The general relativistic two-body problem at 5PN

Gerhard Schäfer

Theoretisch-Physikalisches Institut

Friedrich-Schiller-Universität Jena

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Overview

- up to 4PN: from 1938 (1PN final) – 2014 (4PN first)
- the order 5PN: the years 2019 – 2022
- beyond 5PN: 5.5PN and 6PN: 2020 – 2022

$$n\mathrm{PN}: \quad \left(\tfrac{1}{c^2}\right)^n \sim \left(\tfrac{GM}{rc^2}\right)^l \left(\frac{\mathrm{v}^2}{c^2}\right)^m, \quad l+m=n$$

$$n\mathrm{PM}: \quad G^n \sim \left(\tfrac{GM}{rc^2}\right)^n$$

$$[c^2(g_{00}+1), \quad cg_{0i}, \quad g_{ij}-\delta_{ij}] = c^0 + c^{-2} + \ldots,$$

$$[g_{00}+1, \qquad g_{0i}, \quad g_{ij}-\delta_{ij}] = G + G^2 + \ldots$$

Applied methods for conservative dynamics

- perturbation series: PN, PM/MPM, GSF
- bound-state and scattering-state calculations
- point particles (BHs) [extended bodies through 2PN]
- dimensional regularization [analytic reg. through 2PN]
- ADM-canonical formalism through 4PN
- EOB tool through 6PN
- Fokker-action formalism through 4PN
- EFT formalism through 6PN

REVIEWS

L. Blanchet, Liv. Rev. Rel. **17**, 2 (2014) [arXiv:1310.1528] (Fokker action, harmonic gauge, multipolar expansion)

G. Schäfer & P. Jaranowski, Liv. Rev. Rel. **21**, no.1 7 (2018) [arXiv:1805.07240] (ADM canonical formalism, ADM gauge)

R.A. Porto, Phys. Rept. **633**, 1-104 (2016) [arXiv:1601.04914] (EFT, grav. dyn.)

M. Levi, Rept. Prog. Phys. **83**, 075901 (2020) [arXiv:1807.01699] (EFT, PN)

A. Buonanno et al., Snowmass White Paper, [arXiv:2204.05195] (gravitational waves, scattering amplitudes, double copy, EFT)

W.D. Goldberger, Snowmass White Paper, [arXiv:2206.14249] (EFT, comp. bin.)

harmonic gauge: $\sqrt{-g}g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \qquad \partial_\nu h^{\mu\nu} = 0$

ADM metric variables (N, N^i, γ_{ij}) and gauge conditions:

$$N = (-g^{00})^{-1/2}, \quad N^i = \gamma^{ij} N_j, \quad N_i = g_{0i}, \quad \gamma_{ij} = g_{ij}$$

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(N dx^0)^2 + \gamma_{ij}(dx^i + N^i dx^0)(dx^j + N^j dx^0)$$

$$\sqrt{-g} = N\sqrt{\gamma}, \quad \pi^{ij} = -\sqrt{\gamma}(\gamma^{ik}\gamma^{jl} - \gamma^{ij}\gamma^{kl})K_{kl}, \quad K_{ij} = -N\Gamma^0{}_{ij}$$

$$\boxed{\gamma_{ij} = \psi\delta_{ij} + h^{TT}_{ij}} \qquad \partial_j h^{TT}_{ij} = \partial_i h^{TT}_{ij} = 0, \quad h^{TT}_{ii} = 0$$

$$\pi^{ij} = \tilde{\pi}^{ij} + \pi^{ijTT}, \quad \tilde{\pi}^{ij} = \partial_i V^j + \partial_j V^i - \tfrac{2}{3}\delta_{ij}\partial_k V^k, \quad \tilde{\pi}^{ii} = 0$$

$$A^{TT}_{ij} = P_{ijkl}A_{kl}, \quad P_{ijkl} = P_{ik}P_{jl} - \tfrac{1}{2}P_{ij}P_{kl}, \quad P_{ik} = \delta_{ik} - \frac{\partial_i\partial_k}{\Delta}$$

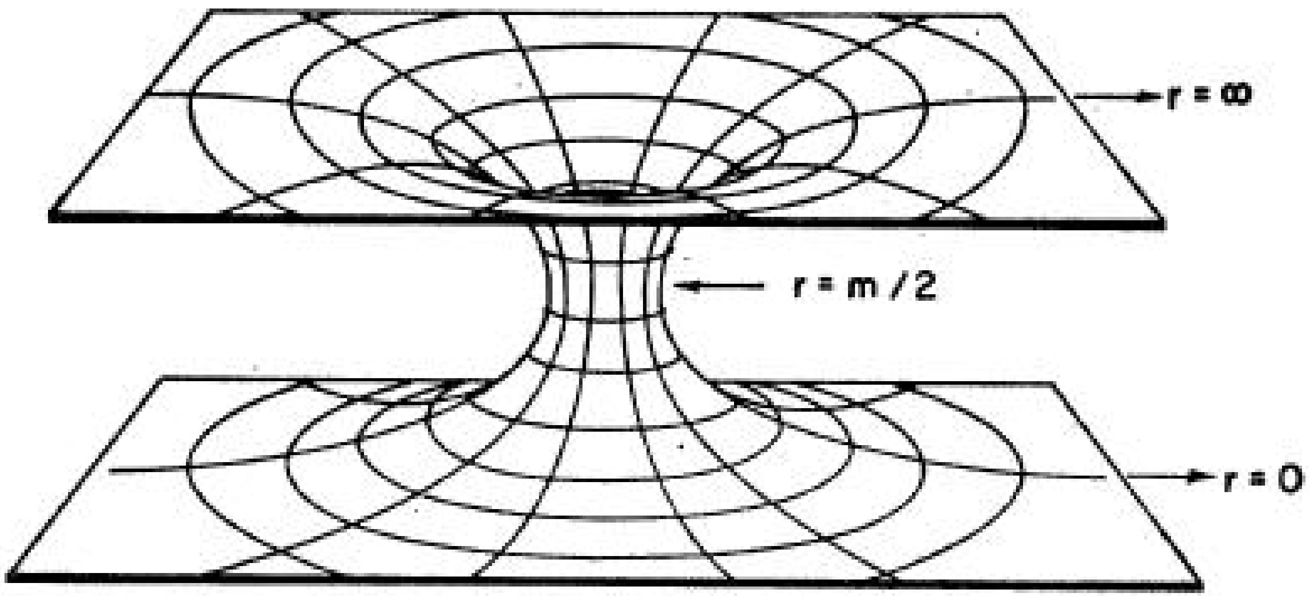
Binary-Black-Hole(BBH) Spacetime

Isolated BH

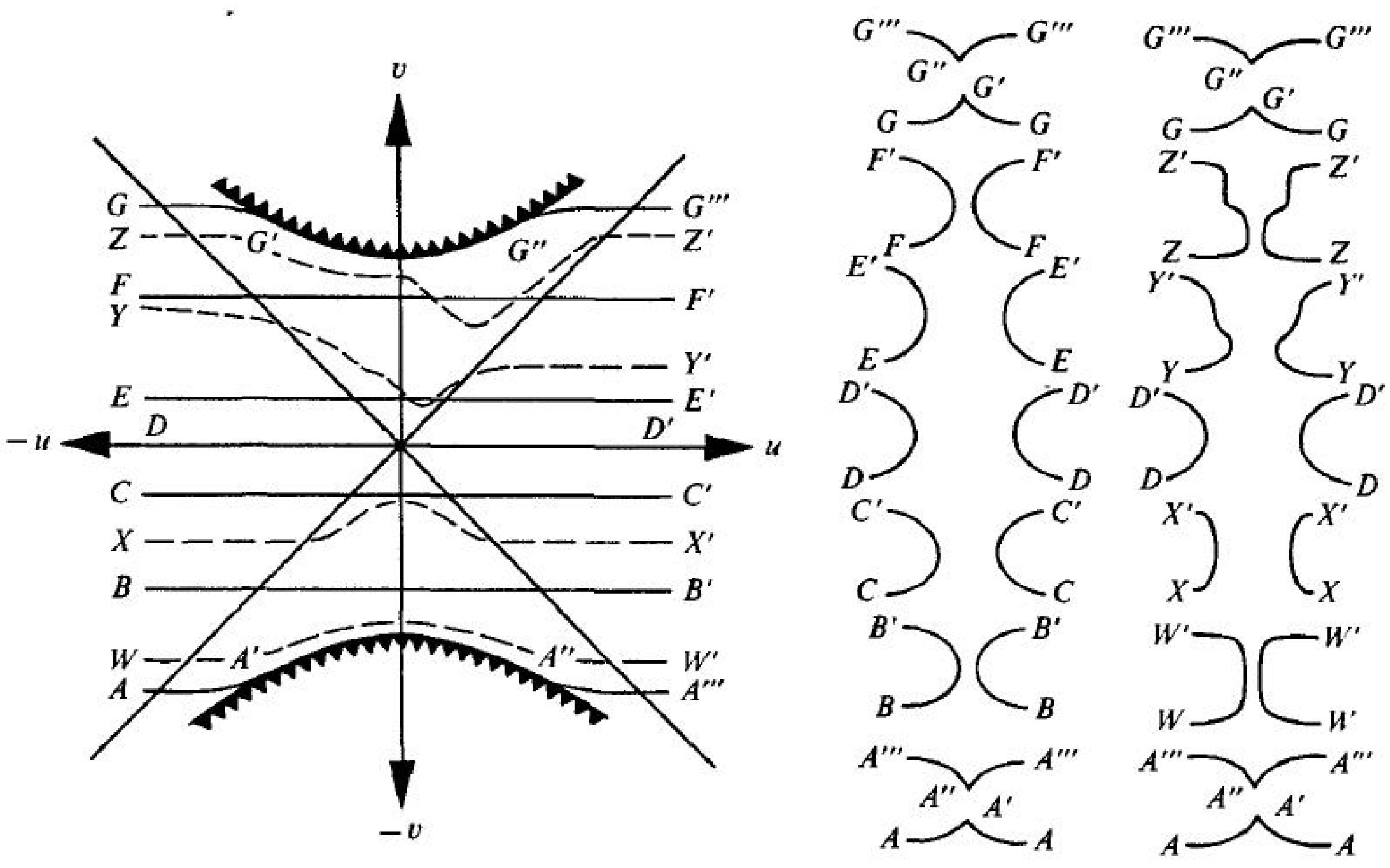
$$
\begin{aligned}
ds^2 &= -\left(\frac{1-\frac{Gm}{2rc^2}}{1+\frac{Gm}{2rc^2}}\right)^2 c^2dt^2 + \left(1+\frac{Gm}{2rc^2}\right)^4 \delta_{ij}dx^i dx^j \\
&= -\left(\frac{1-\frac{Gm}{2r'c^2}}{1+\frac{Gm}{2r'c^2}}\right)^2 c^2dt^2 + \left(1+\frac{Gm}{2r'c^2}\right)^4 \delta_{ij}dx'^i dx'^j
\end{aligned}
$$

