A new generic and structurally stable cosmological model without singularity

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Dynamical Systems in Cosmology

Working cosmological model:

"BB" $\rightarrow$ Inflation $\rightarrow$ RDE $\rightarrow$ MDE $\rightarrow$ de Sitter

Dynamical system theory:

an unstable node $\rightarrow$ a saddle $\rightarrow$ a saddle $\rightarrow$ a saddle $\rightarrow$ a stable node

GR with ξ

We start from the total action of the theory

$$S = S_g + S_\psi \,,$$

consisting of the gravitational part described by the standard Einstein-Hilbert action integral

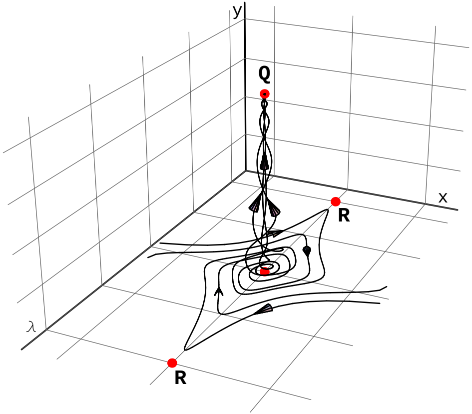
$$S_g = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\,R\,,$$

where $\kappa^2 = 8\pi G$, and the matter part of the theory is in the form of non-minimally coupled scalar field

$$S_\psi = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\Big(\varepsilon\nabla^\alpha\psi\,\nabla_\alpha\psi + \varepsilon\xi R\psi^2 + 2U(\psi)\Big)\,,$$

where $\varepsilon = +1, -1$ corresponds to the canonical and the phantom scalar field, respectively.

Non-minimally coupled scalar field



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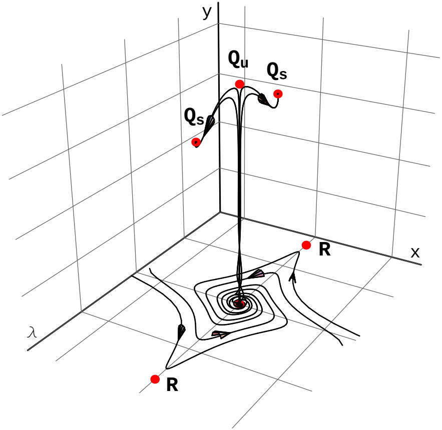


Asymptotically monomial potential functions

Inflationary paradigm:
"plateau-like" potential functions $\frac{U'(\psi)}{U(\psi)} \to 0$ as $\psi \to \infty$

Working assumption:
Let us assume, that starting from some values of the scalar field $\psi > \psi^*$ the potential function can be approximated as

$$U(\psi) = \pm\frac{1}{2}m^2\psi^2 \pm M^{4+n}\psi^{-n}\,,$$

where $n > -2$ and the second term constitutes small, asymptotically vanishing deviation.

Projective coordinates and analysis at infinity

We introduce the following dimensionless phase space variables

$$u = \frac{\dot{\psi}}{H\psi} = \frac{\mathrm{d}\ln\psi}{\mathrm{d}\ln a}\,, \quad v = \frac{\sqrt{6}}{\kappa}\frac{1}{\psi}\,,$$

the energy conservation condition

$$\frac{H^2}{H_0^2} = \frac{\mu + \alpha v^{2+n}}{v^2 - (1-6\xi)u^2 - 6\xi(u+1)^2}$$

where

$$\mu = \pm\frac{m^2}{H_0^2}\,, \quad \alpha = \pm 2\frac{M^{4+n}}{H_0^2}\left(\frac{\kappa}{\sqrt{6}}\right)^{2+n} \quad .$$

Instability of the initial de Sitter state

The asymptotic state under considerations

$$u^* = -\frac{2\xi}{1-4\xi}\,, \quad v^* = 0$$

eigenvalues of the linearisation matrix

$$\lambda_1 = -4 + \frac{1}{1-4\xi}\,, \quad \lambda_2 = \frac{4\xi}{1-4\xi}$$

an unstable node

$$\frac{3}{16} < \xi < \frac{1}{4}$$

the energy conservation condition

$$\left(\frac{H(0)}{H(a_0)}\right)^2\Bigg|^* = -\mu\frac{(1-4\xi)^2}{2\xi(1-6\xi)(3-16\xi)}\,,$$

subject to condition $\mu = -\frac{m^2}{H_0^2}$.

