Canonical structure of f(T) and f(Q) gravity

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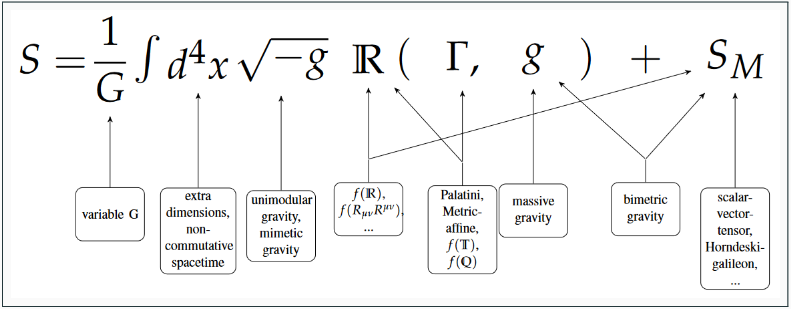
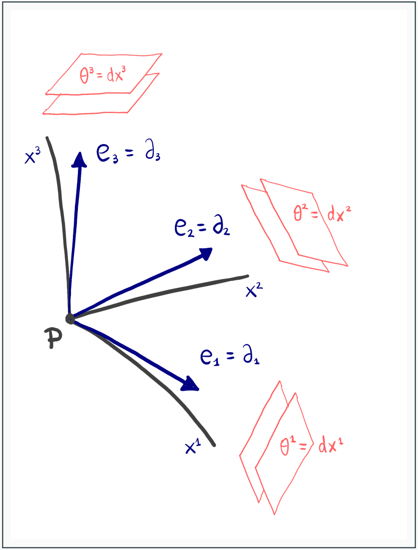
Modifying gravity

The simplest modification of the Einstein-Hilbert action is...



[A. Golovnev, M.J. Guzman (2022) arXiv:2203.16610].

Mathematical framework: (co)frames



- Local coordinates $\{x^\mu\}$ at P.
- They naturally define a basis vectors $e_\mu = \partial/\partial x^\mu = \partial_\mu$
- The basis 1-forms $\theta^\mu = dx^\mu$ are dual to the e_μ.
- In 4D, a linear combination of the θ^μ gives us an arbitrary frame, tetrad, or vierbein $\theta^a = \theta^a_\mu dx^\mu$
- Completeness relation $\theta^a(e_b) = \delta^a_b$, orthonormality condition $\eta_{ab} = g_{\mu\nu} e^\mu_a e^\nu_b$ that defines the metric tensor $g_{\mu\nu} = \eta_{ab} \theta^a_\mu \theta^b_\nu$.

Mathematical framework: linear connection

The connection $\Gamma^{\alpha}{}_{\mu\nu}$ defines the parallel transport of a vector along a curve in a manifold. Generically it has three parts:

$$\Gamma^{\alpha}{}_{\mu\nu} = \underbrace{\bar{\Gamma}^{\alpha}{}_{\mu\nu}}_{\text{(CHRISTOFFEL'S)}} + K^{\alpha}{}_{\mu\nu} + L^{\alpha}{}_{\mu\nu}$$

CONTORTION:

$$K^{\alpha}{}_{\mu\nu} = \frac{1}{2}\left(T^{\alpha}{}_{\mu\nu} + T_{\mu}{}^{\alpha}{}_{\nu} + T_{\nu}{}^{\alpha}{}_{\mu}\right)$$

DISTORTION:

$$L^{\alpha}{}_{\mu\nu} = \frac{1}{2}\left(Q^{\alpha}{}_{\mu\nu} - Q_{\mu}{}^{\alpha}{}_{\nu} - Q_{\nu}{}^{\alpha}{}_{\mu}\right)$$

TORSION:

$$\boxed{T^{\rho}{}_{\mu\nu} = 2\Gamma^{\rho}{}_{[\mu\nu]}}$$

NON-METRICITY:

$$\boxed{Q_{\rho\mu\nu} = \nabla_{\rho}\, g_{\mu\nu}}$$

It is related with the spin connection $\omega^{a}{}_{b\mu}$ by the tetrad postulate

$$\partial_{\mu}\theta^{a}{}_{\nu} + \omega^{a}{}_{b\mu}\theta^{b}{}_{\nu} - \Gamma^{\rho}{}_{\mu\nu}\theta^{a}{}_{\rho} = 0 \tag{1}$$