symmetry transformation (inversion): $r'r = \left(\frac{Gm}{2c^2}\right)^2$

$r'^2 = x'^i x'^i, \quad r^2 = x^i x^i$



Brill/Lindquist, JMP 1963

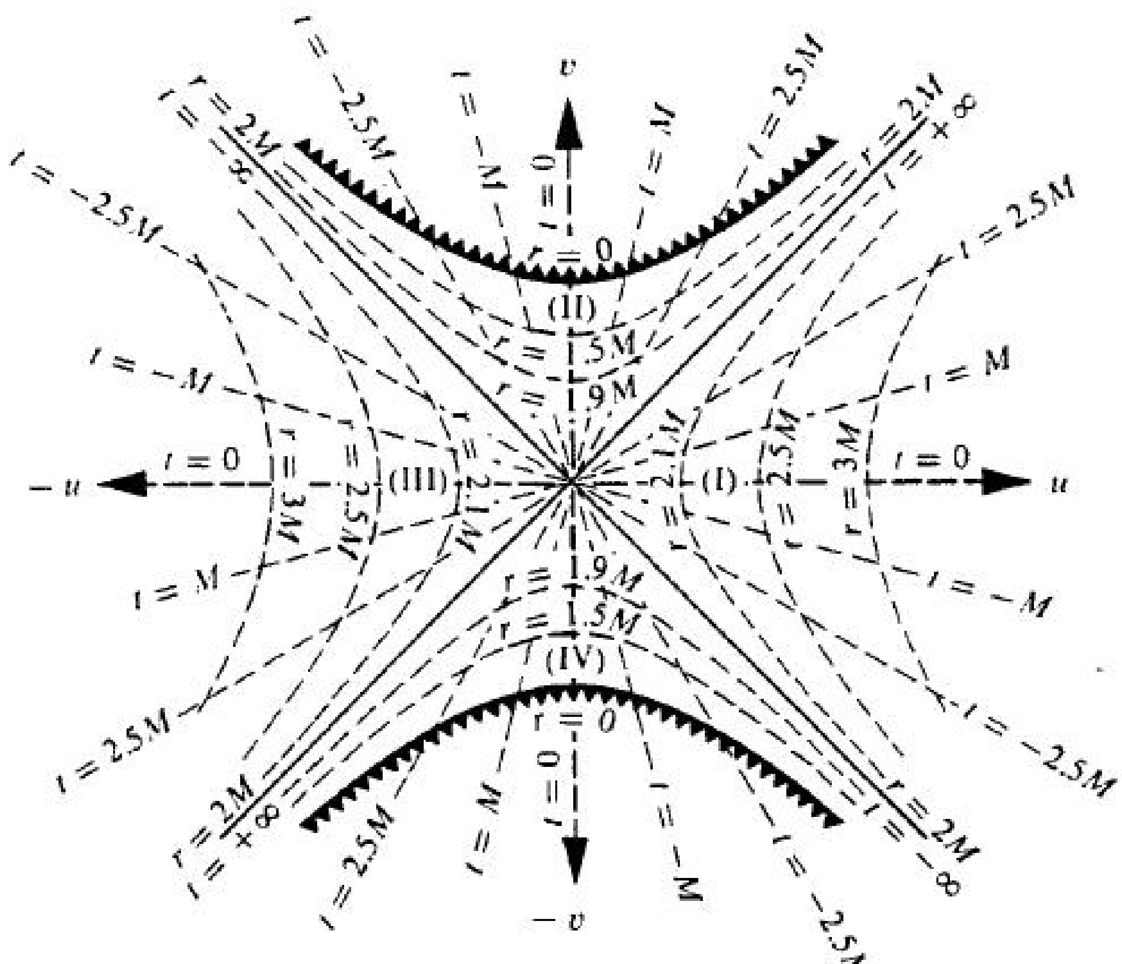


Spacelike slices

History
A-B-C-D-E-F-G

History
A-W-X-D-Y-Z-G

Misner/Thorne/Wheeler Gravitation 1973



Kruskal-Szekeres

MTW Gravitation 1973

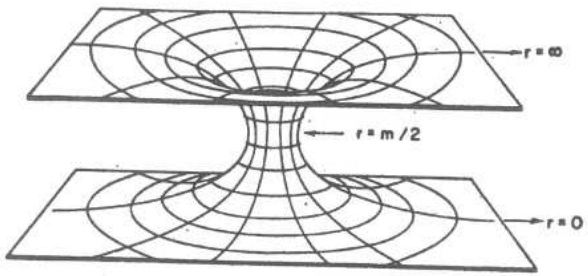


FIG. 1. A two-dimensional analog of the Schwarzschild-Kruskal manifold is shown isometrically imbedded in flat three-space. The figure shows the curvature and topology of the metric

$$ds^2 = (1 + m/2r)^4 (dr^2 + r^2 d\theta^2).$$

The sheets at the top and bottom of the funnel continue to infinity and represent the asymptotically flat regions of the manifold $(r \to 0,\ r \to \infty)$.

BBH: Brill/Lindquist, JMP 1963

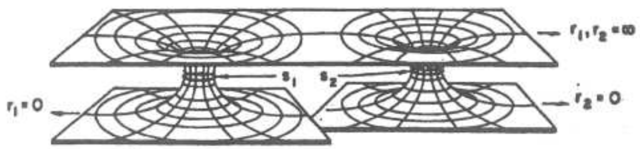


FIG. 2. A two-dimensional analog of the hypersurface of time symmetry of a manifold containing two "throats" is shown isometrically imbedded in flat three-space. The figure illustrates the curvature and topology for a system of two "particles" of equal mass m, and separation large compared to m, described by the metric

$$ds^2 = (1 + m/2r_1 + m/2r_2)^4 ds_F^2.$$

Surface integrals of the shown BBH vacuum solution result in

its proper masses m_1 and m_2 as well as

its total (proper masses plus binding) energy $H_{\rm BL}$.

Point particles as sources of our complete 3-manifold

The formal positions of m_a are located in euclidean coordinate space, obtained via conformal transformation of our 3-metric, with densities

$$m_a \delta_a^{(3)} = m_a \delta^{(3)}(\mathbf{x} - \mathbf{x}_a), \qquad \int m_a \delta_a^{(3)}\ d^3x = m_a.$$

Brill-Lindquist BBH metric: an initial-value metric

$$ds^2 = -\left(\frac{1-\frac{1}{8}\chi}{1+\frac{1}{8}\phi}\right)^2 c^2 dt^2 + \left(1+\frac{1}{8}\phi\right)^4 d\mathbf{x}^2$$

Point Particles in Hamilton Constraint ($h_{ij}^{\mathrm{TT}} = 0,\ p_{ai} = 0$):

$$-\left(1+\frac{1}{8}\,\phi\right)\Delta\phi = \frac{16\pi G}{c^2}\,\left(m_1\delta_1^{(3)} + m_2\delta_2^{(3)}\right) \qquad ???$$

$$\phi = \frac{4G}{c^2}\left(\frac{\alpha_1}{r_1} + \frac{\alpha_2}{r_2}\right)$$

$$\alpha_a = m_a - \frac{m_a + m_b}{2} + \frac{c^2 r_{ab}}{G}\left(\sqrt{1 + \frac{m_a + m_b}{c^2 r_{ab}/G} + \left(\frac{m_a - m_b}{2c^2 r_{ab}/G}\right)^2} - 1\right)$$

$$\boxed{H_{\mathrm{BL}} = (\alpha_1 + \alpha_2)\ c^2 = (m_1 + m_2)\ c^2 - G\ \frac{\alpha_1\alpha_2}{r_{12}}}$$

