UNIWERSYTET MIKOŁAJA KOPERNIKA W TORUNIU

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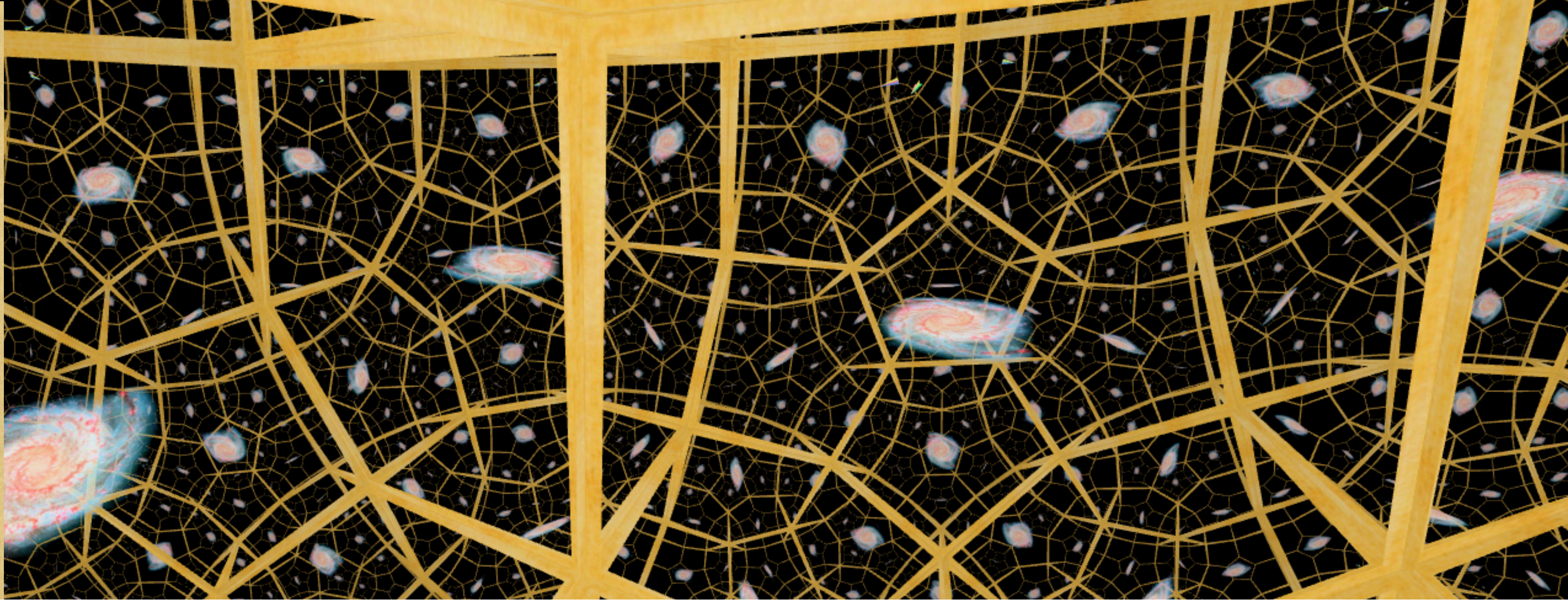
19 September 2022



NON-RELATIVISTIC REGIME AND TOPOLOGY : CONSEQUENCES FOR THE ROLE OF SPATIAL CURVATURE IN COSMOLOGY

Quentin Vigneron

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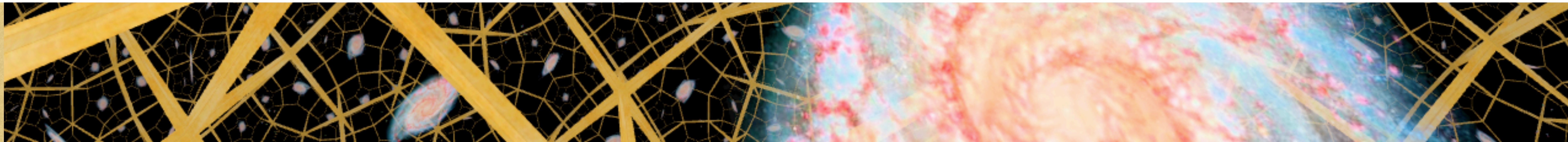
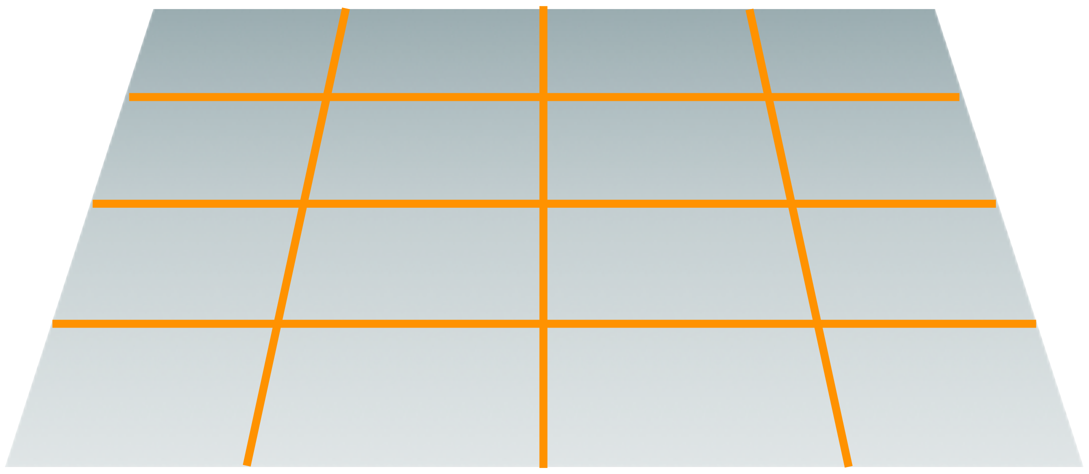


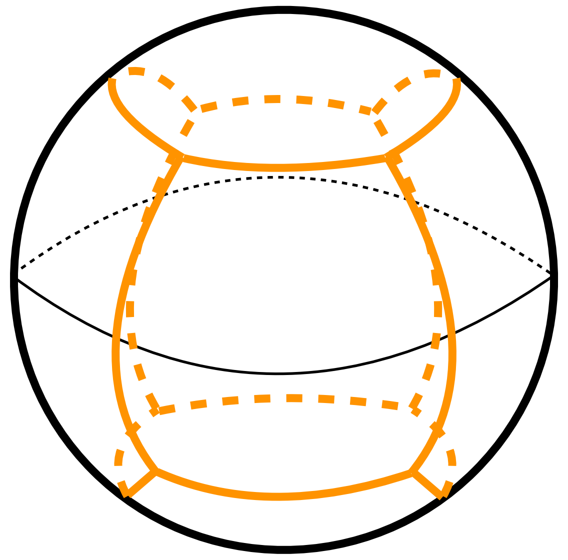
Image: *Curved Spaces*, Jeffrey Weeks

Classification of closed 3-manifolds Σ (Thurston's classification):

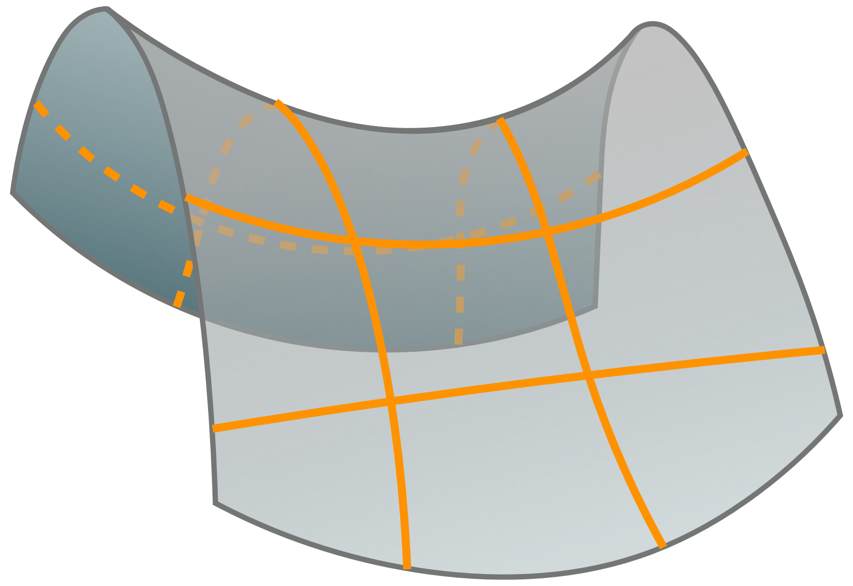
Euclidean **topology**



Spherical **topology**



Hyperbolic **topology**



+ Others

Not characterised by the scalar curvature, but by topological properties:

The covering space $\tilde{\Sigma}$: $\mathbb{E}^3$ $\underbrace{\mathbb{S}^3 \quad \mathbb{H}^3 \quad \cdots}_{\text{"Non-Euclidean topologies"}}$

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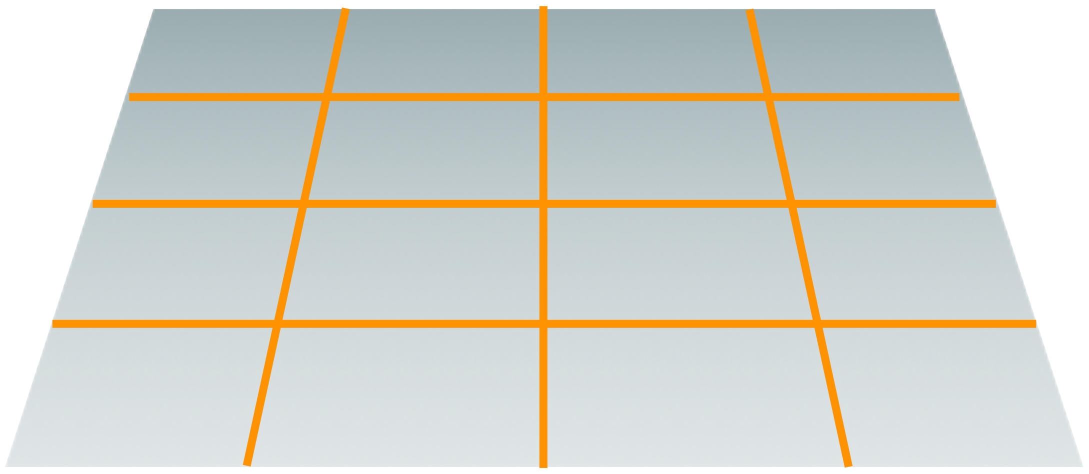

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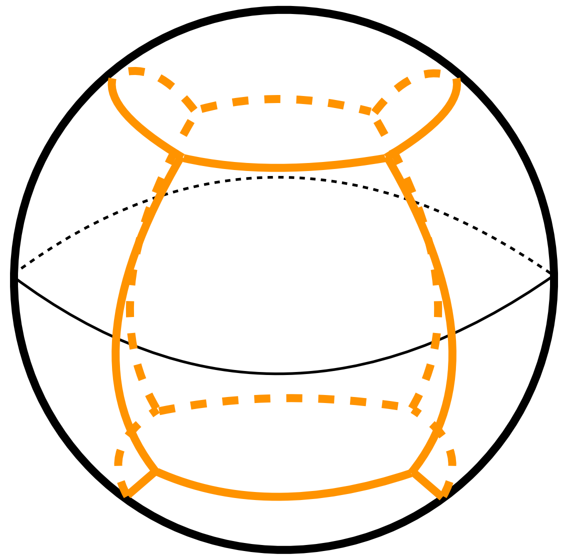

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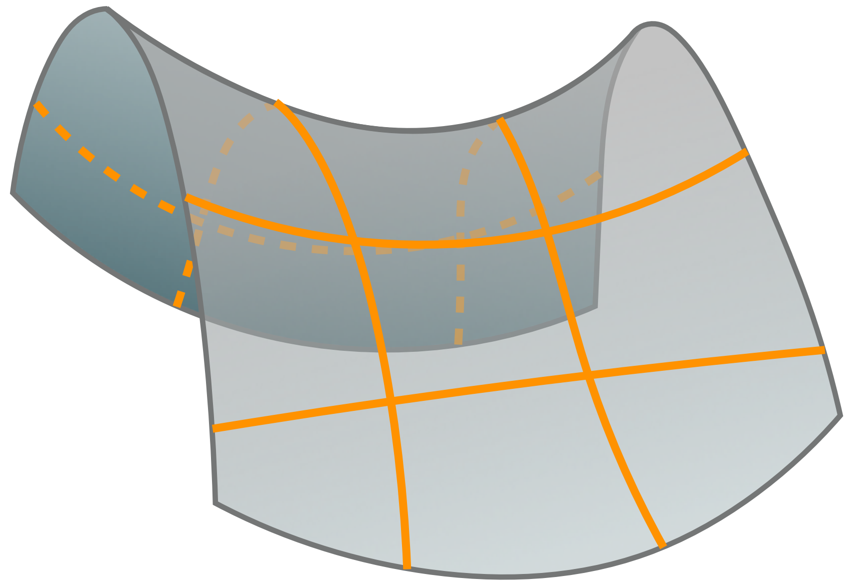