Metric teleparallel framework

- A teleparallel framework is the one for which the linear connection has vanishing Riemann curvature $R^{\mu}{}_{\nu\alpha\beta} = 0$.
- In this setup we can still fix either the non-metricity tensor, or the torsion tensor, to zero. In the former case, the connection can be written in term of Lorentz matrices as

$$\omega^{a}{}_{b\mu} = -\left(\Lambda^{-1}\right)^{c}{}_{b}\partial_{\mu}\Lambda_{c}{}^{a}. \tag{2}$$

- Consequently, the torsion tensor is

$$T^{a}{}_{\mu\nu} = \partial_{\mu}\theta^{a}{}_{\nu} - \partial_{\nu}\theta^{a}{}_{\mu} + \omega^{a}{}_{b\mu}\theta^{b}{}_{\nu} - \omega^{a}{}_{b\nu}\theta^{b}{}_{\mu}. \tag{3}$$

- A useful object for later is the torsion scalar $\mathbb{T}$:

$$\mathbb{T} = -\frac{1}{4}T_{\rho\mu\nu}T^{\rho\mu\nu} - \frac{1}{2}T_{\rho\mu\nu}T^{\mu\rho\nu} + T^{\rho}{}_{\mu\rho}T^{\sigma\mu}{}_{\sigma}.$$

The teleparallel equivalent of general relativity

The action for the teleparallel equivalent of general relativity is [Aldrovandi, Pereira (2013)]

$$S_{\mathrm{TEGR}} = \frac{1}{2\kappa} \int d^4x \; \theta \mathbb{T},$$

contains $\mathbb{T}$, which satisfies the identity

$$\theta \mathbb{T} = -\theta \mathbb{R} + \partial_\mu (\theta T^{\nu}{}_{\nu}{}^{\mu}).$$

Here $\mathbb{R}$ depends exclusively on the metric, which is invariant under local Lorentz transformations of the tetrad

$$\theta^a \longrightarrow \theta^{a'} = \Lambda^{a'}{}_a(x)\theta^a.$$

This is not true for $\partial_\mu (\theta T^{\nu}{}_{\nu}{}^{\mu})$, which is a harmless boundary term. Then, it is said that TEGR is a **Lorentz pseudo-invariant** theory. TEGR encompasses the same degrees of freedom than GR [Ferraro, Guzman (2016) arXiv: 1609.06766].

Symmetric teleparallel gravity

- After setting a vanishing torsion and curvature in the connection we obtain

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial \xi^{\beta}} \partial_{\mu} \partial_{\nu} \xi^{\beta}, \tag{4}$$

  where $\xi^{\alpha}(x^{\mu})$ are a set of invertible functions depending on the coordinates, and $\frac{\partial x^{\alpha}}{\partial \xi^{\rho}}$ is the inverse of the corresponding Jacobian.
- Any coordinates affinely related to $\xi^{\alpha} = x^{\alpha}$ make the connection $\Gamma^{\alpha}_{\mu\nu} = 0$. These coordinates are called the so-called **coincident gauge**. The non-metricity is then trivial

$$Q_{\alpha\mu\nu} = \partial_{\alpha} g_{\mu\nu}. \tag{5}$$

Symmetric teleparallel gravity

- We define the disformation tensor

$$L^{\alpha}{}_{\mu\nu} = \frac{1}{2} Q^{\alpha}{}_{\mu\nu} - Q_{(\mu\nu)}{}^{\alpha}, \tag{6}$$

measures the separation of the symmetric part of the full connection from the Levi-Civita one.

- The non-metricity conjugate $P^{\alpha}{}_{\mu\nu}$ is defined as

$$P^{\alpha}{}_{\mu\nu} = -\frac{1}{2} L^{\alpha}{}_{\mu\nu} + \frac{1}{4}(Q^{\alpha} - \tilde{Q}^{\alpha}) g_{\mu\nu} - \frac{1}{4}\delta^{\alpha}_{(\mu} Q_{\nu)}, \tag{7}$$

where $Q_{\alpha} = g^{\mu\nu} Q_{\alpha\mu\nu}$ and $\tilde{Q}_{\alpha} = g^{\mu\nu} Q_{\mu\alpha\nu}$ are the two independent traces of the non-metricity tensor.