Our 3-metric in d-dimensional space:

$$\gamma_{ij} = \left(1 + \frac{d-2}{4(d-1)}\phi\right)^{\frac{4}{d-2}} \delta_{ij} dx^i dx^j$$

$$\phi = \frac{4\pi G^{(d)}}{c^2} \frac{\Gamma(\frac{d-2}{2})}{\pi^{\frac{d}{2}}} \left(\frac{\alpha_1}{r_1^{d-2}} + \frac{\alpha_2}{r_2^{d-2}}\right), \qquad -\Delta^{-1}\delta^{(d)} = \frac{\Gamma((d-2)/2)}{4\pi^{d/2}} \frac{1}{r^{d-2}}$$

HamCon: $$-\left(1 + \frac{d-2}{4(d-1)}\ \phi\right)\Delta\phi = \frac{16\pi G^{(d)}}{c^2} \sum_a m_a \delta_a^{(d)}$$

$$\left(1 + \frac{G^{(d)}(d-2)\Gamma((d-2)/2)}{c^2(d-1)\pi^{(d-2)/2}} \left(\frac{\alpha_1}{r_1^{d-2}} + \frac{\alpha_2}{r_2^{d-2}}\right)\right) \sum_a \alpha_a \delta_a^{(d)} = \sum_a m_a \delta_a^{(d)}$$

$$\left(1+\frac{G^{(d)}(d-2)\Gamma((d-2)/2)}{c^2(d-1)\pi^{(d-2)/2}}\left(\frac{\alpha_1}{r_1^{d-2}}+\frac{\alpha_2}{r_2^{d-2}}\right)\right)\alpha_1\delta_1^{(d)}=m_1\delta_1^{(d)}$$

dimensional regularization through $\boxed{1<d<2}$:

$$\left(1+\frac{G^{(d)}(d-2)\Gamma((d-2)/2)}{c^2(d-1)\pi^{(d-2)/2}}\frac{\alpha_2}{r_{12}^{d-2}}\right)\alpha_1\delta_1^{(d)}=m_1\delta_1^{(d)}$$

Near-Zone local-in-time PN-expansion ($|\mathbf{x}-\mathbf{x}'|\omega/c << 1$):

$$\int d^3x' \frac{S(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c},\mathbf{x}')}{c^4|\mathbf{x}-\mathbf{x}'|} = \sum_{n=0}^{\infty} \frac{(-1)^n c^{-n}}{c^4 n!} \int \frac{d^3x'}{|\mathbf{x}-\mathbf{x}'|}[|\mathbf{x}-\mathbf{x}'|^n \frac{\partial^n}{\partial t^n} S(t,\mathbf{x}')]$$

Far-Zone local-in-time PN-expansion ($r >> r'$, $r'\omega/c << 1$):

$$\int d^3x' \frac{S(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c},\mathbf{x}')}{c^4|\mathbf{x}-\mathbf{x}'|} = \sum_{m=0}^{\infty} \frac{c^{-m}}{c^4 m! r} \int d^3x' [(\mathbf{x}'\cdot\mathbf{n})^m \frac{\partial^m}{\partial t^m} S(t-\frac{r}{c},\mathbf{x}')] + \mathcal{O}(r^{-2})$$

$$|\mathbf{x}-\mathbf{x}'| = r\sqrt{1+r'^2/r^2-2(\mathbf{x}'\cdot\mathbf{n})/r} = r-(\mathbf{x}'\cdot\mathbf{n})+\mathcal{O}(r^{-1})$$

Problems with non-compact support of S and with locality in time!

Analytic Regularization: Multiplication of source with $(r'/s)^B$ before expansion; at the end $B\to 0$ with droping poles $1/B$ but keeping the remaining $s/c=\alpha$.

Multipole expansion of far-zone field (e.g., L. Blanchet in LRR)

$$h_{ij}^{\mathrm{TT}}(t,\mathbf{x}) = \frac{G}{c^4}\frac{P_{ijkm}(\mathbf{n})}{r}\sum_{l=2}^{\infty}\Bigg\{\left(\frac{1}{c^2}\right)^{\frac{l-2}{2}}\frac{4}{l!}\,\mathrm{M}^{(l)}_{kmi_3\ldots i_l}(t-\frac{r_*}{c})\;N_{i_3\ldots i_l}$$

$$+\left(\frac{1}{c^2}\right)^{\frac{l-1}{2}}\frac{8l}{(l+1)!}\,\epsilon_{pq(k}\,\mathrm{S}^{(l)}_{m)pi_3\ldots i_l}(t-\frac{r_*}{c})\;n_q\;N_{i_3\ldots i_l}\Bigg\}+\mathcal{O}(1/r^2)$$

showing tail term and physical light-cone correction:

$$\mathrm{M}_{ij}^{(2)}(t-\frac{r_*}{c}) = Q_{ij}^{(2)}\left(t-\frac{r_*}{c}\right) + \frac{2GM}{c^3}\int_0^\infty dv\,\ln\left(\frac{v}{2\alpha}\right)Q_{ij}^{(4)}(t-\frac{r_*}{c}-v)+O\left(\frac{1}{c^4}\right)$$

$$r_* = r + \frac{2GM}{c^2}\ln\left(\frac{r}{c\alpha}\right)+O\left(\frac{1}{c^3}\right)$$

$$Q_{ij} = I_{ij} - \tfrac{1}{3}I_{kk}\delta_{ij}, \quad I_{ij} = \textstyle\int d^3x\ \varrho_*\ x^i x^j$$

$$I_{ij}^{(2)} = 2\int d^3x\left[\frac{P_iP_j}{\varrho_*}+\frac{1}{4\pi G}\partial_iU_*\partial_jU_*\right]$$

Near-zone 2.5PN reaction and 4PN tail field, and conservative near-zone Hamiltonian

$$(h_{ij}^{\mathrm{TT}})^{\mathrm{reac/tail}(\alpha)} = -\frac{4G}{5c^5}\left[Q_{ij}^{(3)}(t) + \frac{4MG}{c^3}\int_0^\infty dv\, \ln\left(\frac{v}{2\alpha}\right) Q_{kl}^{(5)}\,(t-v)\right]$$

$$H^{\mathrm{tail}(\alpha)} = -\frac{1}{8}(h_{ij}^{\mathrm{TT}})^{\mathrm{tail}(\alpha)}\; Q_{ij}^{(2)}$$

α is left-over scale from B-regularization in the far zone ($c\alpha = s$).

$$H_{4\mathrm{PN}}^{\mathrm{nz}\,(\alpha')} = H_{4\mathrm{PN}}^{\mathrm{loc}\,0} + \frac{2}{5}\;\frac{G^2M}{c^8} Q_{ij}^{(3)} Q_{ij}^{(3)} \ln\left(\frac{r_{12}}{c\alpha'}\right)$$

α' is left-over scale from B-regularization in the near zone ($c\alpha' = s'$).

total Hamiltonian: $H^{\mathrm{tail}(\alpha)} + H_{4\mathrm{PN}}^{\mathrm{nz}\,(\alpha')}$

e.g., GS & P. Jaranowski in LRR

Damour/Jaranowski/GS 2014 [Jaranowski/GS 2015] (Hamiltonian)

connection of α and α' by refering to GSF(gravitational self-force) calculations in Schwarzschild spacetime

Bernard/Blanchet/Bohé/Faye/Marsat 2017
[Marchand/Bernard/Blanchet/Faye 2018] (Fokker action, harm. coord.)

connection of α and α' through r'^{η} regulator in near-zone expansion within MPM algorithm and dimensional regularization

Foffa/Porto/Rothstein/Sturani 2019 (EFT, harm. coord.)

connection of α and α' by pure dimensional regularization

Blümlein/Maier/Marquard/GS 2020 (EFT, harm. coord.)

connection of α and α' by pure dimensional regularization

4PN Binary BH Conservative Dynamics

Nonlocal-in-time action for the fourth post-Newtonian conservative dynamics of two-body systems

T. Damour, P. Jaranowski, GS

Phys. Rev. D 89:064058 (2014)

$$
\begin{aligned}
H(t) &= m_1 c^2 + m_2 c^2 + H_N + \frac{1}{c^2} H_{[1PN]} \\
&+ \frac{1}{c^4} H_{[2PN]} + \frac{1}{c^5} H_{[2.5PN]}(t) \\
&+ \frac{1}{c^6} H_{[3PN]} + \frac{1}{c^7} H_{[3.5PN]}(t) \\
&+ \frac{1}{c^8} H_{[4PN]} + \dots
\end{aligned}
$$

$\hat{H} = (H - Mc^2)/\mu, \qquad \mu = m_1 m_2 / M, \qquad M = m_1 + m_2$

$\nu = \mu/M, \qquad 0 \leq \nu \leq 1/4$

test-body case: $\nu = 0$, equal-mass case: $\nu = 1/4$

CMF: $\mathbf{p}_1 + \mathbf{p}_2 = 0, \qquad \mathbf{p} \equiv \mathbf{p}_1/\mu, \qquad p_r = (\mathbf{n} \cdot \mathbf{p}),$