Physics from dynamics

The general form of the energy conservation condition

$$\frac{3}{\kappa^2}H^2 = \rho_{\text{eff}} < \rho_{\text{Pl}} = m_{\text{Pl}}^4\,.$$

We have that at the asymptotic unstable de Sitter state

$$\left.\frac{H^2}{H_0^2}\right|^* = -\mu\frac{(1-4\xi)^2}{2\xi(1-6\xi)(3-16\xi)} < \frac{\kappa^2 m_{\text{Pl}}^4}{3H_0^2}\,,$$

and we obtain the following inequality

$$\frac{m^2}{m_{\text{Pl}}^2}\frac{3}{8\pi}\frac{(1-4\xi)^2}{2\xi(1-6\xi)(3-16\xi)} < 1\,.$$

We can observe that even for the mass of the scalar field of order of the Planck mass $m^2 \simeq m_{\text{Pl}}^2$ we can find values of the non-minimal coupling constant that satisfy this relation and potential quantum gravity effects are excluded from the model.

ξ and $U(\psi) = -\frac{1}{2}m^2\psi^2 + M^{4+n}\psi^{-n}$

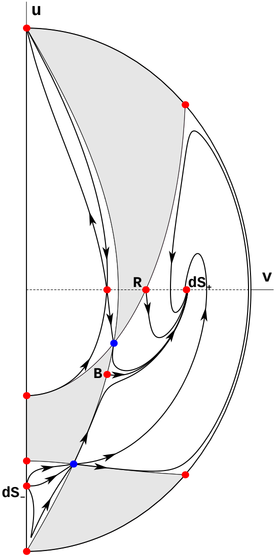


Figure: $\varepsilon = +1$, $\xi = \frac{7}{32}$, $n = 1$, $\mu = -1$, $\alpha = 3$

The special case $\xi = \frac{1}{4}$

New projective coordinate

$$\hat{u} = \frac{1}{u} = H\frac{\psi}{\dot{\psi}}$$

We are interested in dynamics in vicinity of the critical point

$$\hat{u}^* = 0\,, \quad v^* = 0$$

Simple inspection gives that at the asymptotic state the energy conservation condition vanishes together with the cosmological time derivative of the Hubble function

$$\left.\frac{H^2}{H_0^2}\right|^* = 0\,, \quad \left.\frac{\dot{H}}{H_0^2}\right|^* = 0\,,$$

which gives rise to the Einstein static universe. The acceleration equation calculated at this state is

$$\left.\frac{\dot{H}}{H^2}\right|^* = -3\,,$$

which suggest that the Einstein static state under considerations is filled with effective substance in the form of Zeldovich stiff matter with equation of state parameter $w_{\text{eff}} = 1$.

$\xi = \frac{1}{4}$ and asymptotically quadratic potential function

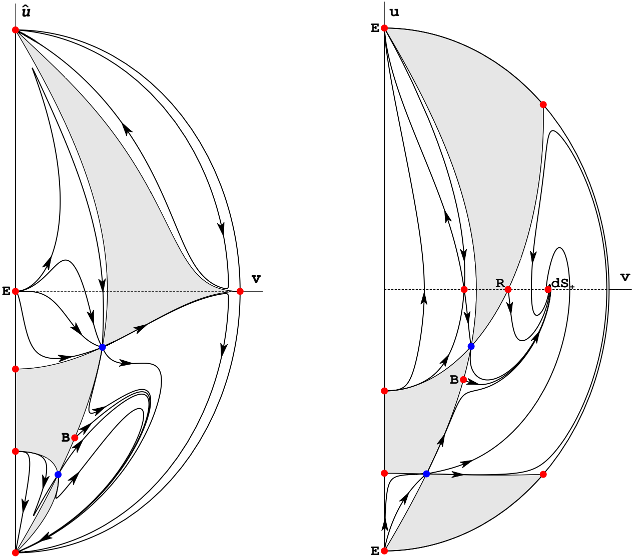


Figure: $\varepsilon = +1$, $\xi = \frac{1}{4}$, $n = 1$, $\mu = -1$, $\alpha = 3$

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} = \left|1 - \xi\kappa^2\psi^2\right| g_{\mu\nu} \,.$$