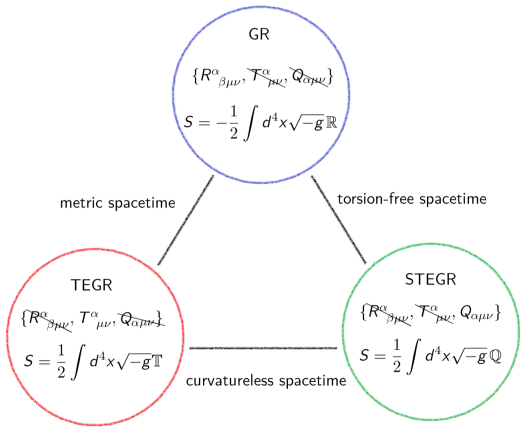
- The non-metricity scalar

$$\mathbb{Q} = -Q_{\alpha\mu\nu} P^{\alpha\mu\nu}, \tag{8}$$

and written as a quadratic combination of non-metricity, it can also read as

$$\mathbb{Q} = -\frac{1}{4} Q_{\alpha\mu\nu} Q^{\alpha\mu\nu} + \frac{1}{2} Q_{\alpha\mu\nu} Q^{\mu\nu\alpha} + \frac{1}{4} Q_{\alpha} Q^{\alpha} - \frac{1}{2} Q_{\alpha} \tilde{Q}^{\alpha}. \tag{9}$$

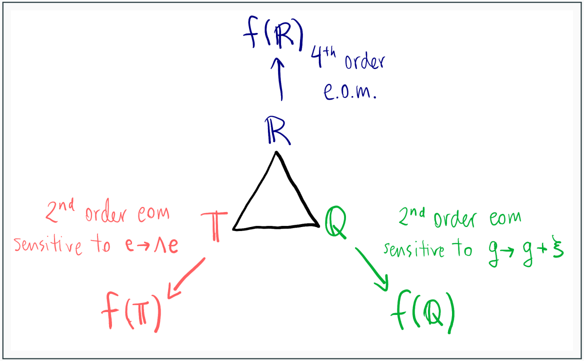
The geometrical trinity of gravity



[adapted from Beltran-Jimenez, Heisenberg, Koivisto (2019)]

Why curvature?

- Curvature is not the unique way of representing the observed gravitational phenomena. There are at least two equivalent representations: TEGR and STEGR
- If we think that $f(\mathbb{R})$ is the simplest modification of GR, then $f(\mathbb{T})$ and $f(\mathbb{Q})$ are on equal footing, using the simplicity argument.

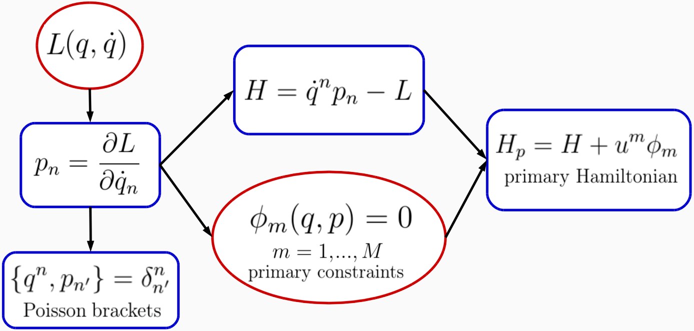


Guzman (2022), EREP2022

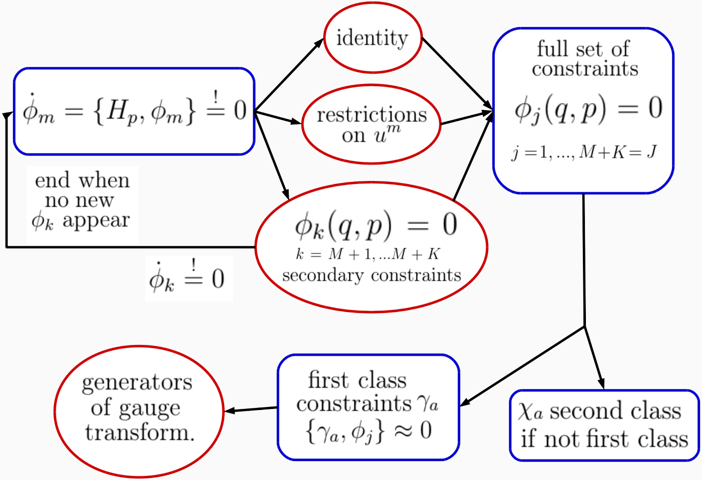
Motivations for Hamiltonian formalism

- Allows a non-ambiguous identification of gauge symmetries and counting of physical degrees of freedom
- Crucial in approaches to canonical quantum gravity
- Assessing the well-posedness of the Cauchy problem, therefore the viability of any theory
- Theoretical basis for numerical modified general relativity