$\mathbf{q} \equiv (\mathbf{x}_1 - \mathbf{x}_2)/GM, \qquad \mathbf{n} = \mathbf{q}/|\mathbf{q}|, \qquad \hat{t} = t/GM$

$$\hat{H}_N = \frac{p^2}{2} - \frac{1}{q}$$

$$\hat{H}_{[1PN]} = \frac{1}{8}(3\nu - 1)p^4 - \frac{1}{2}[(3+\nu)p^2 + \nu p_r^2]\frac{1}{q} + \frac{1}{2q^2}$$

$$\begin{aligned}
\hat{H}_{[2PN]} \quad &= \quad \frac{1}{16}(1 - 5\nu + 5\nu^2)p^6 \\
&+ \quad \frac{1}{8}[(5 - 20\nu - 3\nu^2)p^4 - 2\nu^2 p_r^2 p^2 - 3\nu^2 p_r^4]\frac{1}{q} \\
&+ \quad \frac{1}{2}[(5 + 8\nu)p^2 + 3\nu p_r^2]\frac{1}{q^2} - \frac{1}{4}(1 + 3\nu)\frac{1}{q^3}
\end{aligned}$$

$$
\begin{aligned}
\hat{H}_{[3PN]} &= \frac{1}{128}(-5+35\nu-70\nu^2+35\nu^3)p^8 \\
&+ \frac{1}{16}[(-7+42\nu-53\nu^2-5\nu^3)p^6+(2-3\nu)\nu^2 p_r^2 p^4 \\
&+ 3(1-\nu)\nu^2 p_r^4 p^2 - 5\nu^3 p_r^6]\frac{1}{q} \\
&+ [\frac{1}{16}(-27+136\nu+109\nu^2)p^4+\frac{1}{16}(17+30\nu)\nu p_r^2 p^2 \\
&+ \frac{1}{12}(5+43\nu)\nu p_r^4]\frac{1}{q^2} \\
&+ [\left(-\frac{25}{8}+\left(\frac{1}{64}\pi^2-\frac{335}{48}\right)\nu-\frac{23}{8}\nu^2\right)p^2 \\
&+ \left(-\frac{85}{16}-\frac{3}{64}\pi^2-\frac{7}{4}\nu\right)\nu p_r^2]\frac{1}{q^3}+[\frac{1}{8}+\left(\frac{109}{12}-\frac{21}{32}\pi^2\right)\nu]\frac{1}{q^4}
\end{aligned}
$$

3PN the starting of dimensional regularization

Jaranowski/GS ('98,'99)[in part], Damour/Jaranowski/GS ('01)

Blanchet/Faye ('01)[in part], Blanchet/Damour/Esposito-Farèse ('04) [harmonic gauge, point masses]

Itoh/Futamase ('03), Itoh('04) [EIH surface-integral method, harmonic gauge, point singularities]

Foffa/Sturani ('11) [EFT, harmonic gauge, point masses]

$$\boxed{\text{4PN}}$$

$$
\begin{aligned}
\hat{H}_{[4PN]} &= \left(\frac{7}{256} - \frac{63}{256}\nu + \frac{189}{256}\nu^2 - \frac{105}{128}\nu^3 + \frac{63}{256}\nu^4\right) p^{10} \\
&+ \left\{\frac{45}{128}p^8 - \frac{45}{16}p^8\nu + \left(\frac{423}{64}p^8 - \frac{3}{32}p_r^2p^6 - \frac{9}{64}p_r^4p^4\right)\nu^2\right. \\
&+ \left(-\frac{1013}{256}p^8 + \frac{23}{64}p_r^2p^6 + \frac{69}{128}p_r^4p^4 - \frac{5}{64}p_r^6p^2 + \frac{35}{256}p_r^8\right)\nu^3 \\
&+ \left.\left(-\frac{35}{128}p^8 - \frac{5}{32}p_r^2p^6 - \frac{9}{64}p_r^4p^4 - \frac{5}{32}p_r^6p^2 - \frac{35}{128}p_r^8\right)\nu^4\right\}\frac{1}{q} \\
&+ \left\{\frac{13}{8}p^6 + \left(-\frac{791}{64}p^6 + \frac{49}{16}p_r^2p^4 - \frac{889}{192}p_r^4p^2 + \frac{369}{160}p_r^6\right)\nu\right. \\
&+ \left(\frac{4857}{256}p^6 - \frac{545}{64}p_r^2p^4 + \frac{9475}{768}p_r^4p^2 - \frac{1151}{128}p_r^6\right)\nu^2 \\
&+ \left.\left(\frac{2335}{256}p^6 + \frac{1135}{256}p_r^2p^4 - \frac{1649}{768}p_r^4p^2 + \frac{10353}{1280}p_r^6\right)\nu^3\right\}\frac{1}{q^2}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \left[\frac{105}{32}p^4 + C_{41}\,\nu + C_{42}\,\nu^2 + \left(-\frac{553}{128}p^4 - \frac{225}{64}p_r^2 p^2 - \frac{381}{128}p_r^4\right)\nu^3\right]\frac{1}{q^3} \\
&+ \quad \left\{\frac{105}{32}p^2 + C_{21}\,\nu + C_{22}\,\nu^2\right\}\frac{1}{q^4} \\
&+ \quad \left\{-\frac{1}{16} + c_{01}\,\nu + c_{02}\,\nu^2\right\}\frac{1}{q^5} \\
&- \quad \frac{1}{5}\hat{Q}_{ij}^{(3)}(\hat{t}) \int_{-\infty}^{+\infty} dw \ln\left(\frac{|w|c}{2q}\right)\hat{Q}_{ij}^{(4)}(\hat{t}-w)\,\nu
\end{aligned}
$$

$$
\begin{aligned}
C_{42} &= \left(-\frac{1189789}{28800} + \frac{18491}{16384}\pi^2\right) p^4 + \left(-\frac{127}{3} - \frac{4035}{2048}\pi^2\right) p_r^2 p^2 \\
&+ \left(\frac{57563}{1920} - \frac{38655}{16384}\pi^2\right) p_r^4
\end{aligned}
$$

$$
C_{22} = \left(\frac{672811}{19200} - \frac{158177}{49152}\pi^2\right) p^2 + \left(-\frac{21827}{3840} + \frac{110099}{49152}\pi^2\right) p_r^2
$$

$$
c_{02} = -\frac{1256}{45} + \frac{7403}{3072}\pi^2
$$

$$C_{41} = \left(-\frac{589189}{19200} + \frac{2749}{8192}\pi^2\right) p^4 + \left(\frac{63347}{1600} - \frac{1059}{1024}\pi^2\right) p_r^2 p^2$$

$$+ \left(-\frac{23533}{1280} + \frac{375}{8192}\pi^2\right) p_r^4$$

$$C_{21} = \left(\frac{185761}{19200} - \frac{21837}{8192}\pi^2\right) p^2 + \left(\frac{3401779}{57600} - \frac{28691}{24576}\pi^2\right) p_r^2$$

$$c_{01} = -\frac{169199}{2400} + \frac{6237}{1024}\pi^2$$

4PN analytical

Jaranowski/GS ('12,'13,'15) [in part], Damour/Jaranowski/GS ('14)

Bini/Damour ('13) [in part] [GSF, EOB], Foffa/Sturani ('13) [in part] [EFT]

$$
\begin{aligned}
H_{4\mathrm{PN}}^{\mathrm{near-zone\,(s)}}[\mathbf{x}_a, \mathbf{p}_a] &= H_{4\mathrm{PN}}^{\mathrm{loc}\,0}[\mathbf{x}_a, \mathbf{p}_a] \\
&+ \frac{2}{5}\,\frac{G^2 M}{c^8}\left(Q_{ij}^{(3)}\right)^2 \ln\left(\frac{r_{12}}{s_{nz}}\right) \\
&+ \frac{d}{dt}\,G[\mathbf{x}_a, \mathbf{p}_a]
\end{aligned}
$$

$$
\begin{aligned}
H_{4\mathrm{PN}}^{\mathrm{tailsym\,(s)}}(t) &= -\frac{1}{5}\,\frac{G^2 M}{c^8}\,Q_{ij}^{(3)}(t) \\
&\times \int_{-\infty}^{+\infty} dv \ln\left(\frac{|v|c}{2s_{fz}}\right) Q_{ij}^{(4)}(t-v)
\end{aligned}
$$

Matching GSF-results by Bini/Damour for perturbed Schwarzschild metric from particle in circular motion yields $\ln\left(s_{fz}/s_{nz}\right) = -\frac{1681}{1536}$.

$$H^{\text{tailsym}}_{4\text{PN}}(t) = -\frac{1}{5}\,\frac{G^2 M}{c^8}\,Q^{(3)}_{ij}(t)\,\text{Pf}_{2r_{12}/c}\int_{-\infty}^{+\infty}\frac{dv}{|v|}\,Q^{(3)}_{ij}(t-v)$$

Classical gravitational “Lamb Shift” (orbital average):

$$\Delta E = -\frac{G}{5c^5}\,\frac{GM}{c^3 P}\int_0^P dt\left[Q^{(3)}_{ij}(t)\,\text{Pf}_{2r_{12}/c}\int_{-\infty}^{+\infty}\frac{dv}{|v|}\,Q^{(3)}_{ij}(t-v)\right]$$

$$< \frac{dE}{dt} > = -\frac{G}{5c^5}\,\frac{1}{P}\int_0^P dt\left[Q^{(3)}_{ij}(t)Q^{(3)}_{ij}(t)\right] \quad \text{(Einstein's quad. formula)}$$