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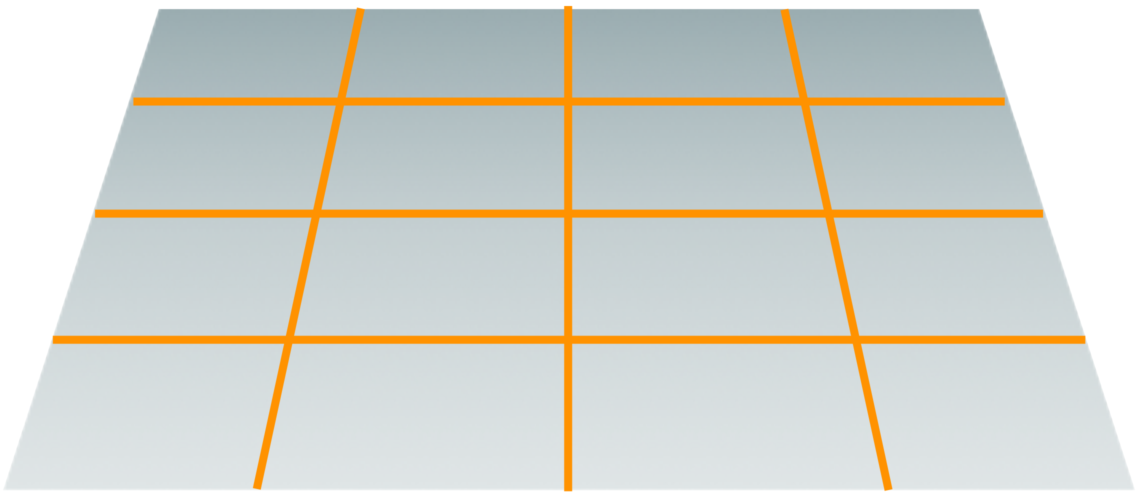
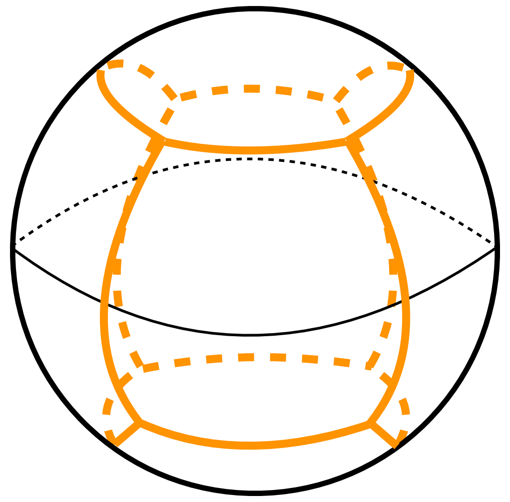
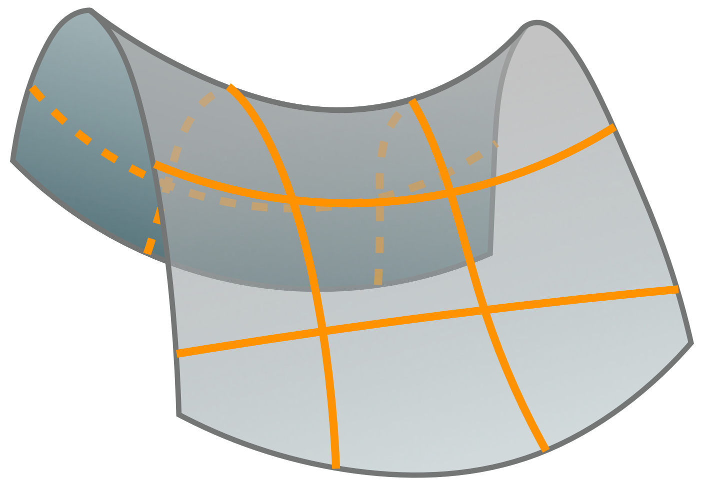
For a homogeneous and isotropic solution:

$$\mathscr{R}_{ij} = 0 \qquad \mathscr{R}_{ij} = 2K\, h_{ij} \qquad \mathscr{R}_{ij} = 2K\, h_{ij}$$

In general:

$$\mathscr{R}_{ij} \neq 0 \qquad \mathscr{R}_{ij} \neq 2K\, h_{ij} \qquad \mathscr{R}_{ij} \neq 2K\, h_{ij}$$

with $\mathscr{R}_{ij} \neq 0$ with $\mathscr{R}_{ij} \neq 0$

Class of topology (i.e. covering space $\tilde{\Sigma}$) / Physical regime	Euclidean ($\tilde{\Sigma} = \mathbb{E}^3$) 	Non-Euclidean ($\tilde{\Sigma} \neq \mathbb{E}^3$): Spherical ($\tilde{\Sigma} = \mathbb{S}^3$)  , Hyperbolic ($\tilde{\Sigma} = \mathbb{H}^3$)  , Others
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$\downarrow$ $c \to \infty$		
Non-relativistic regime (i.e. Newtonian)	$\Delta\Phi = 4\pi G \left(\rho - \langle\rho\rangle\right)$, $\mathscr{R}_{ij} = 0$ $\dots$	

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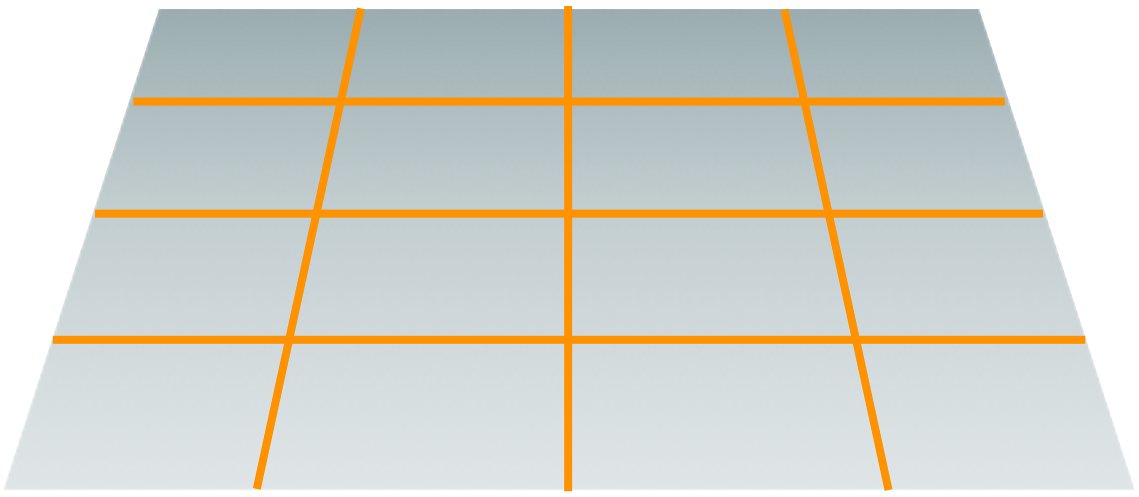
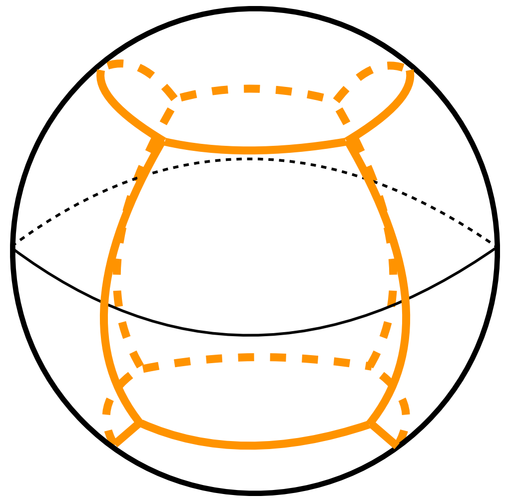
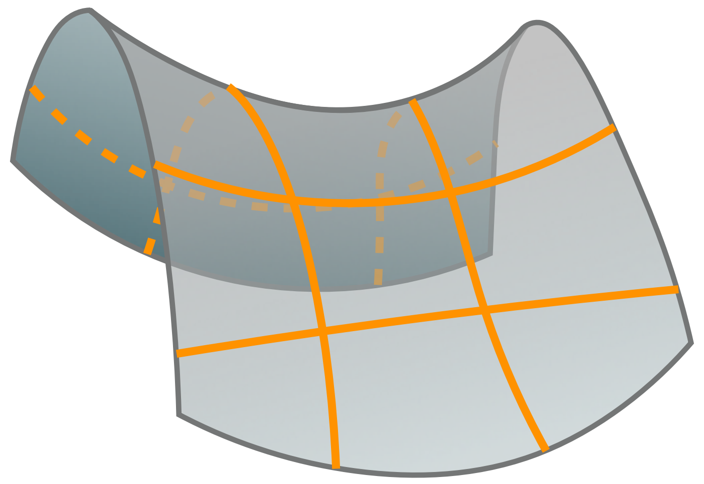
		 ,  , Others
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Questions:

How to define a Newtonian theory on a non-Euclidean topology?

Compatibility with Einstein's equation?

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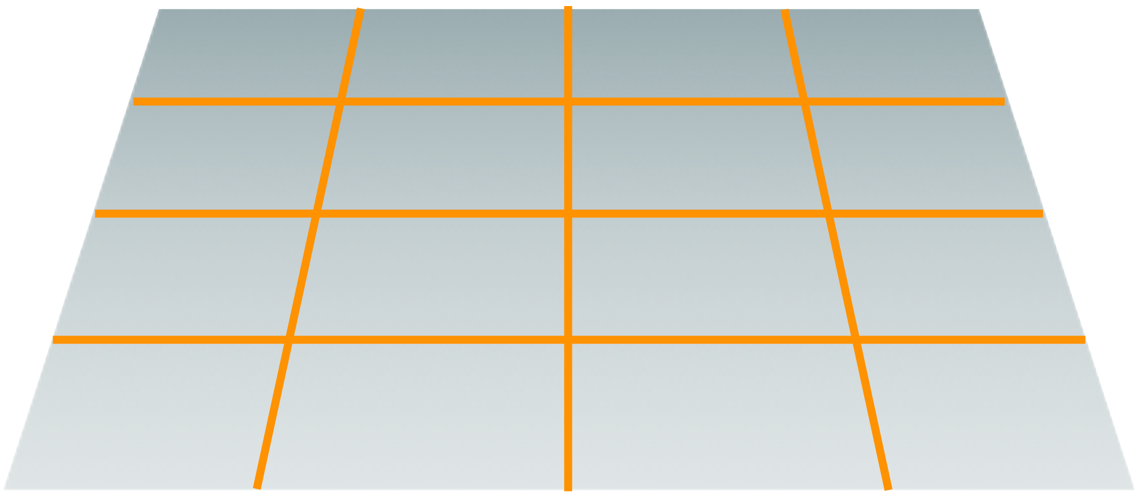
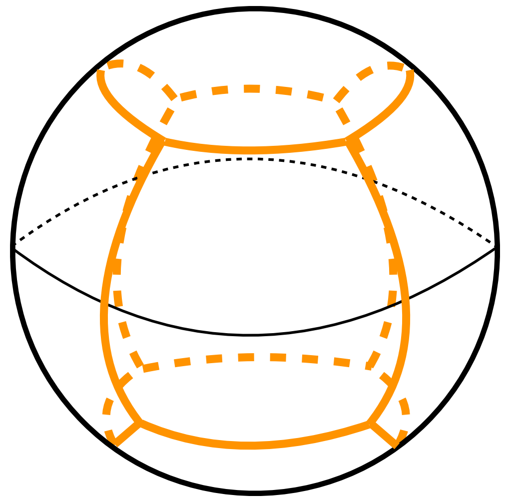
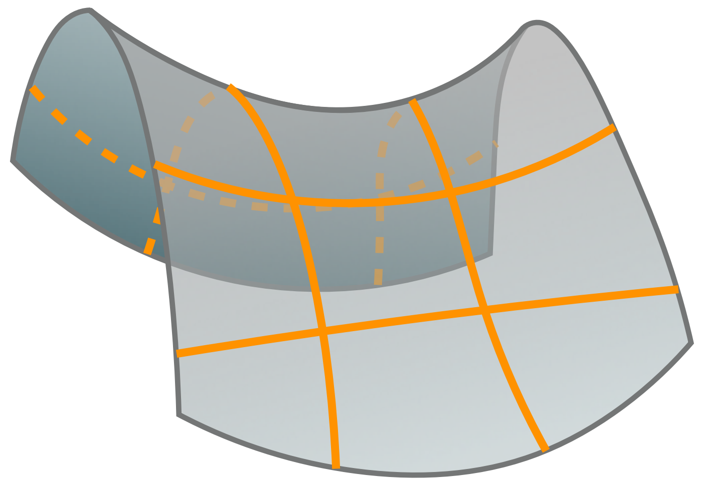
		Spherical ($\tilde{\Sigma} = \mathbb{S}^3$)  , Hyperbolic ($\tilde{\Sigma} = \mathbb{H}^3$)  , Others
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Study of the cosmological backreaction
[Vigneron (a) (2022), 2109.10336]

