Conformal geodesics can not spiral

Maciej Dunajski

Clare College
and
Department of Applied Mathematics and Theoretical Physics
University of Cambridge.

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- Proof: Exponential map/convex geodesics neighbourhoods.

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- Conformal geodesics (Yano 1950s, Penrose 1960s, ...): preferred curves of conformal geometry. Solutions to a system of 3rd order ODEs - specify position, unit velocity u and acceleration $a = \nabla_u u$

$$\nabla_u a = -(|a|^2 + g(u, L(u))u + L(u)$$

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- Conformal invariance with $\Upsilon \equiv \Omega^{-1} d\Omega$

$$\hat{g} = \Omega^2 g, \quad \hat{L}_{ij} = L_{ij} - \nabla_i \Upsilon_j + \Upsilon_i \Upsilon_j + |\Upsilon|^2 g_{ij}$$

$$\hat{u}_i = \Omega u_i, \quad \hat{a}_i = a_i - \Upsilon_i + u^j \Upsilon_j u_i.$$

Example: Conformal Circles

- Assume (M, g) is Einstein: $R_{ij} = (1/n)Rg_{ij}$.

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  3. Paths of a charged particle moving in a constant magnetic field.

Properties of conformal geodesics

- Integrability
  1. Conformal Killing–Yano tensor (CKY): $Y \in \Lambda^2(M)$ such that
  $$\nabla_i Y_{jk} = \nabla_{[i} Y_{jk]} - 2 g_{i[j} K_{k]} \quad \text{for some} \quad K \in \Lambda^1(M)$$
  2. Tod 2012, Gover, Snell, Taghavi-Chabert 2018 : $\mathcal{Y} = Y_{ij} u^i a^j - K_i u^i$ is a first integral.
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- Can conformal geodesics spiral? (convex neighbourhood arguments break down - no local length minimizers).
- Importance: good coordinate systems for local studies of conformal boundaries in Penrose's conformal copactification (Friedrich–Schmidt): time–like geodesics do not pass through null infinity. Spiraling - a new kind of coordinate singularity.

Main Theorem

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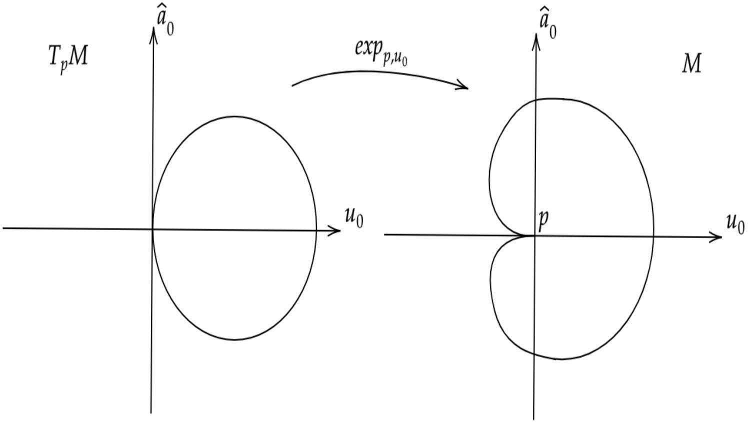
$\hat{a}_0$
T_pM
exp_{p,u_0}
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M
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p
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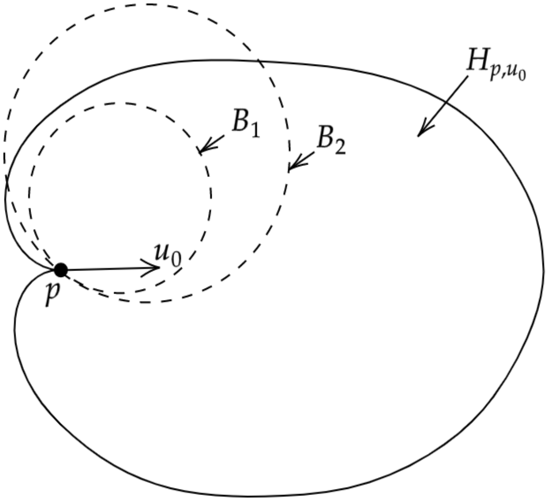
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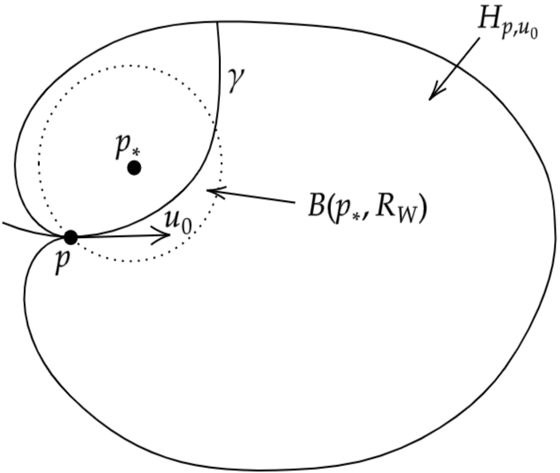
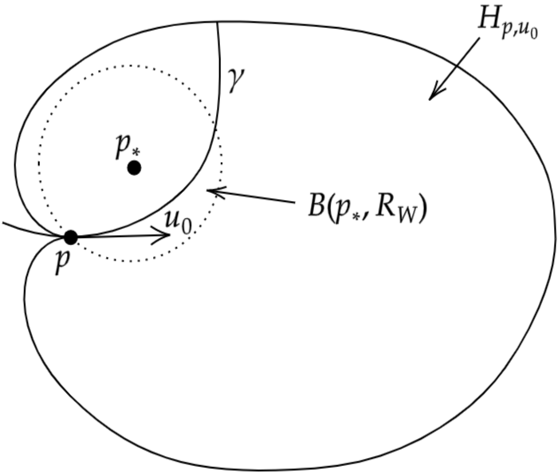
  3. Show that all geodesics with unit tangent u_0 at p_0 intersect the boundary of H_{p_0,u_0}. Choose a neighbourhood entered by γ for the last time, and show that it must lie inside a heart. Reach a contradiction.


H_{p,u_0}
γ
p_*
$B(p_*, R_W)$
u_0
p



Thank You