$\boxed{\text{4PN}}$ EFT, Foffa/Sturani (2013) [Galley/Leibovich/Porto/Ross (2016)]

$$H_{\text{4PN}}^{\text{tail,nonloc}} = -\frac{GM}{c^3}\text{Pf}_{2r_{12}/c}\int_{-\infty}^{\infty}\frac{d\tau}{|\tau|}\mathcal{F}_{\text{N}}^{\text{split}}(t, t+\tau)$$

$$\mathcal{F}_{\text{N}}^{\text{split}}(t, t') = \frac{G}{c^5}\frac{1}{5}Q_{ij}^{(3)}(t)Q_{ij}^{(3)}(t')$$

$$H_{\text{4PN}}^{\text{tail,loc}} = -\frac{GM}{c^3}\frac{41}{30}\mathcal{F}_{\text{N}}^{\text{split}}(t, t)$$

$\frac{41}{30} \longleftrightarrow \frac{5}{6}$ famous numerical constant in Lamb-shift formula

[errors by Feynman and Schwinger, cf., L.S. Brown, arXiv:physics/9911056]

$\boxed{\text{5PN}}$ EFT, $\hbar = 1$,

particles, no anti-particles, gravitons = anti-gravitons

in-out formalism:

$$e^{iW[J]} \equiv \; < 0, \text{out}|0, \text{in} >_J \; = \; (0|\hat{U}_J(\infty, -\infty)|0)$$

$$< \hat{\phi}_H(t, x) >_{\text{in-out}} \; \equiv \; \frac{\delta W}{\delta J(t,x)}|_{J=0} = \frac{< 0, \text{out}|\hat{\phi}_H(t,x)|0, \text{in} >}{< 0, \text{out}|0, \text{in} >} = \bar{\phi}(t, x)$$

$$= \frac{(0|\hat{U}(+\infty, t)\hat{\phi}_I(t, x)\hat{U}(t, -\infty)|0)}{(0|\hat{U}(+\infty, -\infty)|0)}$$

$$e^{iW[J]} = \int \mathcal{D}\phi \exp\left\{ iS[\phi] + i\int d^Dx\ J\ \phi \right\}$$

effective action : $\Gamma[\bar{\phi}] = W[J] - \int d^Dx\ J\ \bar{\phi} \equiv S_{eff}[\bar{\phi}]$

in-in formalism:

$$e^{iW[J_1,J_2]} \equiv (0|\hat{U}_{J_2}(-\infty,\infty)\hat{U}_{J_1}(\infty,-\infty)|0)$$

$$< \hat{\phi}^1_H(t,x) >_{\text{in}-\text{in}} \equiv \frac{\delta W}{\delta J_1(t,x)}|_{J_1=J_2=0}$$

$$< \hat{\phi}^2_H(t,x) >_{\text{in}-\text{in}} \equiv \frac{\delta W}{\delta J_2(t,x)}|_{J_1=J_2=0}$$

$$< \hat{\phi}^1_H(t,x) >_{\text{in}-\text{in}} = < \hat{\phi}^2_H(t,x) >_{\text{in}-\text{in}} = < 0,\text{in}|\hat{\phi}_H(t,x)|0,\text{in} > = \bar{\phi}(t,x)$$

$$= (0|\hat{U}(-\infty,t)\hat{\phi}_I(t,x)\hat{U}(t,-\infty)|0)$$

$$e^{iW[J_1,J_2]} \; = \; \int \mathcal{D}\phi_1 \; \mathcal{D}\phi_2 \; \exp\left\{ iS[\phi_1] - iS[\phi_2] + i \int d^D x \; J_1 \; \phi_1 - i \int d^D x \; J_2 \; \phi_2 \right\}$$

boundary conditions: $\phi_1 = \phi_2$ at $t = \infty$

$$\Gamma[\bar{\phi}_1, \bar{\phi}_2] \; = \; W[J_1, J_2] - \int d^D x \; J_1 \; \bar{\phi}_1 - \int d^D x \; J_2 \; \bar{\phi}_2 \; \equiv \; S_{eff}[\bar{\phi}_1, \bar{\phi}_2]$$

in our case:

$$J_1 \quad \longrightarrow \quad \mathbf{j}_{a1}, \quad J_{\mu\nu 1}, \qquad J_2 \longrightarrow \quad \mathbf{j}_{a2}, \quad J_{\mu\nu 2}$$

$$\phi_1 \quad \longrightarrow \quad \mathbf{x}_{a1}, \quad \tilde{h}_1^{\mu\nu}, \qquad \phi_2 \longrightarrow \quad \mathbf{x}_{a2}, \quad \tilde{h}_2^{\mu\nu}$$

$$i\int d^Dx\ J_1\ \phi_1 - i\int d^Dx\ J_2\ \phi_2 \qquad \rightarrow$$

$$i\sum_{a=1,2}\int dt(\mathbf{j}_{a1}\cdot\mathbf{x}_{a1} - \mathbf{j}_{a2}\cdot\mathbf{x}_{a2}) + i\int d^Dx(J_{\mu\nu 1}\tilde{h}_1^{\mu\nu} - J_{\mu\nu 2}\tilde{h}_2^{\mu\nu})$$

$$\Gamma[<\hat{\mathbf{x}}_{a1}>, <\hat{\mathbf{x}}_{a2}>, J_{\mu\nu 1}, J_{\mu\nu 2}] = W[\mathbf{j}_{a1}, \mathbf{j}_{a2}, J_{\mu\nu 1}, J_{\mu\nu 2}]$$

$$-\sum_{a=1,2}\int dt(\mathbf{j}_{a1}\cdot <\hat{\mathbf{x}}_{a1}> \ - \ \mathbf{j}_{a2}\cdot <\hat{\mathbf{x}}_{a2}>)$$

$$= \ S_{eff}[<\hat{\mathbf{x}}_{a1}>, <\hat{\mathbf{x}}_{a2}>, J_{\mu\nu 1}, J_{\mu\nu 2}]$$

Keldysh representation: $J_- = J_1 - J_2, \quad J_+ = \frac{1}{2}(J_1 + J_2)$

Equations of motion in the classical limit:

$$0 = \left(\frac{\delta\Gamma(\bar{\mathbf{x}}_{a\pm}, J_{\mu\nu\pm})}{\delta\bar{\mathbf{x}}_{a-}} \right) |(\bar{\mathbf{x}}_{a-} = 0,\ \bar{\mathbf{x}}_{a+} = \bar{\mathbf{x}}_a,\ J_{\mu\nu\pm} = 0)$$

The gravitational point-particle action reads, in harmonic gauge, $S = S_{\mathrm{GR}} + S_{\mathrm{pp}}$, with

$$S_{\mathrm{GR}} = 2\Lambda^2 \int d^D x \, \sqrt{-g} \left(R - \frac{1}{2} \Gamma^\mu \Gamma_\mu \right), \qquad S_{\mathrm{pp}} = - \sum_{a=1}^{2} m_a \int d\tau_a,$$

where $\Lambda = c^2 \mu_1^\epsilon / \sqrt{32\pi G_N}$ and $\Gamma^\mu = \Gamma^\mu_{\alpha\beta} g^{\alpha\beta}$.

Ansatz: $\sqrt{-g} g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \quad h^{\mu\nu} = \frac{1}{\Lambda}(H^{\mu\nu} + \tilde{h}^{\mu\nu})$.

$H^{\mu\nu}$: potential (near-zone) modes, $\quad \tilde{h}^{\mu\nu}$: ultrasoft (radiation) modes

$$\exp\{iS_{\mathrm{pot}}[\mathbf{x}_a, \tilde{h}]\} = \int \mathcal{D}H \exp\{iS\}.$$

Hamiltonian: $H = H_{\mathrm{pot}} + H_{\mathrm{tail}} = H_{\mathrm{loc}} + H_{\mathrm{nl}}$

Free-field generating functional for the radiation gravitons:

$$Z_0[J_{\mu\nu 1}, J_{\mu\nu 2}] = \int \mathcal{D}\tilde{h}_1^{\mu\nu}\mathcal{D}\tilde{h}_2^{\mu\nu}\exp\left\{iS^{(2)}[\tilde{h}_1] - iS^{(2)}[\tilde{h}_2] + i\int d^D x(J_{\mu\nu 1}\tilde{h}_1^{\mu\nu} - J_{\mu\nu 2}\tilde{h}_2^{\mu\nu})\right\}$$

$$e^{iW[\mathbf{j}_{a1},\mathbf{j}_{a2},J_{\mu\nu 1},J_{\mu\nu 2}]} = \int \prod_{a=1}^{2} \mathcal{D}\mathbf{x}_{a1}\mathcal{D}\mathbf{x}_{a2}\exp\Bigg\{i\sum_{a=1}^{2}(S_{\rm pp}(\mathbf{x}_{a1}) - S_{\rm pp}(\mathbf{x}_{a2}))$$

$$+i\sum_{a=1}^{2}\int dt\ (\mathbf{j}_{a1}\cdot\mathbf{x}_{a1} - \mathbf{j}_{a2}\cdot\mathbf{x}_{a2})$$

$$+iS_{\rm pot}\left[\mathbf{x}_{a1}, -i\frac{\delta}{\delta J_{\alpha\beta 1}}\right] - iS_{\rm pot}\left[\mathbf{x}_{a2}, -i\frac{\delta}{\delta J_{\alpha\beta 2}}\right]\Bigg\}Z_0[J_{\mu\nu 1}, J_{\mu\nu 2}]$$

