality}$$

- Algebraically general pKE-spaces

$$\text{Class 4: } [\text{I}] \otimes [\text{D}]^{nn} \quad \text{no examples are known}$$

Para-Kähler Einstein spaces: Summary

- Algebraically special pKE-spaces

$$\left.\begin{array}{l} \text{Class 1: } \{[\text{deg}]^n \otimes [\text{D}]^{nn}, [--,--]\} \\ \text{Class 2: } \{[\text{deg}]^e \otimes [\text{D}]^{nn}, [++,--]\} \\ \text{Class 3: } \{[\text{deg}]^e \otimes [\text{D}]^{nn}, [++,++]\} \end{array}\right\} \text{all known in all the generality}$$

- Algebraically general pKE-spaces

$$\text{Class 4: } [\text{I}] \otimes [\text{D}]^{nn} \quad \text{no examples are known}$$