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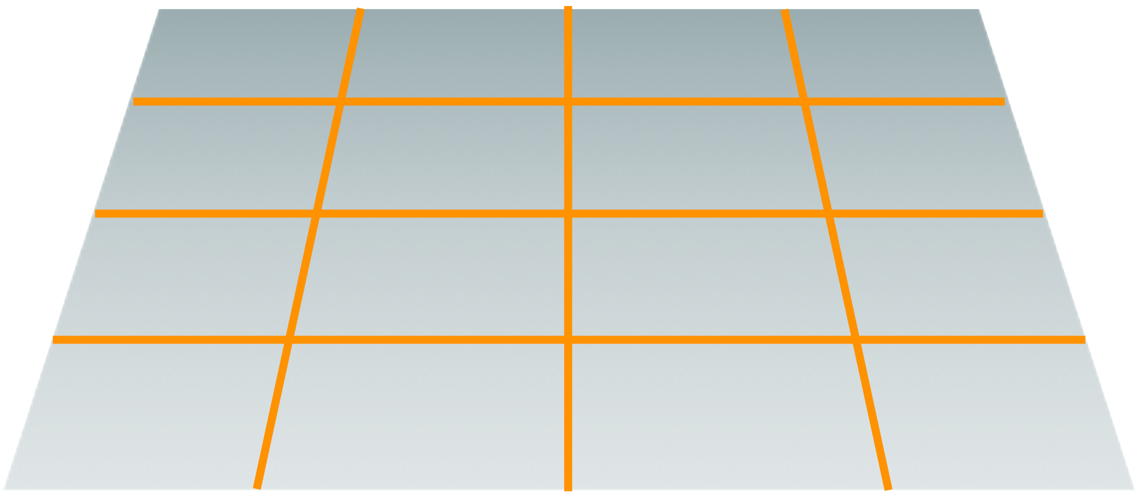
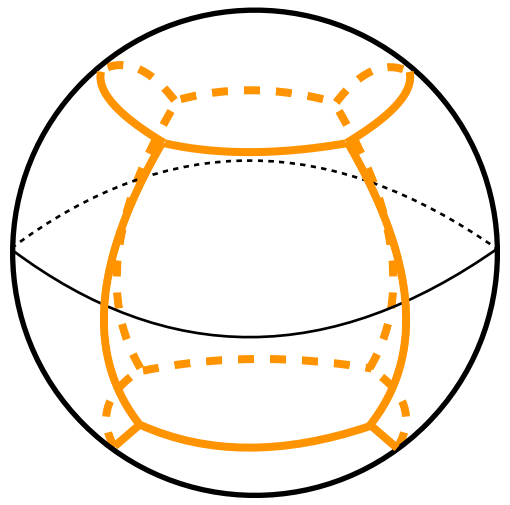
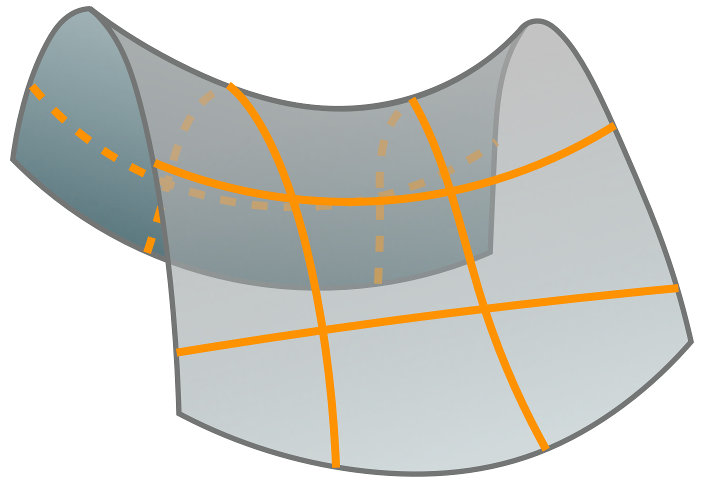
		Spherical ($\tilde{\Sigma} = \mathbb{S}^3$)  , Hyperbolic ($\tilde{\Sigma} = \mathbb{H}^3$)  , Others
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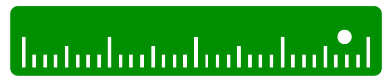
Lorentzian manifold

4-manifold $\mathscr{M}$ + Lorentzian structure $(\boldsymbol{g}, {}^4\nabla)$:

- $\boldsymbol{g}$ =  + 
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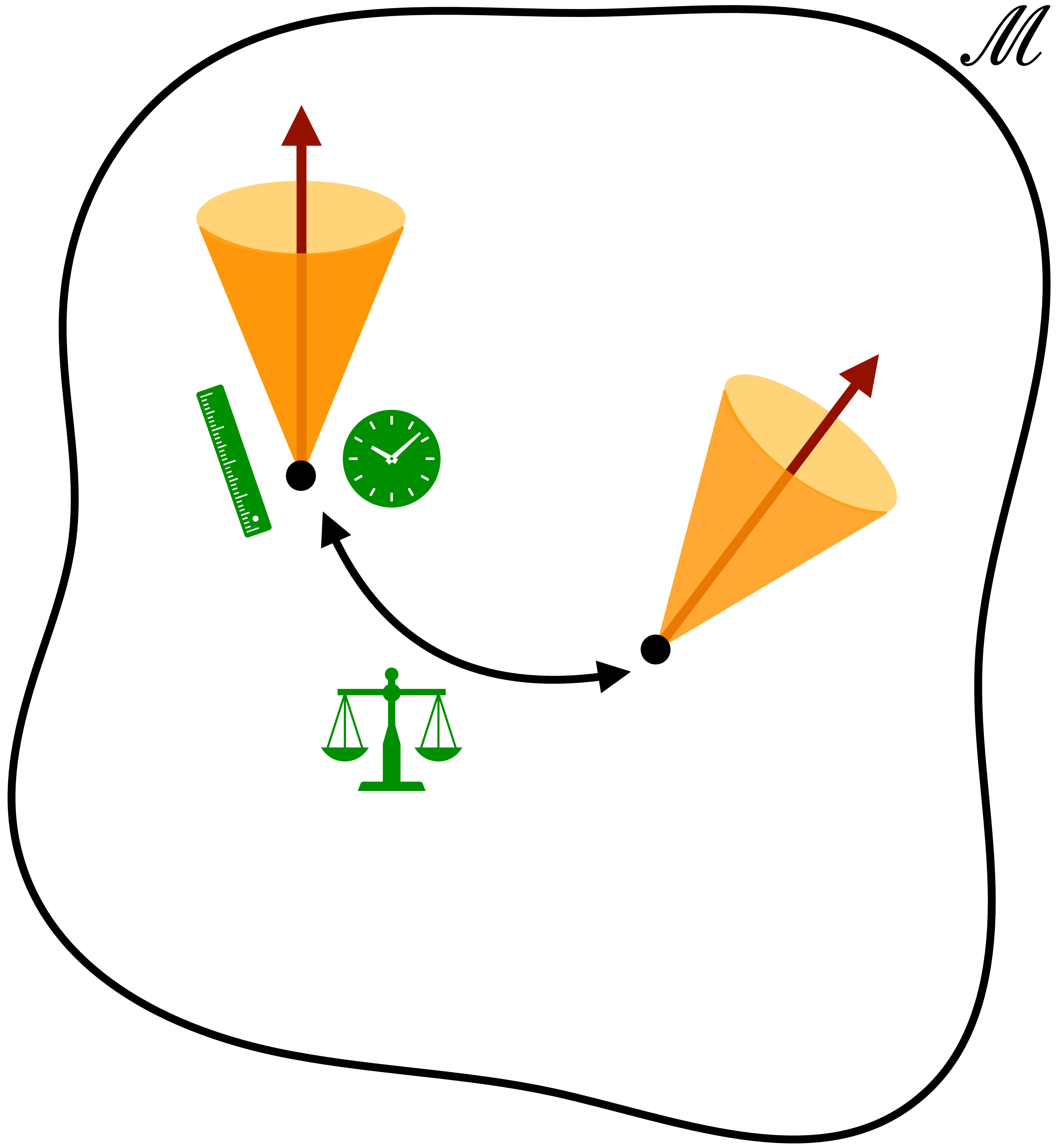

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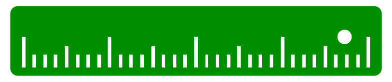
- $\boldsymbol{g}$ =  + 
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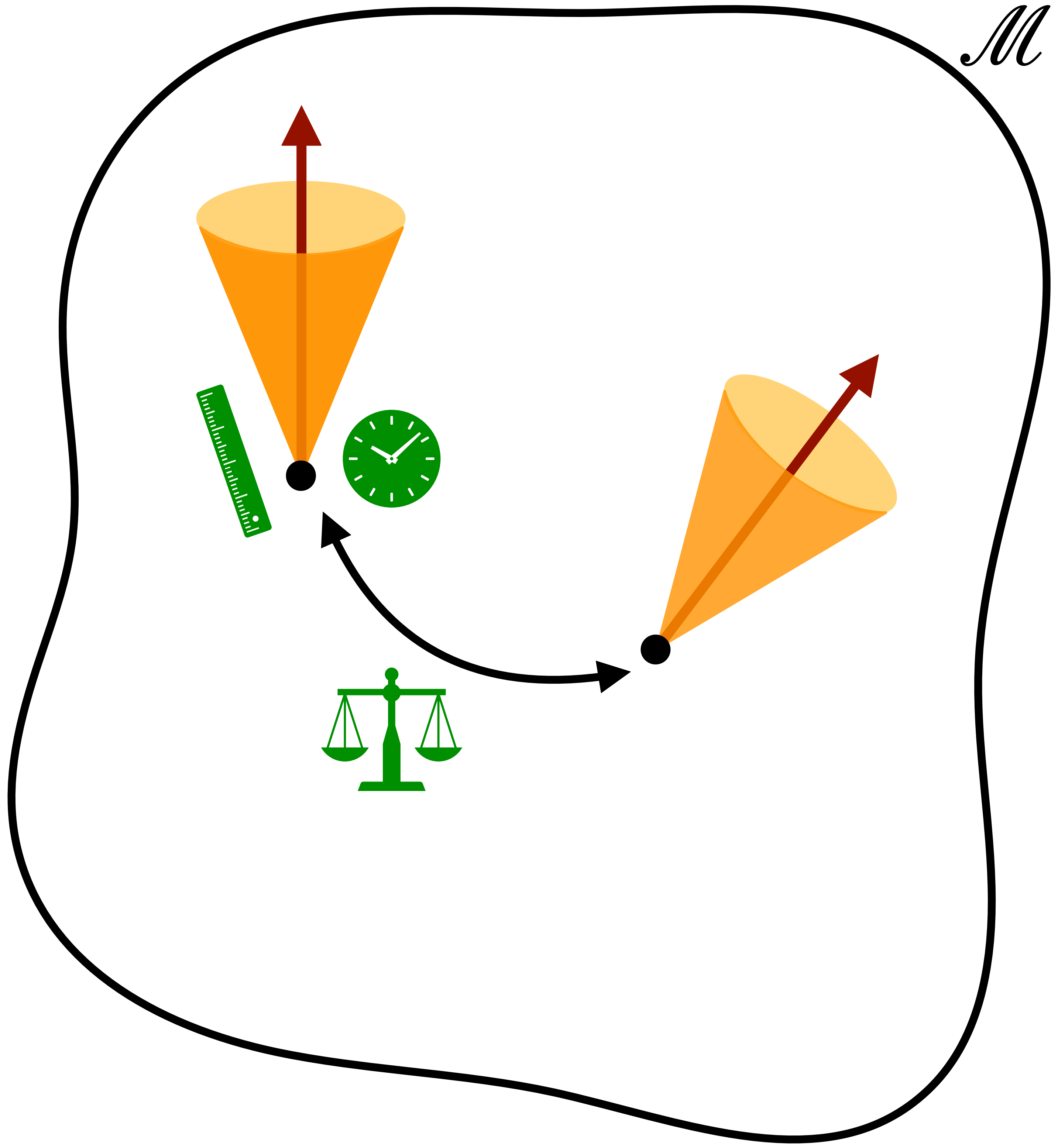

Galilean manifold [e.g. Künzle (1972)]