Now, the total action integral for the theory in the Jordan frame is

$$S = \mathrm{sgn}\left(1 - \xi\kappa^2\psi^2\right)\left(\tilde{S}_{\tilde{g}} + \tilde{S}_{\psi}\right) = \mathrm{sgn}\left(1 - \xi\kappa^2\psi^2\right)\tilde{S}_E \,,$$

where sgn(.) stands for the sign function and action integrals for the theory in the Jordan frame transform to

$$\tilde{S}_{\tilde{g}} = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-\tilde{g}}\,\tilde{R}\,,$$

and

$$\tilde{S}_{\phi} = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-\tilde{g}}\left(\frac{\omega(\psi)}{\psi}\tilde{\nabla}^{\alpha}\psi\,\tilde{\nabla}_{\alpha}\psi + 2\tilde{U}(\psi)\right)\,,$$

where

$$\omega(\psi) = \frac{1 - \xi(1-6\xi)\kappa^2\psi^2}{\left(1 - \xi\kappa^2\psi^2\right)^2}\psi\,,$$

and the scalar field potential function

$$\tilde{U}(\phi) = \mathrm{sgn}\left(1 - \xi\kappa^2\psi^2\right)\frac{U(\psi)}{\left(1 - \xi\kappa^2\psi^2\right)^2}\,.$$

The Einstein frame dynamical analysis

We introduce the following dimensionless phase space variables

$$\tilde{u} = \frac{\dot{\psi}}{\tilde{H}\psi}, \quad v = \frac{\sqrt{6}}{\kappa}\frac{1}{\psi},$$

where now a dot denotes differentiation with respect to time $\tilde{t}$.
With the dimensionless parameters

$$\tilde{\mu} = \pm\frac{m^2}{\tilde{H}_0^2}, \quad \tilde{\alpha} = \pm 2\frac{M^{4+n}}{\tilde{H}_0^2}\left(\frac{\kappa}{\sqrt{6}}\right)^{2+n},$$

the Einstein frame energy conservation condition is

$$\frac{\tilde{H}^2}{\tilde{H}_0^2} = \frac{\mathrm{sgn}\left(v^2 - 6\xi\right)\ v^2\left(\tilde{\mu} + \tilde{\alpha}v^{n+2}\right)}{\left(v^2 - 6\xi\right)^2 - \left(v^2 - 6\xi(1-6\xi)\right)\tilde{u}^2}.$$

The Einstein frame dynamical analysis

The asymptotic state under considerations

$$\tilde{u}^* = -\frac{2\xi}{1-6\xi}\,, \quad v^* = 0\,,$$

gives the vanishing energy conservation condition and the cosmological time derivative of the Hubble function

$$\frac{\tilde{H}^2}{\tilde{H}_0^2}\bigg|^* = 0\,, \quad \frac{\dot{\tilde{H}}}{\tilde{H}_0^2}\bigg|^* = 0\,,$$

and give rise to the Einstein static universe.
The eigenvalues of the linearisation matrix

$$\lambda_1 = -\frac{3-16\xi}{1-6\xi}\,, \quad \lambda_2 = \frac{2\xi}{1-6\xi}\,,$$

which corresponds to a stable node for $\xi > \frac{3}{16}$.

Dynamical variables transformation between the frames

The phase space variables in the Einstein and the Jordan frame are

$$\tilde{u} = \frac{\frac{\mathrm{d}\psi}{\mathrm{d}\tilde{t}}}{\tilde{H}\psi}\,, \qquad u = \frac{\frac{\mathrm{d}\psi}{\mathrm{d}t}}{H\psi}\,,$$

using the following transformations between the frames

$$\mathrm{d}\tilde{t} = \Omega \mathrm{d}t\,, \qquad \tilde{H} = \frac{1}{\Omega}\left(H + \frac{1}{\Omega}\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right),$$

we find the relation between phase space variables in the Jordan and the Einstein frames

$$\frac{u}{\tilde{u}} = 1 - \frac{6\xi}{v^2 - 6\xi}u\,.$$

At the asymptotic states under considerations with $v^* \equiv 0$ we have

$$u^* = -\frac{2\xi}{1-4\xi} \quad \Longleftrightarrow \quad \tilde{u}^* = -\frac{2\xi}{1-6\xi}\,,$$

and we can conclude that for the non-minimal coupling constant $\frac{3}{16} < \xi < \frac{1}{4}$ the unstable de Sitter state in the Jordan frame corresponds to the stable Einstein static state in the Einstein frame.