Dirac algorithm part 1



Dirac algorithm part 2



Nonlinear TEGR: $f(\mathbb{T})$ gravity

The $f(\mathbb{T})$ gravity action is the simplest nonlinear generalization of the TEGR action, given by [Ferraro, Fiorini (2006) arXiv:gr-qc/0610067]

$$S = \frac{1}{2\kappa}\int d^4x\theta f(\mathbb{T}). \tag{10}$$

Denoting $f_T \equiv \frac{df}{d\mathbb{T}}$ and $f_{TT} \equiv \frac{d^2 f}{d\mathbb{T}^2}$, the equations of motion can be conveniently written as

$$\begin{aligned} -\kappa\Theta_\lambda^\nu \equiv 2f_{TT}(\mathbb{T})S_\lambda{}^{\mu\nu}\partial_\mu\mathbb{T} - \frac{1}{2}\delta_\lambda^\nu f(\mathbb{T}) \\ + 2e\theta_\lambda^a f_T(\mathbb{T})\mathscr{D}_\mu[\theta e_a^\sigma S_\sigma{}^{\mu\nu}] + 2T^\rho{}_{\mu\lambda}S_\rho{}^{\mu\nu}f_T(\mathbb{T}) \end{aligned} \tag{11}$$

where $\mathscr{D}$ is the Lorentz-covariant derivative, in particular $\mathscr{D}_\mu e_a^\sigma = \partial_\mu e_a^\sigma - \omega^b{}_{a\mu}e_b^\sigma$. In the Weitzenböck gauge it coincides with the usual partial derivative.

In the covariant form, equations read:

$$\kappa\Theta_{\mu\nu} \equiv f_T(\mathbb{T})\overset{(0)}{G}_{\mu\nu} + 2f_{TT}(\mathbb{T})S_{\mu\nu\alpha}\partial^\alpha\mathbb{T} + \frac{1}{2}\left[f(\mathbb{T}) - f_T(\mathbb{T})\mathbb{T}\right]g_{\mu\nu}. \tag{12}$$

$f(\mathbb{T})$ gravity

The $f(\mathbb{T})$ gravity action

$$S = \frac{1}{2\kappa}\int d^4x\ \theta f(\mathbb{T}) \tag{13}$$

has a Jordan-frame representation

$$S = \frac{1}{2\kappa}\int d^4x\ \theta[\phi\mathbb{T} - V(\phi)] \tag{14}$$

and an "Einstein-frame"-like one

$$\boxed{S = \frac{1}{2\kappa}\int d^4x\ E[\hat{T} + \psi e\partial_\mu(E\hat{T}^\mu) - \frac{1}{2}\partial_\mu\psi\partial^\mu\psi - U(\psi)]}$$

Hamiltonian formulation has only been attempted in the Jordan-frame like.

A digression on "covariance"

- Teleparallel theories of gravity built in terms of the torsion tensor are invariant under the simultaneous transformation (covariant approach)

$$\theta^a_\mu \longrightarrow \Lambda^a{}_b \theta^b_\mu, \tag{15}$$

$$\omega^a{}_{b\mu} \longrightarrow \omega^a{}_{b\mu} = \Lambda^a{}_c \omega^c{}_{d\mu} (\Lambda^{-1})^d{}_b - \partial_\mu \Lambda^a{}_c (\Lambda^{-1})^c{}_b \tag{16}$$

  ($\omega^a{}_{b\mu}$ vanishes both $R^\rho{}_{\sigma\mu\nu}$ and $Q_{\alpha\mu\nu}$). These are represented by **primary first class constraints**, therefore are pure gauge.
- On top of it, an alternative Lorentz transformations only on the tetrad (pure-tetrad approach) appears as additional symmetry in TEGR

$$\theta^a_\mu \longrightarrow \Lambda^a{}_b \theta^b_\mu, \qquad \omega^a{}_{b\mu} = 0. \tag{17}$$

- In $f(T)$ not all Λ's produce tetrads solving the e.o.m. (unless they are remnant symmetries). This is the kind of Lorentz symmetry that is lost in $f(T)$ gravity. It is not restored by introducing the previous "covariant approach". [Golovnev, Guzman 2110.11273]

Constraints in $f(\mathbb{T})$ gravity

The constraint structure in the Jordan-frame like representation of $f(\mathbb{T})$ gravity consists of