Keldysh representation:

$$Z_0[J_{\mu\nu\pm}] = \exp\left\{-\frac{1}{2}\int d^D x \int d^D x' J_{\alpha\beta A}(x) D^{\alpha\beta\gamma\delta}_{AB}(x-x') J_{\gamma\delta B}(x')\right\}$$

where $A, B = \pm$.

$$D^{\alpha\beta\gamma\delta}_{++}(x-x') \;=\; 0, \qquad D^{\alpha\beta\gamma\delta}_{+-}(x-x') \;=\; -\,iD^{\alpha\beta\gamma\delta}_{\text{adv}},$$

$$D^{\alpha\beta\gamma\delta}_{-+}(x-x') \;=\; -\,iD^{\alpha\beta\gamma\delta}_{\text{ret}}, \qquad D^{\alpha\beta\gamma\delta}_{--}(x-x') \;=\; 0.$$

The last zero holds because of the final classical limit.

Galley, Tiglio, PRD **79** 124027 (2009) arXiv:0903.1122

Illuminating EXAMPLE for in-in (or closed-time) formalism

The forced-damped-harmonic-oscillator equation of motion reads

$$M\ddot{x}(t) + R\dot{x}(t) + kx(t) = f(t)$$

Making use of

$$\delta(M\ddot{x} + R\dot{x} + kx - f) = \int \mathcal{D}y \, \exp\left[\frac{i}{\hbar}\int dt \; y \, (-M\ddot{x} - R\dot{x} - kx + f)\right],$$

we find the Lagrangian

$$L = y\,(-M\ddot{x} - R\dot{x} - kx + f) = M\dot{x}\dot{y} + \frac{R}{2}(x\dot{y} - y\dot{x}) - kxy + fy \; + \; \mathrm{TTD}$$

resulting in the equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0 \;\rightarrow\; M\ddot{x} + R\dot{x} + kx = f,$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \;\rightarrow\; M\ddot{y} - R\dot{y} + ky = 0.$$

The replacement in L of y with x and taking 1/2 of the quadratic terms we get

$$L = \frac{1}{2}M\dot{x}^2 - \frac{1}{2}kx^2 + fx$$

which is just the conservative part of our original dynamics.

The transition to the in-in formalism is given by the substitutions

$$y = x_1 - x_2 \;=\; x_-, \qquad x = \frac{1}{2}(x_1 + x_2) \;=\; x_+,$$

where x_1 and x_2 are our forward and backward in time going position variables, respectively.

In the classical limit

$$y = x_- = 0 \ \rightarrow x_1 = x_2, \qquad x = x_+ = x_1 = x_2$$

hold.

In the new variables, our in-in Lagrangian reads

$$L = \frac{M}{2}\ (\dot{x}_1^2 - \dot{x}_2^2) - \frac{k}{2}\ (x_1^2 - x_2^2) + \frac{R}{2}\ (\dot{x}_1 x_2 - \dot{x}_2 x_1) + f\ (x_1 - x_2)$$

The corresponding Hamiltonian is given by

$$H = \frac{1}{2M}\ \left(p_1 - \frac{R}{2} x_2\right)^2 - \frac{1}{2M}\ \left(p_2 + \frac{R}{2} x_1\right)^2 + \frac{k}{2}\ (x_1^2 - x_2^2) - f\ (x_1 - x_2)$$

With even a stochastic force $< f(t) f(t') >_{\text{noise}} \ = \ 2k_B T R\ \delta(t - t')$,

$$H = \frac{1}{2M}\ \left(p_1 - \frac{R}{2} x_2\right)^2 - \frac{1}{2M}\ \left(p_2 + \frac{R}{2} x_1\right)^2 + \frac{k}{2}\ (x_1^2 - x_2^2) - i\ \frac{k_B T R}{\hbar}\ (x_1 - x_2)^2.$$

For $k = 0$ see, Blasone et al., Phase Coherence in Quantum Brownian Motion, Ann. Phys. (N.Y.) **267**, 61 (1998) [arXiv:quant-ph/9707048]

EOB Formalism (without spin for simplicity)

Damour, arXiv:1312.3505

Standard representation

$$\frac{H_{\rm eff}}{\mu c^2} \equiv \frac{H^2 - m_1^2 c^4 - m_2^2 c^4}{2 m_1 m_2 c^4} = 1 + \frac{H_{\rm NR}}{\mu c^2} + \frac{\nu}{2} \left(\frac{H_{\rm NR}}{\mu c^2} \right)^2$$

$$H_{\rm NR} \equiv H - M c^2 , \qquad \hat{H}_{\rm eff} = \frac{H_{\rm eff}}{\mu c^2}$$

$$H = M c^2 \sqrt{1 + 2\nu \left(\hat{H}_{\rm eff} - 1 \right)}$$

EOB representation with modified Schwarzschild metric

$$g_{\rm eff}^{\mu\nu} P_\mu P_\nu + Q_4(P_i) = -\mu^2 c^2 , \qquad H_{\rm eff}^{\rm EOB} \equiv -P_0 c$$

$$H_{\rm eff}^{\rm EOB} = N_{\rm eff}\, c \sqrt{\mu^2 c^2 + \gamma_{\rm eff}^{ij} P_i P_j + Q_4(P_i)}$$

Canonical transformation to connect:

$$H_{\text{eff}}^{\text{EOB}}(X,P) = H_{\text{eff}}(x,p)$$

$$H^{\text{EOB}} = Mc^2\sqrt{1+2\nu\left(\hat{H}_{\text{eff}}^{\text{EOB}}-1\right)}$$

$(X,P) \to (x,p)$:

$$\hat{H}_{\text{eff}}^{\text{EOB}} = \sqrt{A(1+AD\eta^2 p_r^2+\eta^2(p^2-p_r^2)+Q)}$$

$(\eta = 1/c)$

$$A = 1+\sum_{k=1}^{6} a_k(\nu)\eta^{2k}u^k, \quad a_2 = 0 \text{ (1PN)},$$

$$D = 1 + \sum_{k=2}^{5} d_k(\nu)\eta^{2k}u^k,$$

$$Q = \eta^4 p_r^4[q_{42}(\nu)\eta^4 u^2 + q_{43}(\nu)\eta^6 u^3 + q_{44}(\nu)\eta^8 u^4] + \eta^6 p_r^6[q_{62}(\nu)\eta^4 u^2$$
$$+q_{63}(\nu)\eta^6 u^3] + \eta^{12} p_r^8 q_{82}(\nu)u^2$$

$u = \frac{GM}{rc^2}$

N: $a_1 = -2,$

2PN: $d_2 = 6\nu, \quad a_3 = 2\nu,$

3PN: $q_{42} = 8\nu - 6\nu^2, \quad d_3 = 52\nu - 6\nu^2, \quad a_4 = \left(\frac{94}{3} - \frac{41}{32}\pi^2\right)\nu,$

4PN: $q_{62} = -\frac{9}{5}\nu - \frac{27}{5}\nu^2 + 6\nu^3, \quad q_{43} = 20\nu - 83\nu^2 + 10\nu^3,$
$d_4 = \left(\frac{1679}{9} - \frac{23761}{1536}\pi^2\right)\nu + (-260 + 123\pi^2)\nu^2$
$a_5 = \left(-\frac{221}{6} + \frac{41}{32}\pi^2\right)\nu^2$

5PN:

EFT Foffa et al. (2019), Blümlein, Maier, Marquard (2020)

Tutti-Frutti Bini, Damour, Geralico (2019):

post-Newtonian (PN) expansion: $\frac{1}{c^2}(G, \mathrm{v}^2)$, $G \sim \mathrm{v}^2$
(virial-theorem-based expansion)

post-Minkowskian (PM) expansion: G

multipolar post-Minkowskian (MPM) expansion (in part PN type)

effective field-theory (EFT)

gravitational self-force (GSF) [test body gets gravitating]

effective one-body (EOB) [gener'zed. strong-field test-body dynamics]

Delaunay averaging (DA) [averaging under action-angle variables].