Physical nonequivalence of the frames

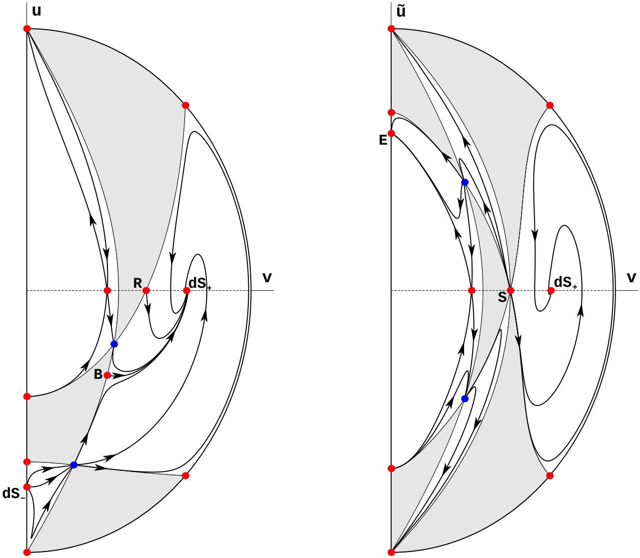


Figure: $\varepsilon = +1$, $\xi = \frac{7}{32}$, $n = 1$, $\mu = -1$, $\alpha = 3$

Non-minimal coupling and the conformal invariance

The Klein-Gordon equation for the scalar field with a monomial potential function is in the following form

$$\Box\psi - \xi R\psi - n\,U_0\psi^{n-1} = 0\,.$$

Using appropriate conformal or Weyl transformation we find

$$\begin{aligned}\tilde{\Box}\tilde{\psi} - \xi\tilde{R}\tilde{\psi} - n\,U_0\tilde{\psi}^{n-1} &= \\ \Omega^{-\frac{D+2}{2}}\left(\Box\psi - \xi R\psi - n\,U_0\psi^{n-1}\right) &= 0\,,\end{aligned}$$

and this equation holds iff the parameters are

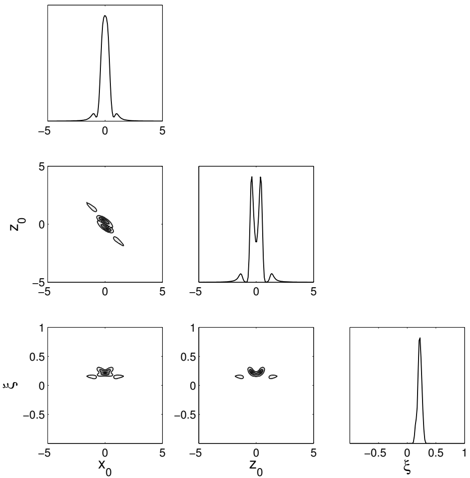
$$\xi = \xi_{\rm conf} = \frac{1}{4}\frac{D-2}{D-1}\,, \quad n = n_{\rm conf} = \frac{2D}{D-2}\,.$$

Non-minimal coupling and the conformal invariance

Hence, we obtain the following discrete set of theoretically motivated values of the non-minimal coupling constant and the exponent of a monomial scalar field potential function suggested by the conformal invariance condition in $D \geq 2$ space-time dimensions

$$\Big\{(D,\xi,n)\Big\} = \Big\{\,(2,0,\infty)\,,\left(3,\frac{1}{8},6\right),\left(4,\frac{1}{6},4\right), \left(5,\frac{3}{16},\frac{10}{3}\right),\ldots,\left(\infty,\frac{1}{4},2\right)\Big\}\,.$$

Observational constraints



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Observational data:
Union2.1+H(z)+Alcock-Paczyński test

Conclusions

- We have found that for generic scalar field potential functions which asymptotically tend to a quadratic form at infinite values of the scalar field there is an unstable critical point corresponding to: the de Sitter state for non-minimal coupling constant $\frac{3}{16} < \xi < \frac{1}{4}$; and static Einstein universe filled with effective matter in the form of Zeldovich stiff matter for the non-minimal coupling constant $\xi = \frac{1}{4}$.
- Using dynamical systems methods we were able to directly compare dynamics of the model in original Jordan frame and conformally transformed Einstein frame. We have shown that both descriptions are physically nonequivalent since the initial unstable de Sitter state in the Jordan frame is transformed into the stable Einstein static state in the Einstein frame.

Healthy speculations ?

- Does $\xi = \frac{1}{4}$ allow us to suggest conformal invariance of the theory ?
- Naturalness of $\xi = \frac{1}{4}$?
- Can pseudo-Riemannian manifold of a gravitational theory be infinite dimensional ?
- Do we have to abandon current ontological interpretation of pseudo-Riemannian geometry as a language for gravitational interactions ?
- EFT of gravitation and (a)temporal emergence of space-time ?
- Revival of John A. Wheeler program of pre-geometric (non-geometric) approach to gravitational interaction ?
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or maybe no dimensions at all.