- 17 pairs of canonical variables $(\theta^a_\mu, \Pi^\mu_a)$ plus (ϕ, π)
- Four primary constraints $\Pi^0_a \approx 0$
- Six extra primary constraints

$$C_{ab} = 2\eta_{e[b}\Pi^i_{a]}\theta^e_i + 4\theta\phi\partial_i\theta^c_j(e^0_{[b}e^i_{a]}e^j_c + e^i_{[b}e^j_{a]}e^0_c + e^j_{[b}e^0_{a]}e^i_c) \quad (18)$$

- One (secondary) Hamiltonian constraint

$$C_0 = \frac{\kappa\sqrt{\gamma}\phi}{8\kappa}\left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2\right) - \frac{\sqrt{\gamma}}{2\kappa}{}^{(3)}\mathbb{T} + \frac{\sqrt{\gamma}}{2\kappa}V(\phi) - n^a\partial_i\Pi^i_a \quad (19)$$

  and three (secondary) momenta constraints $C_i \approx 0$
- An extra secondary constraint
  $\chi = (\Pi^{(mn)}g^{il}T^{0jk} - \Pi^{0m}g\,in\,T^{ljk})\epsilon_{ijk}\partial_m\phi\partial_n\phi\partial_l\phi \approx 0$

Theoretical aspects of interest

The Hamiltonian analysis of $f(\mathbb{T})$ presents several theoretical challenges as:

- Shifted momenta due to modified pseudo-invariance, modified primary constraints
- Bifurcations on Dirac algorithm
- It has been found [Li, Miao, Miao, 1105.5934], [Ferraro, Guzman 1810.07171 (wrongly in 1802.02130)], [Blagojevic, Nester, 2006.15303] that

$$\{C_{ab}, C_{cd}\} = -\eta_{ac}G_{bd} + \eta_{bd}G_{ac} - \eta_{bc}G_{ad} + \eta_{ad}G_{bc} \qquad (20)$$

with

$$G_{bd} = 2\theta(\theta^i_b\theta^0_d - \theta^i_d\theta^0_b)\partial_i\phi \qquad (21)$$

The (non)vanishing of the last expression is related with the remnant symmetries of $f(\mathbb{T})$ [Ferraro, Fiorini 1412.3424].

$f(\mathbb{Q})$ gravity Lagrangian

- $f(\mathbb{Q})$ gravity Lagrangian can be conveniently rewritten as

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left[f(\chi) + \varphi(\mathbb{R} + \partial_\mu J^\mu - \chi)\right], \tag{22}$$

  where two auxiliary scalar fields χ and φ have been introduced.

- Integrating out χ from its eom $f'(\chi) = \varphi$ yields

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left[\varphi\mathbb{R} - U(\varphi) - \partial_\alpha J^\alpha\right], \tag{23}$$

  where $U(\varphi) = [\varphi\chi - f]_{\chi=\chi(\varphi)}$

- To recover an Einstein frame-like Lagrangian, a conformal transformation $g_{\mu\nu} = \frac{1}{\phi}q_{\mu\nu}$ is performed, also it is introduced a field redefinition $\varphi = e^{2\phi}$, and it is necessary to conformally transform the current J^α

With all this, the Einstein frame representation of $f(\mathbb{Q})$ gravity is

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-q}\left[\mathbb{R}(q) - 2(q^{\alpha\beta}q^{\mu\nu} - q^{\alpha\mu}q^{\beta\nu})\partial_\alpha\phi\nabla_\beta q_{\mu\nu} \right. \\ \left. +6(\partial\phi)^2 - \tilde{U}(\phi)\right] \tag{24}$$

Up to now the gauge has not been fixed, however in the coincident gauge the action looks like

$$S \propto \int d^4x\alpha\sqrt{\gamma}\left(\mathbb{R}(q) + 6(\partial\phi)^2 + \tilde{U}(\phi) - 2\left(q^{\alpha\beta}q^{\mu\nu} - q^{\alpha\mu}q^{\beta\nu}\right)\partial_\alpha\phi\partial_\beta q_{\mu\nu}\right).$$

It features a diffeo-breaking scalar field coupled to first order derivatives of the metric. [Beltran-Jimenez and Koivisto, 2104.05566 (2021)]