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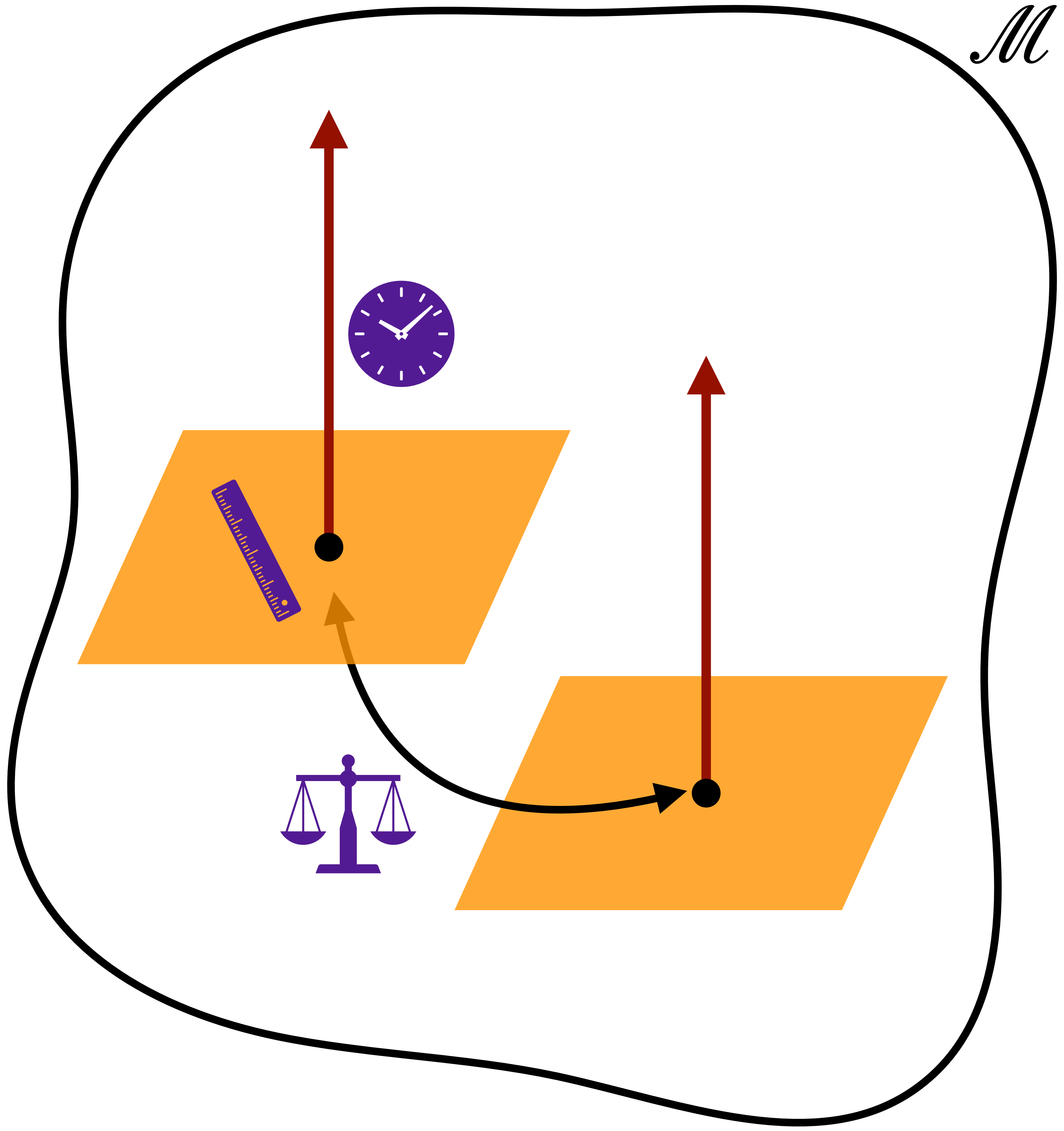

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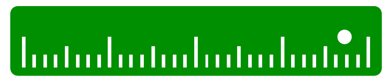
- $\boldsymbol{g}$ =  + 
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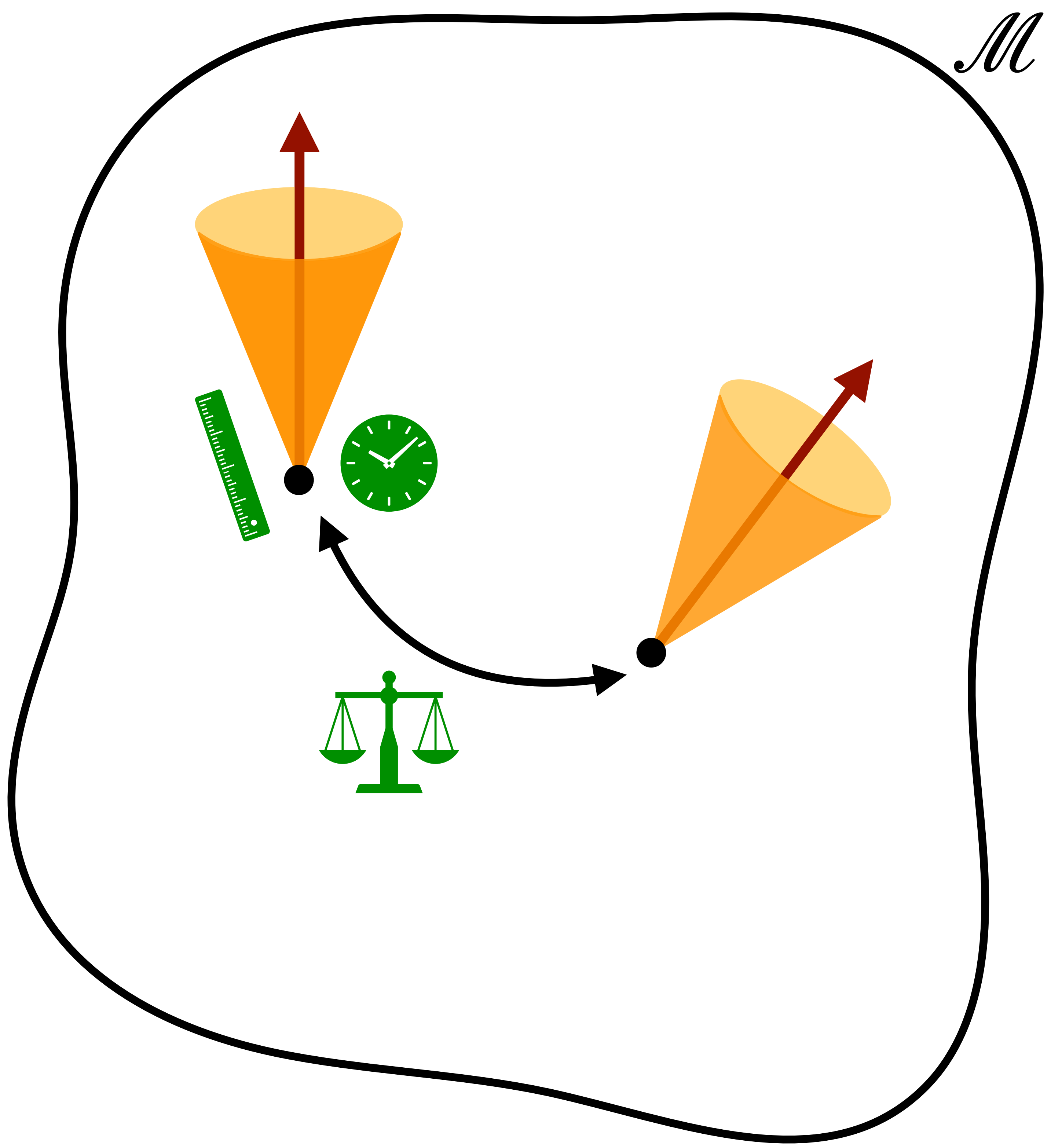
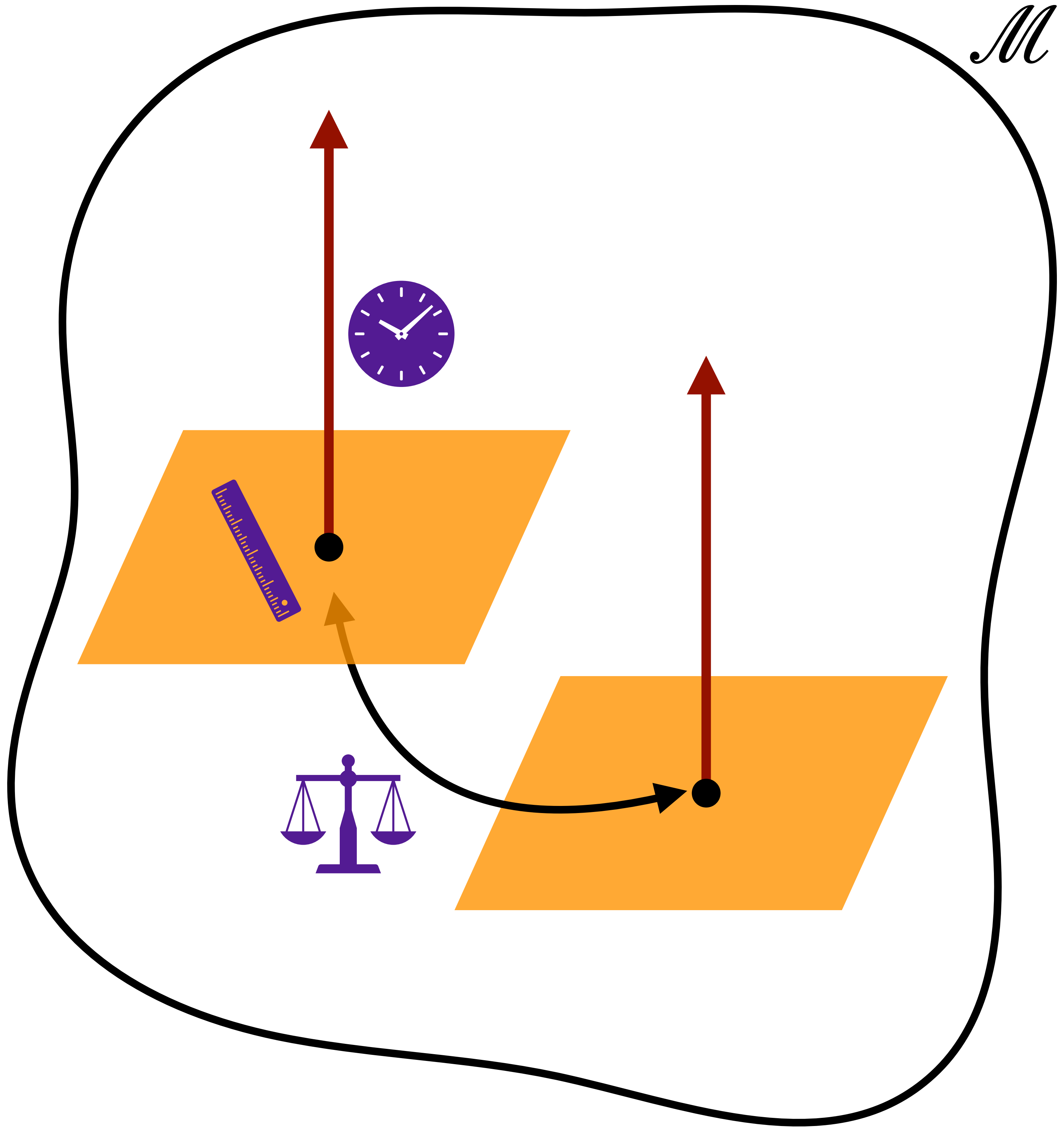


$$R^\mu{}_{\alpha\mu\beta} \neq \hat{R}^\mu{}_{\alpha\mu\beta}$$

Spacetime curvatures



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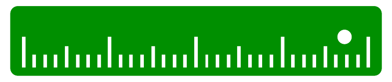
- $\boldsymbol{g}$ =  + 
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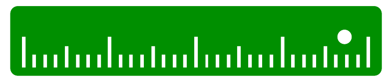
- $\boldsymbol{g}$ =  + 
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Spatial projection

Newton's equations

with expansion

[Vigneron (2021), 2012.10213]

$$\begin{cases} g^a = \dot{v}^a + 2Hv^a - (a_{\neq \mathrm{grav}})^a \\ \Delta\Phi = 4\pi G \left(\rho - \langle \rho \rangle\right) \\ \dot{\rho} + \rho\left(3H + \nabla_i v^i\right) = 0 \\ q = \Omega_\mathrm{m}/2 - \Omega_\Lambda \\ \mathscr{R}_{ij} = 0 \end{cases}$$