From TF BDG (2020):

$$q_{82} = \frac{6}{7}\nu + \frac{18}{7}\nu^2 + \frac{24}{7}\nu^3 - 6\nu^4$$

$$q_{63} = \frac{123}{10}\nu - \frac{69}{5}\nu^2 + 116\nu^3 - 14\nu^4$$

$$q_{44} = \left(\frac{1580641}{3150} - \frac{93031}{1536}\pi^2\right)\nu + \left(-\frac{9367}{15} + \frac{31633}{512}\pi^2\right)\nu^2 + \left(640 - \frac{615}{32}\pi^2\right)\nu^3$$

$$d_5 = \left(\frac{331054}{175} - \frac{63707}{512}\pi^2\right)\nu + d_5^{\nu^2} + \left(\frac{1069}{3} - \frac{205}{16}\pi^2\right)\nu^3$$

$$a_6 = \left(-\frac{1026301}{1575} + \frac{246367}{3072}\pi^2\right)\nu + a_6^{\nu^2} + 4\nu^3$$

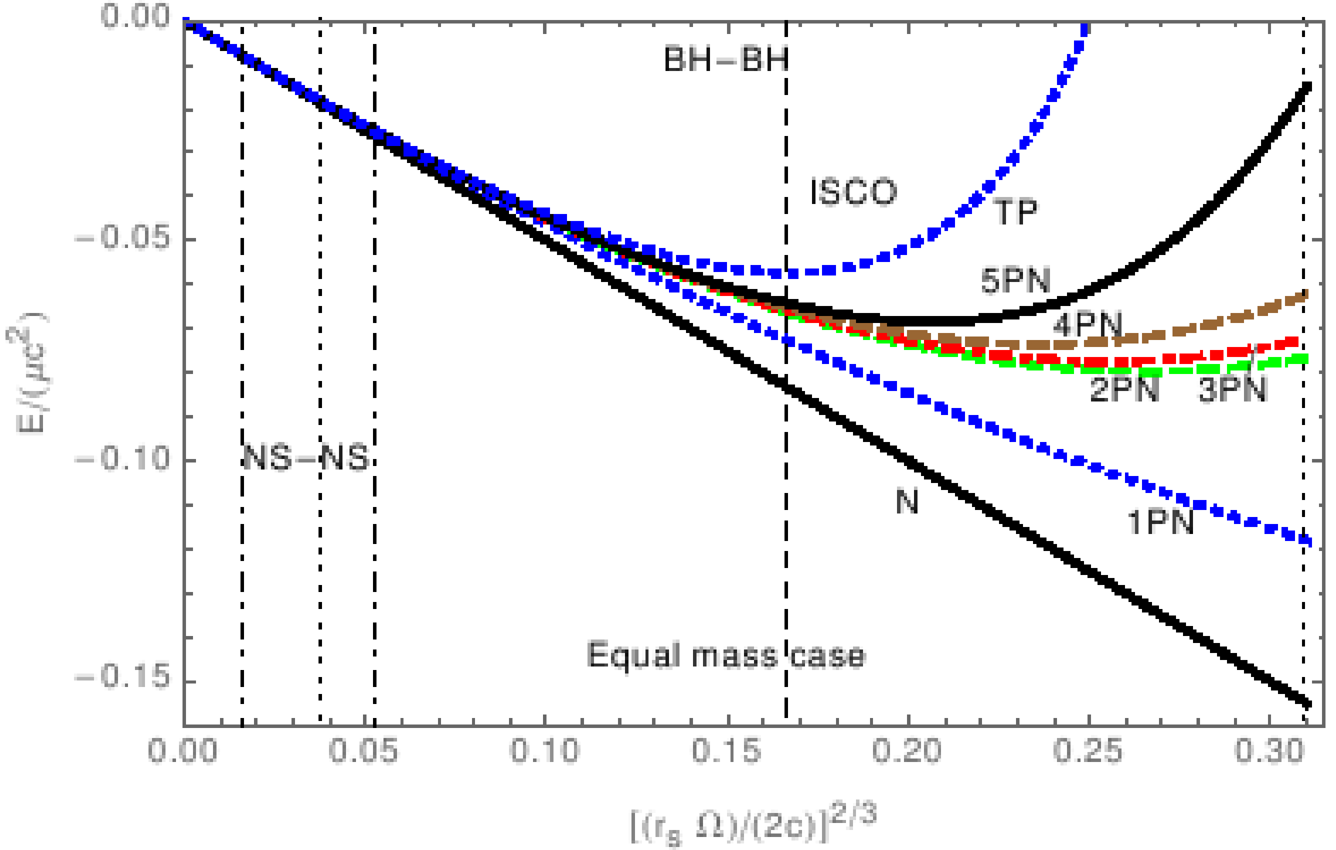
From EFT Blümlein, Maier, Marquard, GS (2021):

$$d_5^{\pi^2\nu^2} = \frac{306545}{512}\pi^2\nu^2, \quad a_6^{\pi^2\nu^2} = \frac{25911}{256}\pi^2\nu^2$$

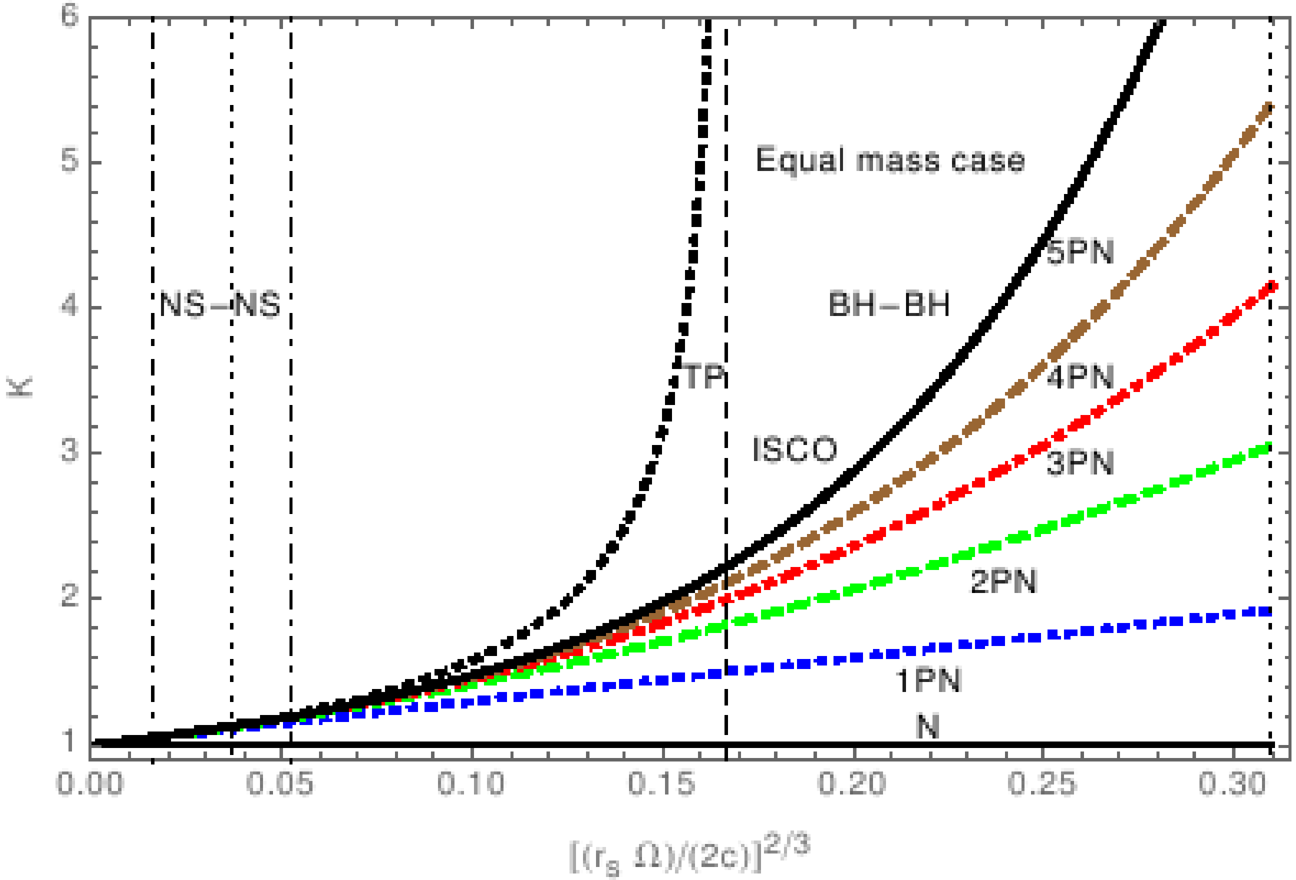
From EFT BMMS (2022) arXiv:2110.13822

$$d_5^{\nu^2} = \left(-\frac{31244704}{4725} + \frac{306545}{512}\pi^2\right)\nu^2, \quad a_6^{\nu^2} = \left(-\frac{1750163}{1575} + \frac{25911}{256}\pi^2\right)\nu^2$$

$$q_{44} = \left(\frac{1580641}{3150} - \frac{93031}{1536}\pi^2\right)\nu + \left(-\frac{532676}{675} + \frac{31633}{512}\pi^2\right)\nu^2 + \left(640 - \frac{615}{32}\pi^2\right)\nu^3$$



Binding Energy for circular orbits (BMMS (2022), arXiv:2110.13822)



Periastron Advance for circular orbits (BMMS (2022), arXiv:2110.13822)