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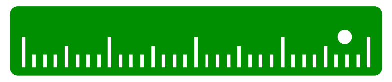
- $\boldsymbol{g}$ =  + 
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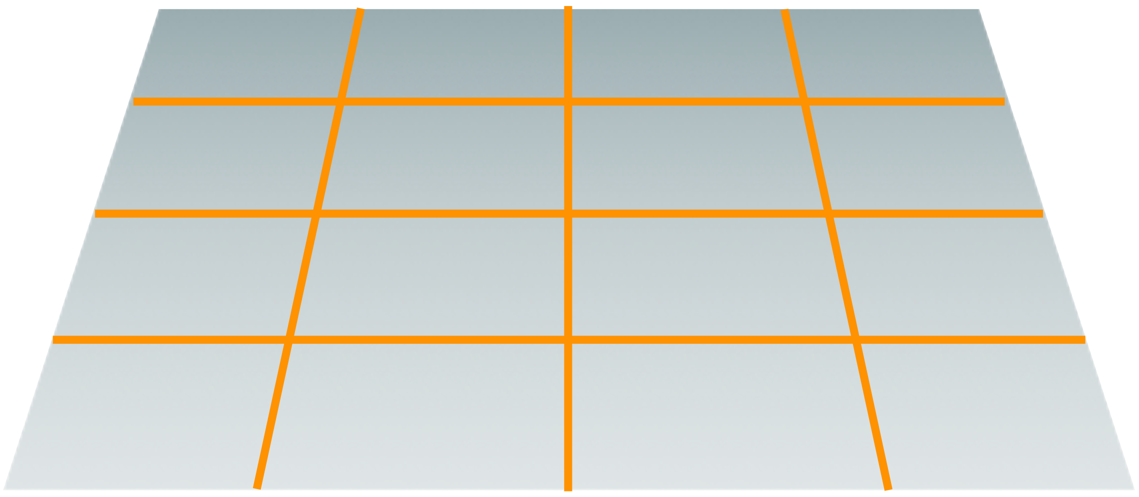
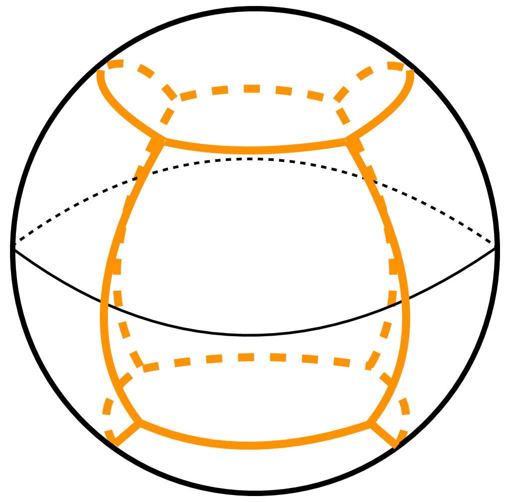
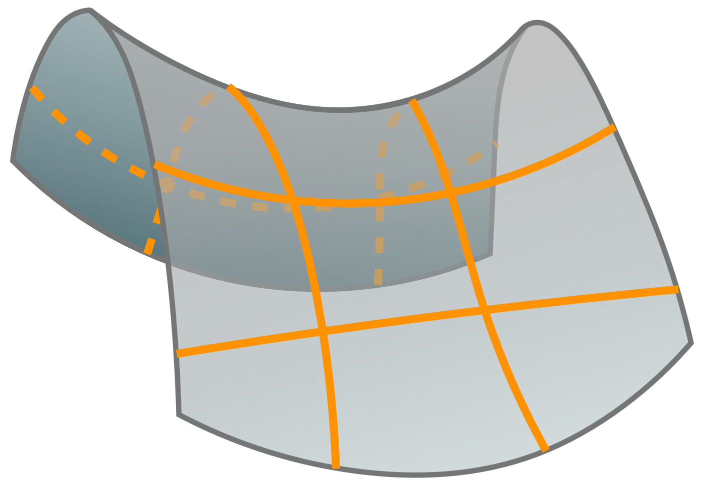
Definition: A **non-Euclidean Newtonian theory** is a theory defined on a *Galilean manifold* whose spatial sections have a *non-Euclidean topology*. It is given by equations relating the Riemann tensor of the Galilean structure to the energy content of the manifold. [Vigneron (b) (2022), 2201.02112]

Class of topology (i.e. covering space $\tilde{\Sigma}$) / Physical regime	Euclidean ($\tilde{\Sigma} = \mathbb{E}^3$) 	Non-Euclidean ($\tilde{\Sigma} \neq \mathbb{E}^3$): Spherical ($\tilde{\Sigma} = \mathbb{S}^3$) , Hyperbolic ($\tilde{\Sigma} = \mathbb{H}^3$) , Others
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?

NON-EUCLIDEAN NEWTONIAN THEORY

Newton-Cartan equation: $\hat{R}_{\mu\nu} = \kappa T^{*}_{\mu\nu} - \Lambda \tau_\mu \tau_\nu$

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Newton-Cartan equation: $\hat{R}_{\mu\nu} = \kappa T^{*}_{\mu\nu} - \Lambda \tau_\mu \tau_\nu \longrightarrow \mathscr{R}_{ij} = 0$

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Hypothesis for the additional term: **[**Künzle (1976); Vigneron (b) (2022), 2201.02112**]**

- Zero for Euclidean topology
- Present in vacuum
- Second order or less in the metrics

$$\hat{R}_{\mu\nu} - 2Kb_{\mu\nu} = \kappa T^{*}_{\mu\nu} - \Lambda \tau_{\mu} \tau_{\nu} \longrightarrow \mathscr{R}_{ij} = 2Kh_{ij}$$

(Spherical or hyperbolic)

$b_{\mu\nu}$:= orthogonal projector to a reference 4-vector $\boldsymbol{G}$, (Galilean observer)

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Spatial equations of the non-Euclidean Newtonian theory: with $\mathscr{R}_{ij} = 2Kh_{ij} \neq 0$

[Vigneron (b) (2022), 2201.02112]
[Vigneron & Roukema (2022), preprint: 2201.09102]

2nd law of Newton:

$$g^i = \dot{v}^i + 2Hv^i - (a_{\neq \mathrm{grav}})^i$$

Constraints on the gravitational field:

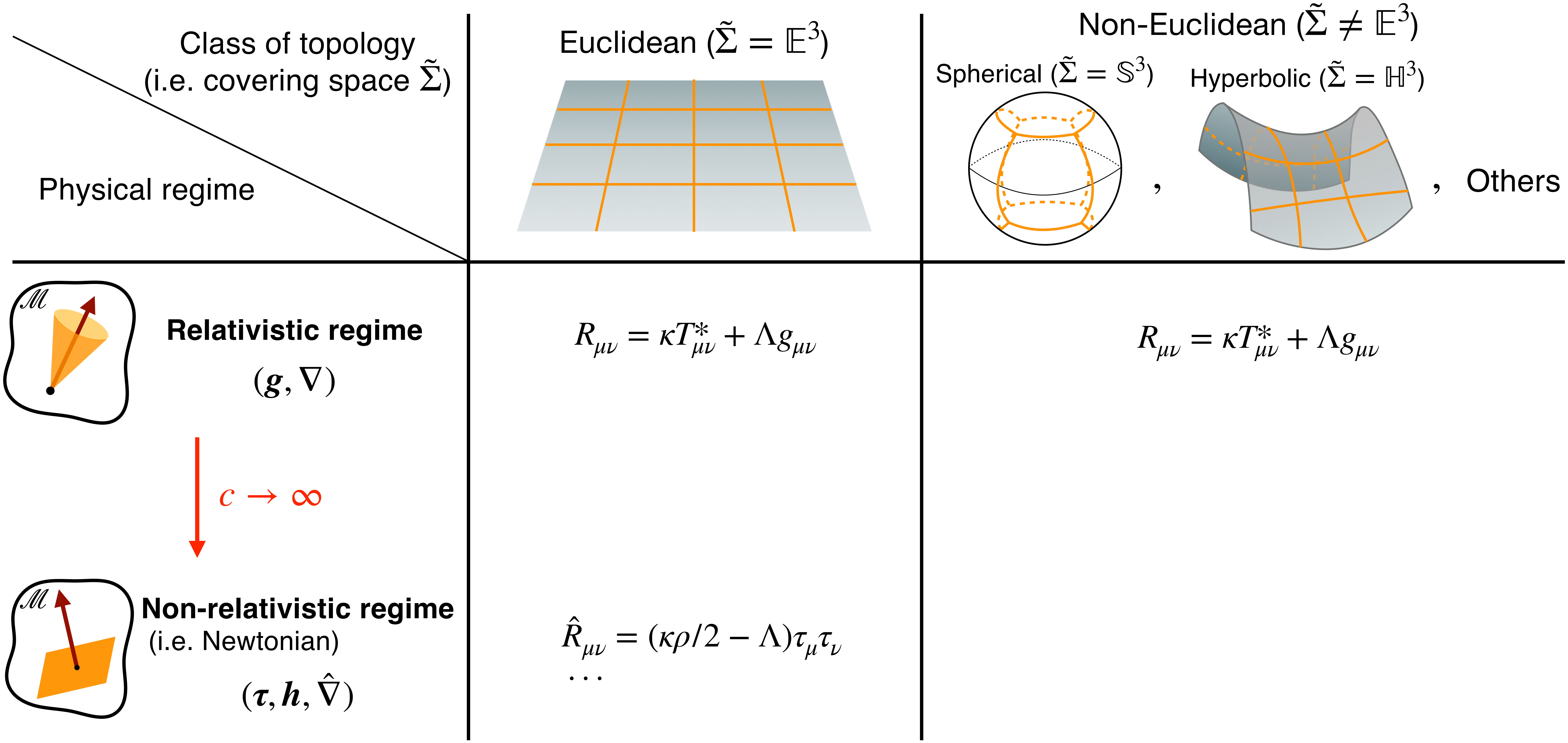
$$\nabla_c g^c = -4\pi G\left(\rho - \langle\rho\rangle\right), \quad \nabla_{[a} g_{b]} = 0 \longrightarrow \Delta\Phi = 4\pi G\left(\rho - \langle\rho\rangle\right)$$

Mass conservation:

$$\dot{\rho} + 3H\rho + \rho \nabla_i v^i = 0$$

Expansion laws:

$$\begin{cases} \Omega_{\neq \mathrm{cst}} + \Omega_{\mathrm{cst}} = 1, \\ q = \Omega_{\mathrm{m}}/2 - \Omega_\Lambda \end{cases}$$



Class of topology
(i.e. covering space $\tilde{\Sigma}$)
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$R_{\mu\nu} = \kappa T^*_{\mu\nu} + \Lambda g_{\mu\nu}$
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$\mathscr{M}$
Non-relativistic regime
(i.e. Newtonian)
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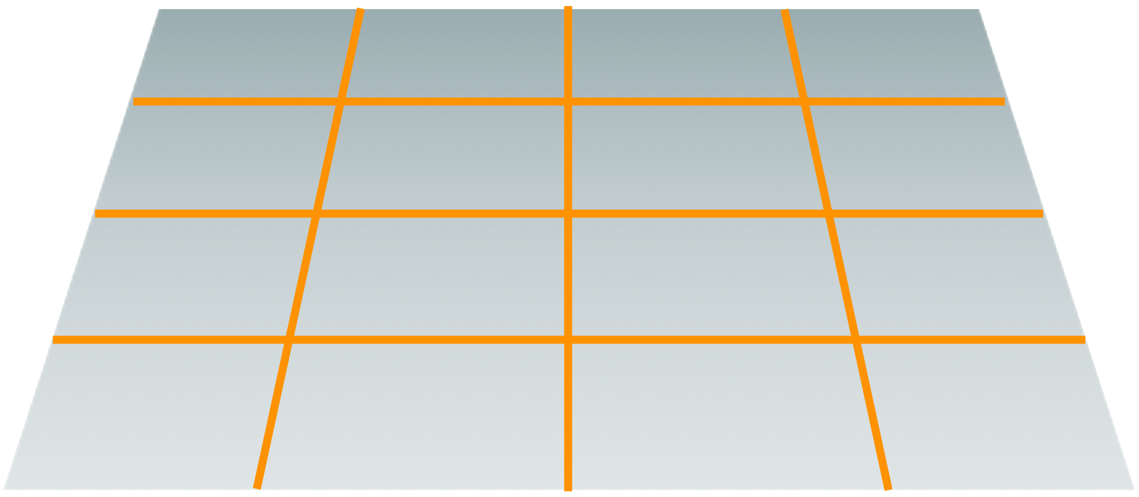
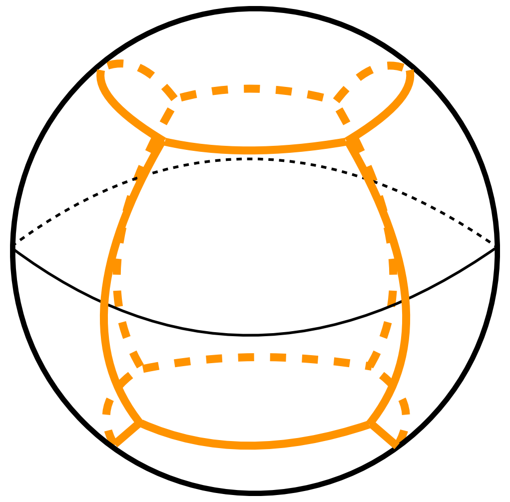
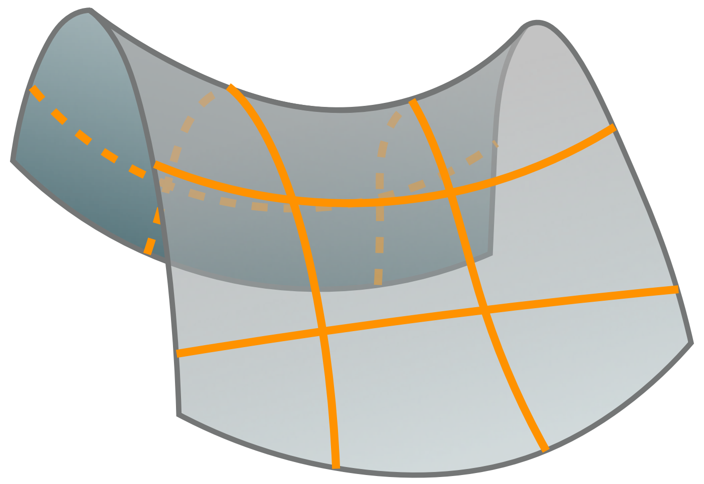
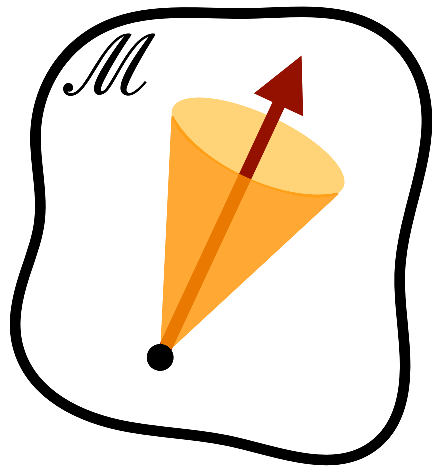
Class of topology (i.e. covering space $\tilde{\Sigma}$) Physical regime	Euclidean ($\tilde{\Sigma} = \mathbb{E}^3$)	Non-Euclidean ($\tilde{\Sigma} \neq \mathbb{E}^3$)
		Spherical ($\tilde{\Sigma} = \mathbb{S}^3$)  , Hyperbolic ($\tilde{\Sigma} = \mathbb{H}^3$)  , Others
 **Relativistic regime** $(\boldsymbol{g}, \nabla)$	$R_{\mu\nu} = \kappa T^*_{\mu\nu} + \Lambda g_{\mu\nu}$	$R_{\mu\nu} = \kappa T^*_{\mu\nu} + \Lambda g_{\mu\nu}$
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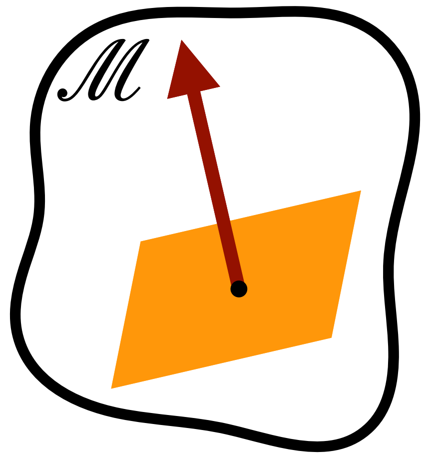 **Non-relativistic regime** (i.e. Newtonian) $(\boldsymbol{\tau}, \boldsymbol{h}, \hat{\nabla})$	$\hat{R}_{\mu\nu} = (\kappa\rho/2 - \Lambda)\tau_\mu\tau_\nu$ $\dots$	$\hat{R}_{\mu\nu} - 2Kb_{\mu\nu} = (\kappa\rho/2 - \Lambda)\tau_\mu\tau_\nu$ $\dots$

Vigneron Q., 2021, *1+3 -Newton-Cartan system and Newton-Cartan cosmology*, Phys. Rev. D, (arXiv:2012.10213)

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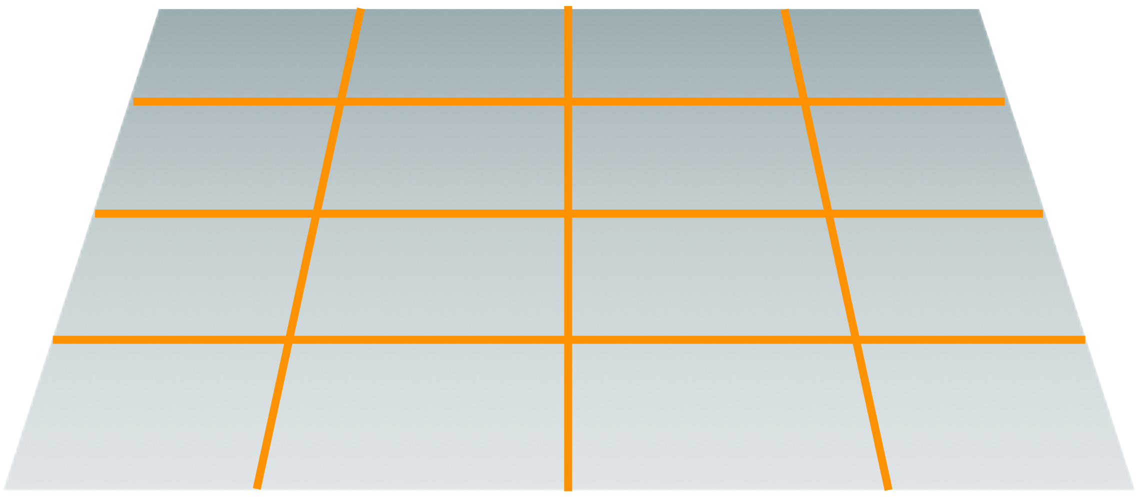
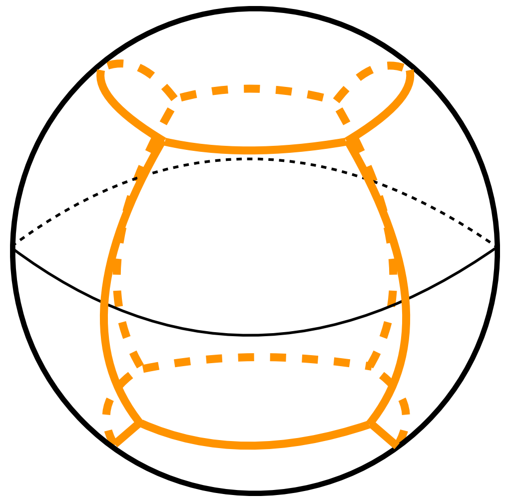
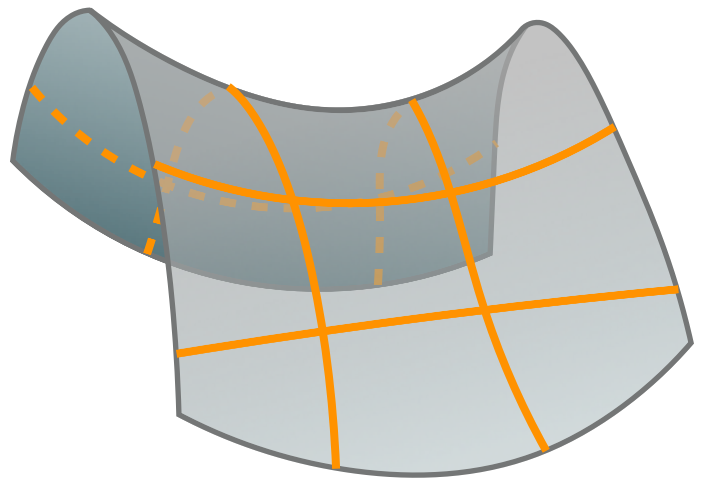
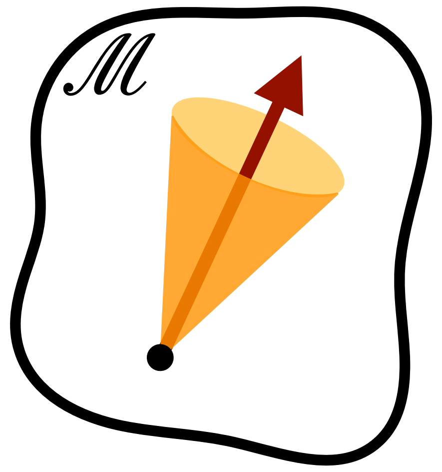
Class of topology (i.e. covering space $\tilde{\Sigma}$) / Physical regime	Euclidean ($\tilde{\Sigma} = \mathbb{E}^3$)	Non-Euclidean ($\tilde{\Sigma} \neq \mathbb{E}^3$): Spherical ($\tilde{\Sigma} = \mathbb{S}^3$), Hyperbolic ($\tilde{\Sigma} = \mathbb{H}^3$), Others
		 ,  , Others
 **Relativistic regime** $(\boldsymbol{g}, \nabla)$	$R_{\mu\nu} = \kappa T^*_{\mu\nu} + \Lambda g_{\mu\nu}$	$R_{\mu\nu} = \kappa T^*_{\mu\nu} + \Lambda g_{\mu\nu}$
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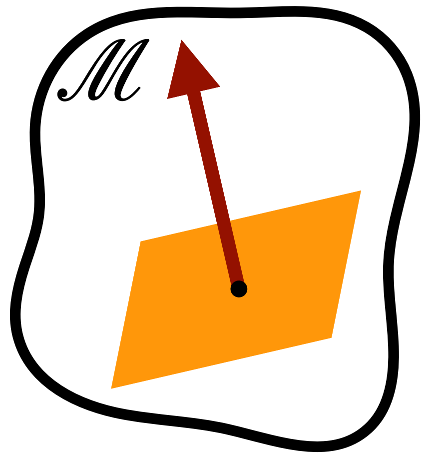 **Non-relativistic regime** (i.e. Newtonian) $(\boldsymbol{\tau}, \boldsymbol{h}, \hat{\nabla})$	$\hat{R}_{\mu\nu} = (\kappa\rho/2 - \Lambda)\tau_\mu\tau_\nu$ ...	$\hat{R}_{\mu\nu} - 2Kb_{\mu\nu} = (\kappa\rho/2 - \Lambda)\tau_\mu\tau_\nu$ ...

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The Galilean limit: a **limit of structures** (from a Lorentzian structure to a Galilean structure) [Künzle (1976)].

We consider a family of Lorentzian structures $\left\{({}^{\lambda}\boldsymbol{g}, {}^{\lambda}\Gamma^{\gamma}{}_{\alpha\beta})\right\}_{\lambda>0}$ on a 4-manifold $\mathscr{M}$ such that:

$$\begin{cases} {}^{\lambda}g^{\alpha\beta} = h^{\alpha\beta} + \mathcal{O}(\lambda) \\ {}^{\lambda}g_{\alpha\beta} = -\frac{1}{\lambda}\tau_\alpha\tau_\beta + \mathcal{O}(1) \end{cases} \longrightarrow \quad {}^{\lambda}\Gamma^{\gamma}{}_{\alpha\beta} = \hat{\Gamma}^{\gamma}{}_{\alpha\beta} + \mathcal{O}(\lambda = 1/c^2)$$

$\mathscr{M}$ equipped with $({}^{\lambda}\boldsymbol{g}, {}^{\lambda}\Gamma^{\gamma}{}_{\alpha\beta}) \xrightarrow{c\to\infty}$ $\mathscr{M}$ equipped with $(\boldsymbol{h}, \boldsymbol{\tau}, {}^{0}\Gamma^{\gamma}{}_{\alpha\beta})$

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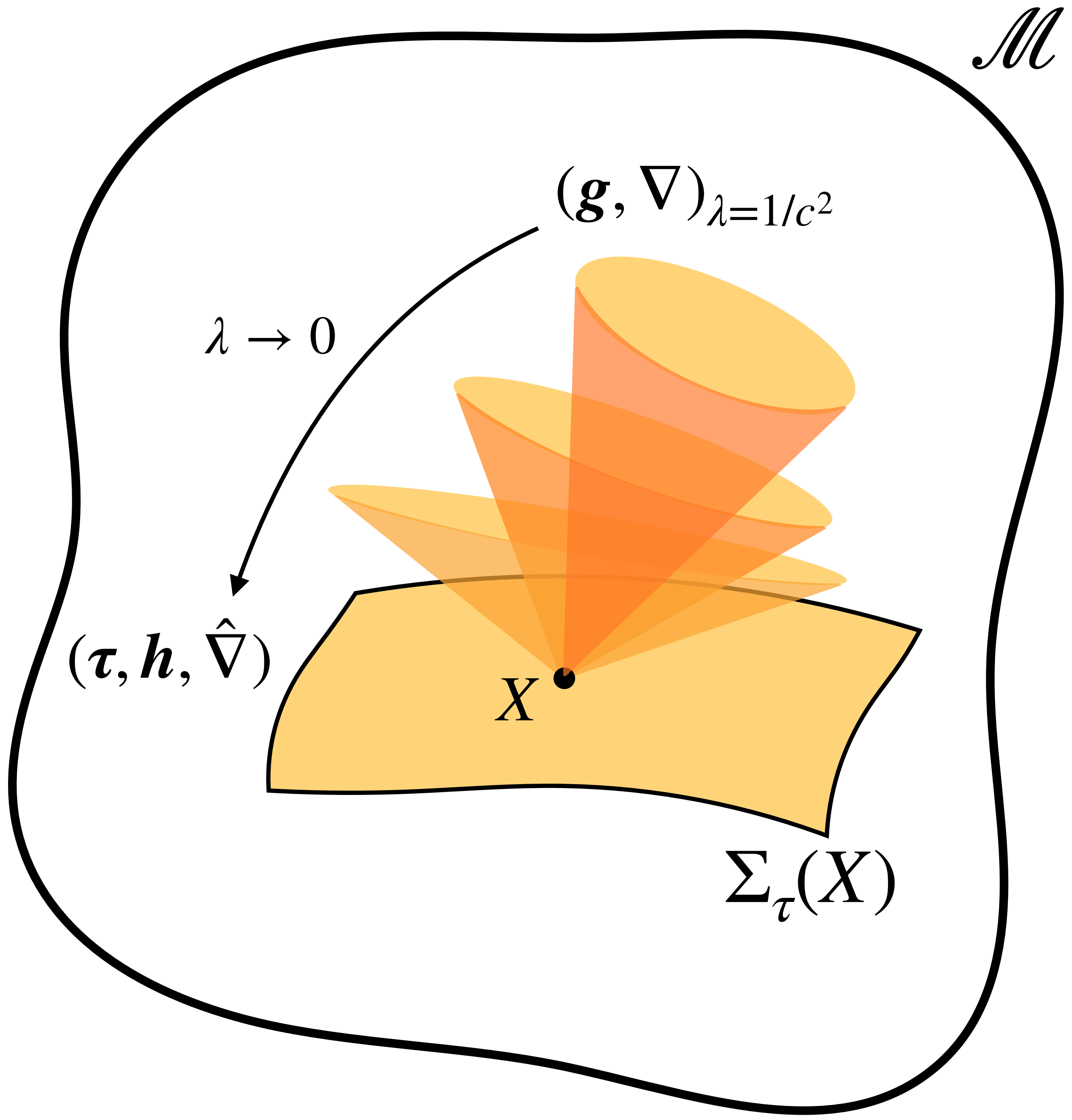