EFT Foffa, Sturani (2020)

$$H_{\text{5PN}}^{\text{tail,nl}} = -\frac{G(E/c^2)}{c^3}\text{Pf}_{2r_{12}/c}\int_{-\infty}^{\infty}\frac{d\tau}{|\tau|}\mathcal{F}_{\text{1PN}}^{\text{split}}(t,t+\tau)$$

$$\mathcal{F}_{\text{1PN}}^{\text{split}}(t,t') = \frac{G}{c^5}\frac{1}{c^2}\left(\frac{1}{189}O_{ijk}^{(4)}(t)O_{ijk}^{(4)}(t') + \frac{16}{45}J_{ij}^{(3)}(t)J_{ij}^{(3)}(t')\right)$$

$$H_{\text{5PN}}^{\text{tail,loc}} = -\frac{G(E/c^2)}{c^3}\left(\frac{82}{35}\mathcal{F}_{\text{1PN}}^{\text{split},MO^2}(t,t) + \frac{127}{60}\mathcal{F}_{\text{1PN}}^{\text{split},MJ^2}(t,t)\right)$$

TF BDG (2021) arXiv:2107.08896

$$R_{\text{oct,e}}^{\text{TF}} = R_{\text{oct,e}}^{\text{FS}} = R_{\text{oct,e}}^{\text{BMMS}} = \frac{82}{35},$$
$$R_{\text{quad,m}}^{\text{TF}} = \frac{147}{60} = \frac{147}{127}R_{\text{quad,m}}^{\text{FS}} = R_{\text{quad,m}}^{\text{BMMS}}.$$

Entering of the current-type quadrupole J_{ij} in most delicate form:

$$\frac{1}{2} R_{0iab} \epsilon_{abj} J_{ij}, \qquad J_{ij} = J_{ji}.$$

d-dimensional generalization via its avatar $J_{i|ab}$:

$$\epsilon_{abj} J_{ij} \equiv J_{b|ia}, \quad J_{i|ab} = -J_{b|ai}, \quad J_{i|ab} + J_{a|bi} + J_{b|ia} = 0.$$

Thus, BDG (2021), also see Almeida, Foffa, Sturani (2021),

$$J_{ij}^{(3)} J_{ij}^{(3)} \;\to\; \frac{1}{2} J_{i|ab}^{(3)} J_{i|ab}^{(3)},$$

with

$$J_{i|ab} = \nu(m_2 - m_1)[(x^i x^a - \frac{x \cdot x}{d-1}\delta^{ia}) v_b - (x^a x^b - \frac{x \cdot x}{d-1}\delta^{ab}) v_i$$

$$-\frac{x \cdot v}{d-1}(x^i \delta^{ab} - x^b \delta^{ia})].$$

Henry, Faye, Blanchet, arXiv:2105.10876

Additionally, BDG arXiv:2107.08896 with confirming BMMS in part,

$$a_6^{\nu^2} = \frac{25911}{256}\pi^2\nu^2 + \nu^2 R_{a_6}(R_{\mathrm{oct,e}}, R_{\mathrm{quad,m}}, C_{QQL}, C_{QQQ_1}, C_{QQQ_2})$$
$$d_5^{\nu^2} = \frac{306545}{512}\pi^2\nu^2 + \nu^2 R_{d_5}(R_{\mathrm{oct,e}}, R_{\mathrm{quad,m}}, C_{QQL}, C_{QQQ_1}, C_{QQQ_2})$$

$$S_{QQL} = C_{QQL} G^2 \int dt Q_{il}^{(4)} Q_{jl}^{(3)} \epsilon_{ijk} L_k$$

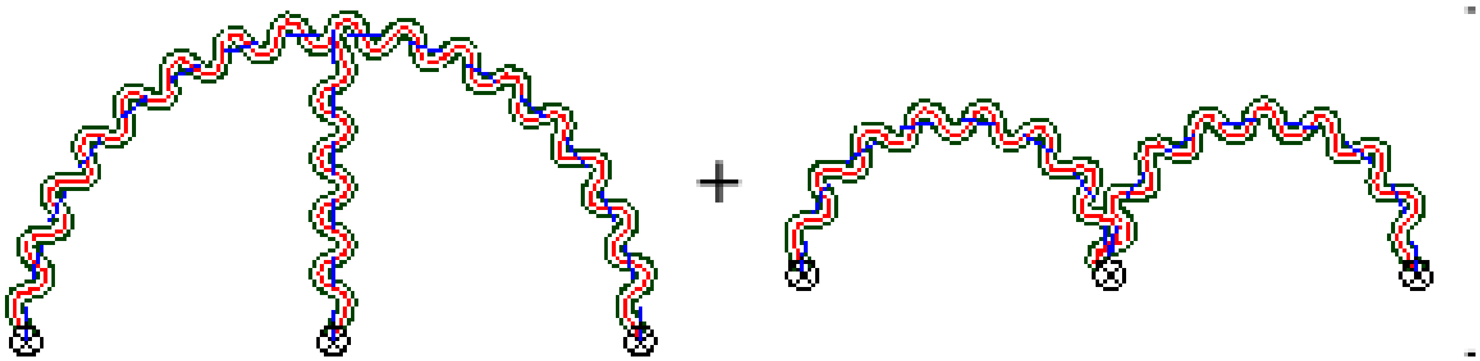
Memory terms:

$$S_{QQQ_1} = C_{QQQ_1} G^2 \int dt Q_{il}^{(4)} Q_{jl}^{(4)} Q_{ij}$$
$$S_{QQQ_2} = C_{QQQ_2} G^2 \int dt Q_{il}^{(3)} Q_{jl}^{(3)} Q_{ij}^{(2)}$$
$$C_{QQL}^{\mathrm{FS}} = -\frac{8}{15} = C_{QQL}^{\mathrm{BMMS}}$$
$$C_{QQQ_1}^{\mathrm{FS}} = -\frac{1}{15} = 8\ C_{QQQ_1}^{\mathrm{BMMS}} \ (\text{without contact graph} = \frac{4}{3}\ C_{QQQ_1}^{\mathrm{BMMS}})$$
$$C_{QQQ_2}^{\mathrm{FS}} = -\frac{4}{105} = \frac{4}{3}\ C_{QQQ_2}^{\mathrm{BMMS}}$$

Feynman graphs for memory terms with three Q_{ij}-sources.

The second graph is a contact graph, identified for the first time.

BMMS (2022) arXiv:2110.13822 and (2022) arXiv:2208.04552



BDG (2021): constraint from scattering $\chi_4^{\text{cons,EFT}} - \chi_4^{\text{cons,TF}} = 0$:

$$0 = \frac{2973}{350} - \frac{69}{2} C_{QQL} + \frac{253}{18} C_{QQQ_1} + \frac{85}{9} C_{QQQ_2},$$

depending on q_{44}.

Fulfilled by neither FS nor BMMS.

TF BDG (2021) arXiv:2107.08896:

Missing conservative quadratic radiation reaction (anti-symmetric)^2 terms, possibly leading to, see BMMS (2022) arXiv:2110.13822,

$$\delta H_{\rm rad}^{({\rm reac})^2} = a\ (\eta^5)^2\ \nu^2 \frac{p_r^4}{r^4}$$

$$q_{44}^{BMMS} \to q_{44}^{\rm BDG}, \quad a = \frac{65801}{1350}$$

$$G_{\rm ret} = G_{\rm sym} + \tfrac{1}{2} G_{\rm reac}, \qquad G_{\rm sym} = \tfrac{1}{2}(G_{\rm ret} + G_{\rm adv}),$$

$$G_{\rm adv} = G_{\rm sym} - \tfrac{1}{2} G_{\rm reac}, \qquad G_{\rm reac} = G_{\rm ret} - G_{\rm adv}.$$

$$G_{\rm sym}(x - x') = G_{\rm sym}(x' - x), \qquad G_{\rm reac}(x - x') = -G_{\rm reac}(x' - x)$$

5.5PN and 6PN

TF BDG (2020)

$$H^{\text{tail}^2,\text{nl}}_{5.5\text{PN}} = -\frac{107}{210}\frac{G^2(E/c^2)^2}{c^6}\int_{-\infty}^{\infty}\frac{d\tau}{\tau}\left[\mathcal{G}^{\text{split}}(t,t+\tau) - \mathcal{G}^{\text{split}}(t,t-\tau)\right]$$

$$\mathcal{G}^{\text{split}}(t,t') = \frac{G}{c^5}\frac{1}{5}Q^{(3)}_{ij}(t)Q^{(4)}_{ij}(t')$$

$$H^{\text{tail,nl}}_{6\text{PN}} = -\frac{G(E/c^2)}{c^3}\text{Pf}_{2r_{12}/c}\int_{-\infty}^{\infty}\frac{d\tau}{|\tau|}\mathcal{F}^{\text{split}}_{2\text{PN}}(t,t+\tau)$$

$$\mathcal{F}^{\text{split}}_{2\text{PN}}(t,t') = \frac{G}{c^5}\frac{1}{c^4}\left(\frac{1}{9072}I^{(5)}_{ijkl}(t)I^{(5)}_{ijkl}(t') + \frac{1}{84}J^{(4)}_{ijk}(t)J^{(4)}_{ijk}(t')\right)$$

The not yet approved (at 5PN) or calculated (at 6PN)
EOB numerical constants:

3 rational parts at 5PN: $a_6^{\nu^2,\mathrm{ratpart}}$, $d_5^{\nu^2,\mathrm{ratpart}}$, $q_{44}^{\nu^2,\mathrm{ratpart}}$

8 parts (non-rational & rational) at 6PN: $a_7^{\nu^2}$, $a_7^{\nu^3}$, $d_6^{\nu^2}$, $q_{45}^{\nu^2}$