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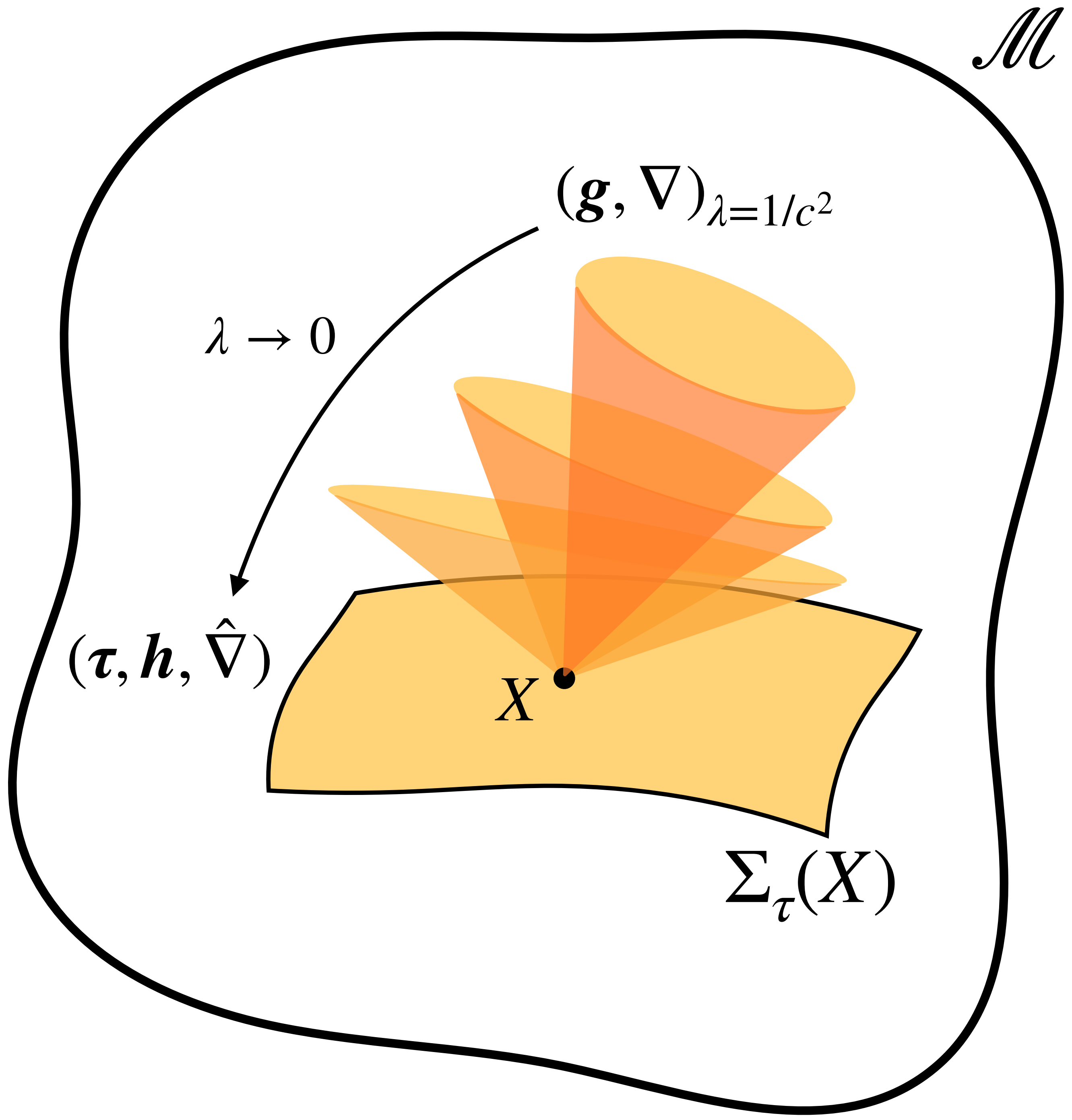

Property: The differentiable manifold $\mathscr{M}$ is unchanged, **and so its topology** (only the structure changes).

Non-relativistic limit of the Einstein equation:

$$\overset{\lambda}{R}_{\alpha\beta} = \lambda^2 \kappa \overset{\lambda}{T}{}^*{}_{\alpha\beta} + \lambda\Lambda \overset{\lambda}{g}_{\alpha\beta} \quad ; \quad \kappa = 8\pi G,$$

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(Newton-Cartan equation) $$\hat{R}_{\mu\nu} = \left(\kappa\rho - \Lambda\right)\tau_\mu \tau_\nu / 2$$

Validity of the result:

- for vacuum,
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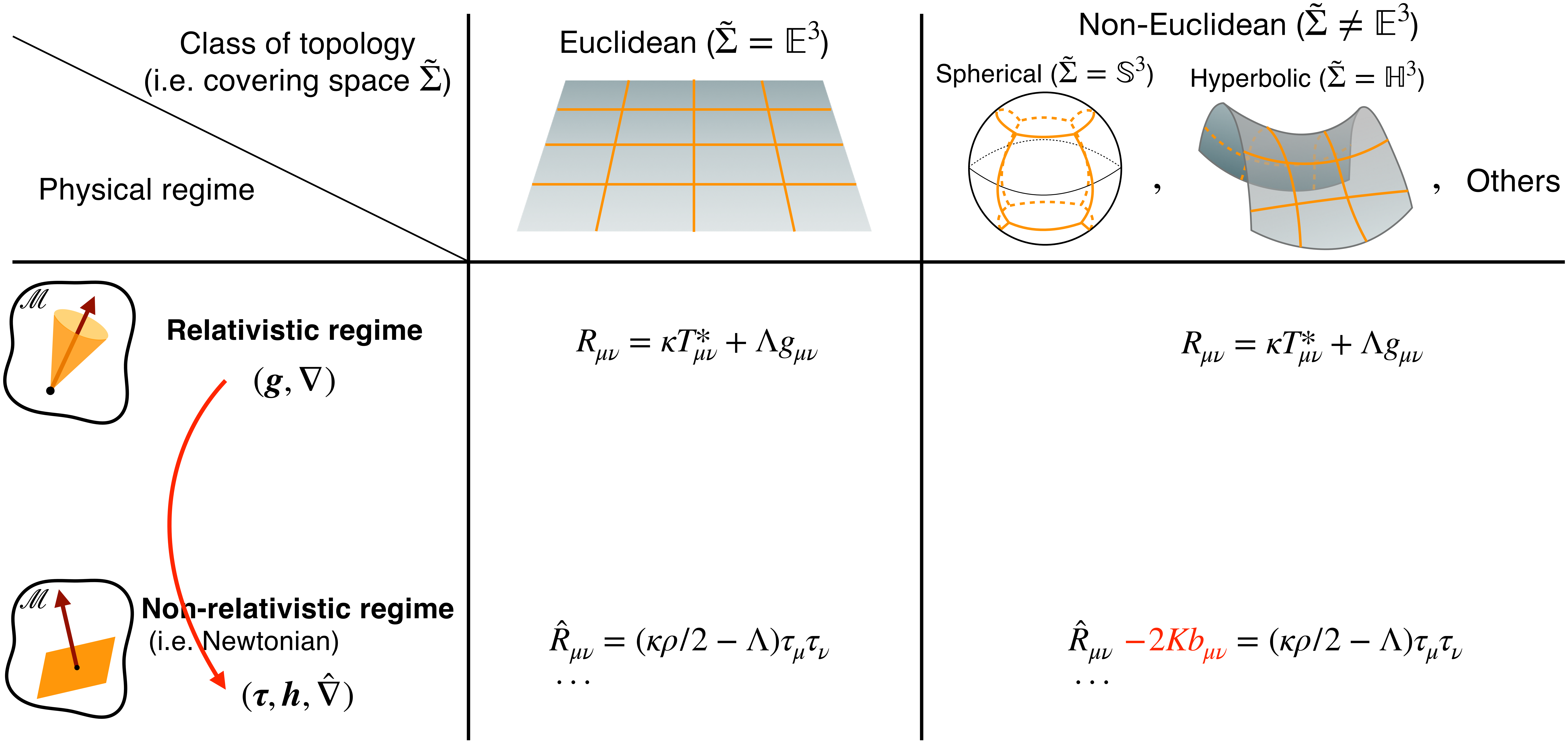
A solution of the Einstein equation having a non-relativistic limit necessarily has a Euclidean spatial topology, and the limit is Newton's theory. [Vigneron (c) (2022), preprint: 2204.13980]



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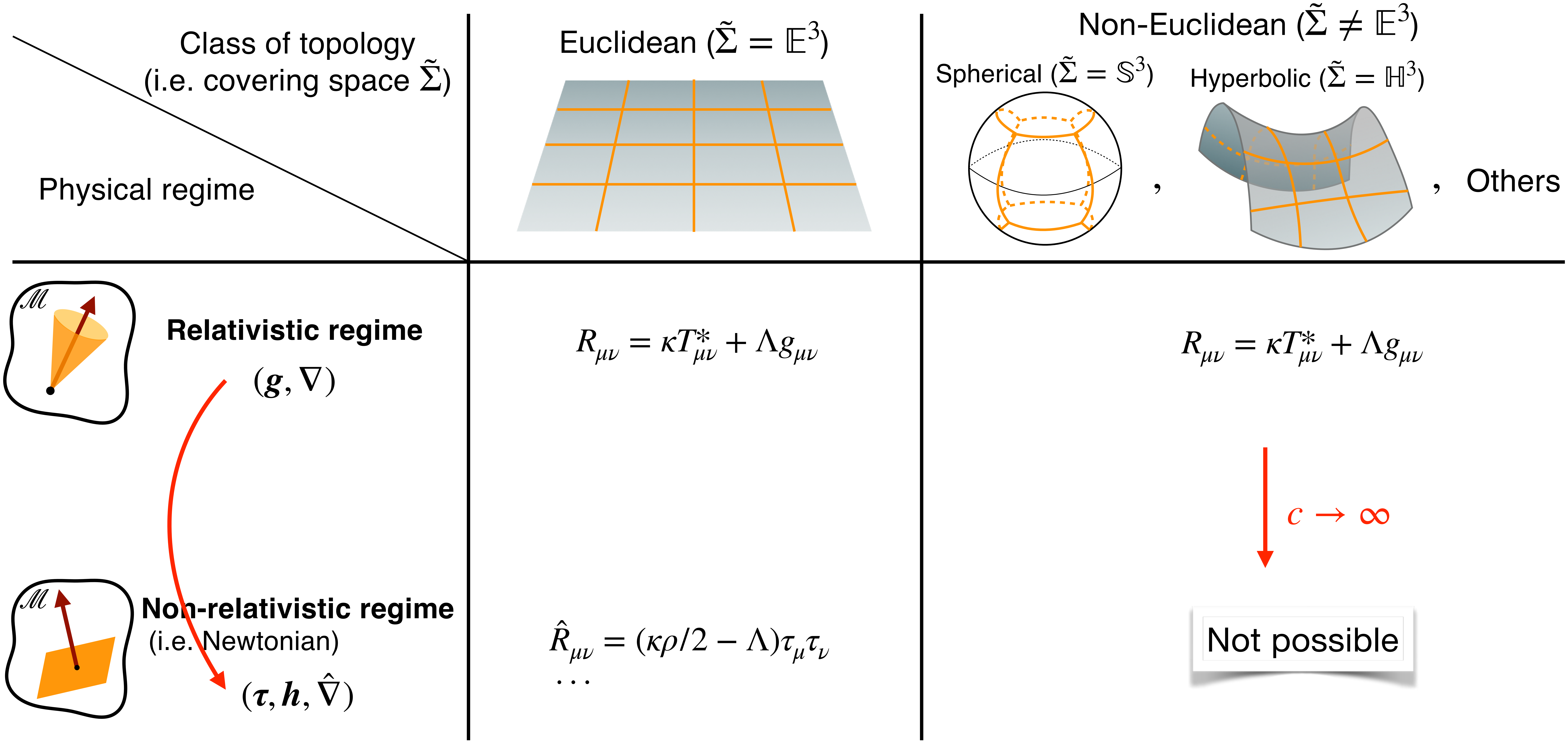

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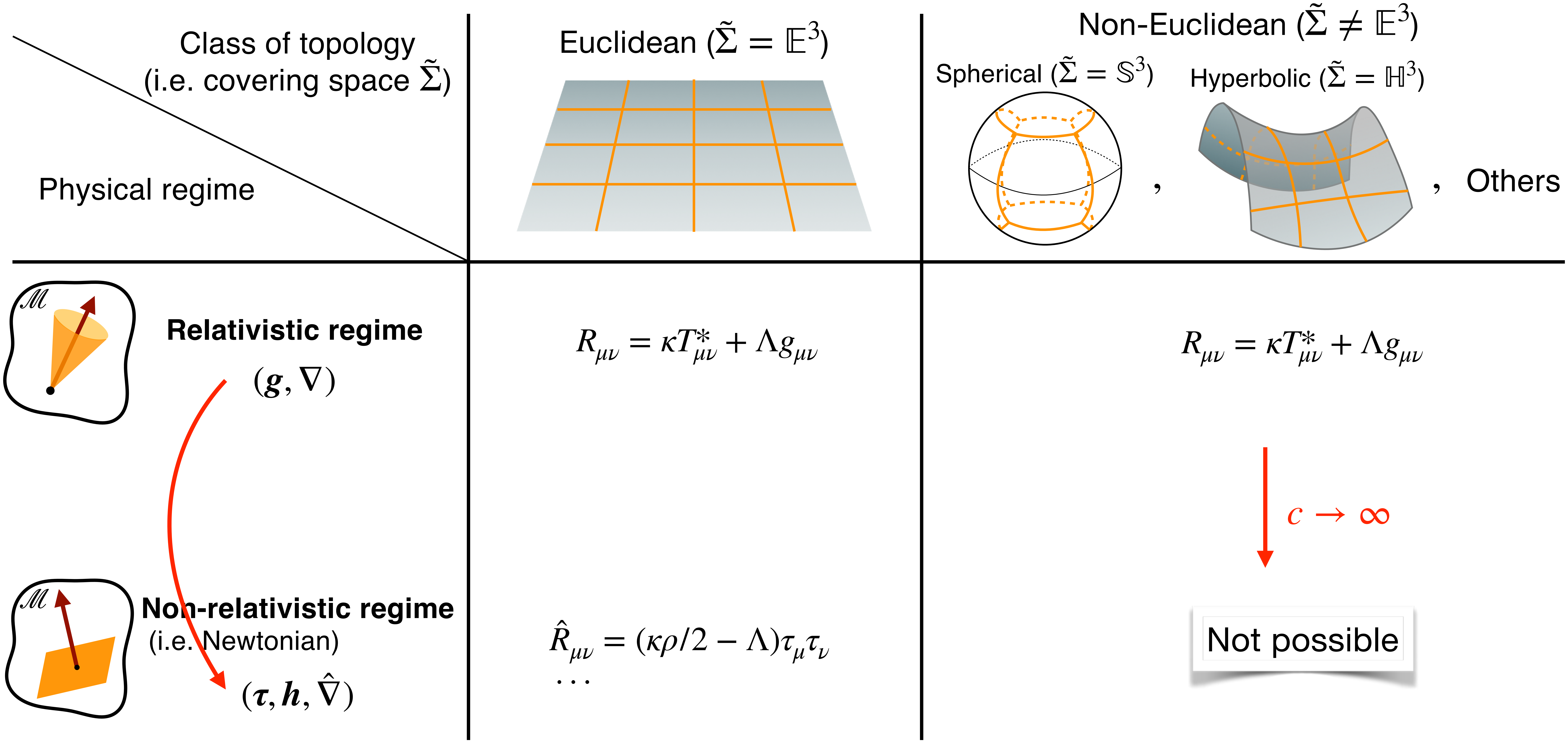
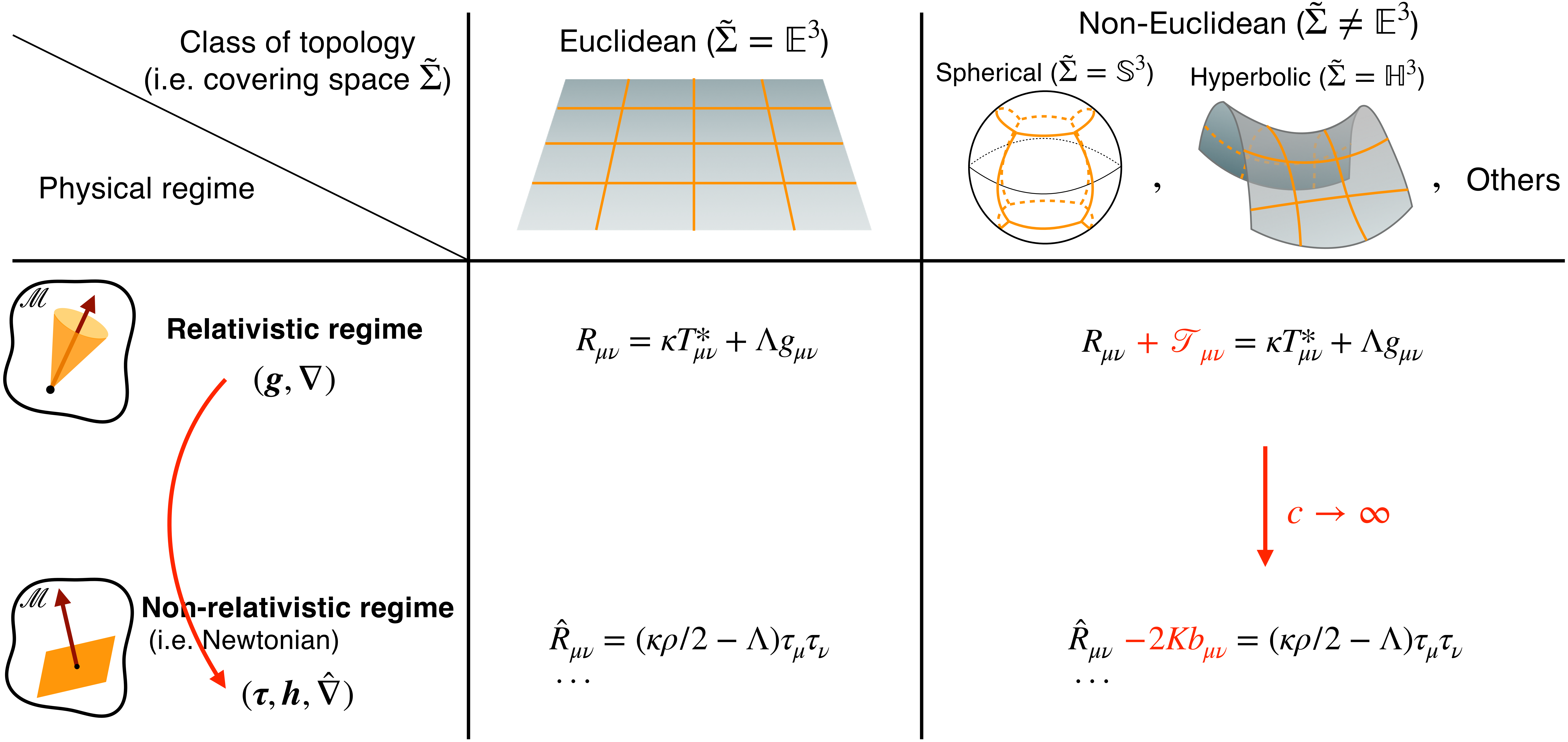


Hypothesis: **Requiring the relativistic equation describing our Universe to be compatible with the non-relativistic regime in any topologies.**



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A SIMPLE SOLUTION: ROSEN'S BI-METRIC THEORY

Requirements for $\mathscr{T}_{\mu\nu}$:

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Our choice of reference metric: **the reference metric determines the spacetime topology**

most symmetric Riemannian metric given the spacetime topology $\tilde{\mathscr{M}} = \mathbb{R} \times \tilde{\Sigma}$.

$$\bar{g}_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \bar{h}_{ij}(x^k) \end{pmatrix}$$ where $\bar{h}_{ij}$ = most symmetric metric (homogeneous) on the spatial covering space $\tilde{\Sigma}$.

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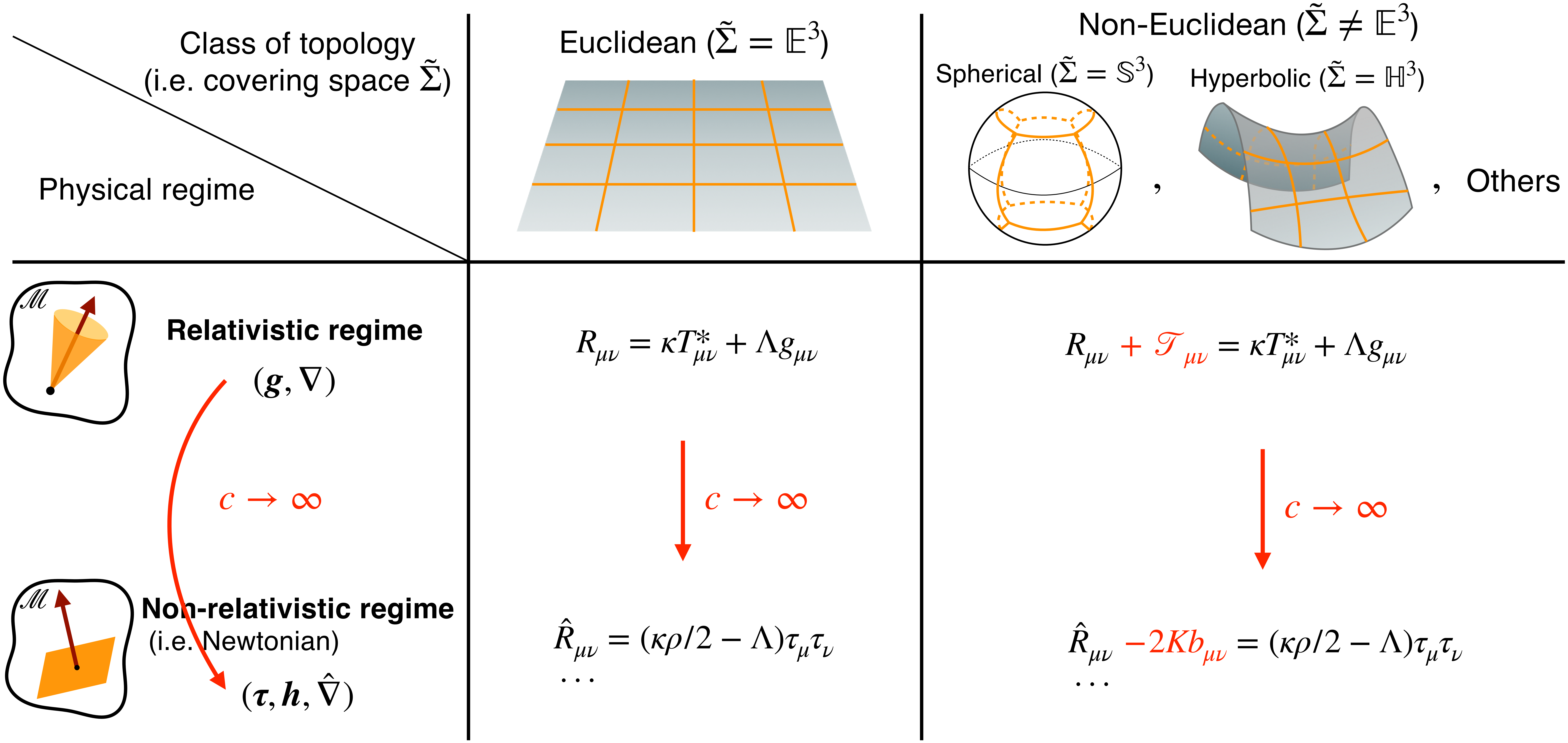

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Vigneron Q., 2022, *Non-relativistic regime and topology: consequences for the role of spatial curvature in cosmology*, soon in preprint.

1. Exact homogeneous and isotropic solution:

Λ**CDM:** $$\begin{cases} \Omega_{\neq K} + \Omega_K = 1, \\ q = \Omega_\mathrm{m}/2 + \Omega_\mathrm{rad} - \Omega_\Lambda \end{cases}$$

Bi-metric: $$\begin{cases} \Omega_{\neq K} = 1, \quad \forall\ \Omega_K \\ q = \Omega_\mathrm{m}/2 + \Omega_\mathrm{rad} - \Omega_\Lambda \end{cases}$$

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2. Weak field limit: for $\Pi = 0 = \Pi_i$

ΛCDM:

$$(\Delta + 3K)\,\Psi = 4\pi G a^2 \delta\rho_\Delta,$$
$$(\Delta + 2K)\,Q_i = -16\pi G a^2 q_i,$$
$$f''_{ij} + 2\mathscr{H} f'_{ij} + (2K - \Delta) f_{ij} = 8\pi G a^2 \Pi_{ij},$$
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Bi-metric: $$\begin{cases} \Omega_{\neq K} = 1, \quad \forall\ \Omega_K \\ q = \Omega_{\mathrm{m}}/2 + \Omega_{\mathrm{rad}} - \Omega_\Lambda \end{cases}$$

2. Weak field limit: for $\Pi = 0 = \Pi_i$

Λ**CDM:**
$$(\Delta + 3K)\Psi = 4\pi G a^2 \delta\rho_\Delta,$$
$$(\Delta + 2K)Q_i = -16\pi G a^2 q_i,$$
$$f''_{ij} + 2\mathscr{H} f'_{ij} + (2K - \Delta) f_{ij} = 8\pi G a^2 \Pi_{ij},$$
$$\dots$$

Bi-metric:
$$\Delta\Psi = 4\pi G a^2 \delta\rho_\Delta,$$
$$(\Delta - 2K)Q_i = -16\pi G a^2 q_i,$$
$$f''_{ij} + 2\mathscr{H} f'_{ij} + \left(6K - \Delta\right) f_{ij} = 8\pi G a^2 \Pi_{ij},$$
$$\dots$$

Reevaluation of the cosmological scenario **from inflation to current time**:

—> change in the inferred value of Ω_K from cosmological data??

Vigneron Q., 2022, *Non-relativistic regime and topology: consequences for the role of spatial curvature in cosmology*, soon in preprint.

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Strengths of this cosmological model:

1. Origin of the modifications not related to cosmology or tensions of ΛCDM.
2. Has the same number of free-parameters than the ΛCDM model.