3+1 Lagrangian

We perform ADM split of the **conformally transformed metric** $q_{\mu\nu}$ as

$$q_{\mu\nu} = \begin{bmatrix} \alpha^2 + \beta^i \beta^j \gamma_{ij} & \beta_i \\ \beta_i & \gamma_{ij} \end{bmatrix}, \qquad q^{\mu\nu} = \begin{bmatrix} -\frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} \\ \frac{\beta^i}{\alpha^2} & \gamma^{ij} - \frac{\beta^i \beta^j}{\alpha^2} \end{bmatrix}. \tag{25}$$

leading to the following Lagrangian

$$\begin{aligned} L_{f(Q)} \propto\, & K_{ij}K^{ij} - (K^k_k)^2 + {}^{(3)}R - \frac{6}{\alpha^2}\dot{\phi}^2 + \frac{2\beta^i}{\alpha^2}\dot{\phi}\partial_i\phi + \frac{2\gamma^{ij}}{\alpha^2}\dot{\phi}\dot{\gamma}_{ij} \\ & - \frac{2\gamma^{ij}}{\alpha^2}\dot{\phi}\partial_j\beta_i - \frac{2\gamma^{ij}}{\alpha^2}\partial_i\phi\dot{\beta}_j - \frac{1}{\alpha^2}\left(2\beta^i\gamma^{jk} - \beta^k\gamma^{ij} - \beta^j\gamma^{ik}\right)\partial_i\phi\dot{\gamma}_{jk} \\ & + \underbrace{\cdots}_{\partial_i\phi,\ \partial_j\beta^i,\ \partial_i\alpha} \end{aligned} \tag{26}$$

Canonical momenta

From the previous Lagrangian it is easily obtained the canonical momenta

$$\begin{aligned}
\pi^{00} &= \frac{\partial L}{\partial \dot{\alpha}} = 0, \\
\pi^{0i} &= \frac{\partial L}{\partial \dot{\beta}_i} = -\frac{2\sqrt{\gamma}}{\alpha}\gamma^{ij}\partial_j\phi \\
\pi^{ij} &= \frac{\partial L}{\partial \dot{\gamma}_{ij}} = \sqrt{\gamma}\left[K^k_k\gamma^{ij} - K^{ij} + 2\frac{\gamma^{ij}}{\alpha}\dot{\phi} - \frac{\partial_k\phi}{\alpha}(2\beta^k\gamma^{ij} - 2\beta^{(i}\gamma^{j)k})\right] \\
\pi^{\phi} &= -\frac{12\sqrt{\gamma}}{\alpha}(\dot{\phi} - \beta^i\partial_i\phi) - 4\sqrt{\gamma}K^i_i + \frac{2\sqrt{\gamma}}{\alpha}\partial_k\beta^k.
\end{aligned} \tag{27}$$

The kinetic mixings $\pi^{ij}(\dot{\phi})$ and $\pi^{\phi}(K^i_i)$ nontrivialize the computation of the Hamiltonian...

Primary constraints

- Since π^{ij} and π^{ϕ} are not fully independent, we obtain an extra primary constraint. Therefore, $f(\mathbb{Q})$ gravity is endowed with a set of five primary constraints

$$\boxed{\begin{aligned} C^0 &= \pi^{00} \approx 0 \\ C^i &= \pi^{0i} + \frac{2\sqrt{\gamma}}{\alpha}\gamma^{ij}\partial_j\phi \approx 0 \\ C^{\phi} &= \pi^{ij}\gamma_{ij} + \frac{\pi^{\phi}}{2} - \frac{\sqrt{\gamma}}{\alpha}(2\beta^k\partial_k\phi + \partial_l\beta^k) \approx 0 \end{aligned}}$$

Constraint algebra (preliminary...)

$$
\begin{aligned}
&\{C^0, C^0\} = 0 \\
&\{C^0(x), C^i(y)\} \propto \gamma^{ij}\partial^y_j \phi \\
&\{C^0(x), C^\phi(y)\} \propto 2\beta^k \partial^y_k \phi + \partial^y_k \beta^k \\
&\{C^i(x), C^j(y)\} = 0 \\
&\{C^i(x), C^\phi(y)\} \propto \gamma^{ij}\partial^y_j \phi \\
&\{C^\phi(x), C^\phi\} \propto \beta^k(y)\partial^y_k \delta(x,y) - \beta^k(x)\partial^x_k \delta(x,y)
\end{aligned}
\tag{28}
$$

The $f(\mathbb{Q})$ Hamiltonian can be written purely in terms the π^{ij}, π^ϕ depends on them via the 5th primary constraint. Next step is to finish Dirac's algorithm, identify first and second class constraints, and compute dof and physical interpretation.

Comparison with previous works

- D'Ambrosio, Garg, Heisenberg, Zentarra (2020): It is hypothetized that in $f(\mathbb{Q})$ there should be only four primary constraints (like GR). However, breaking of diffeomorphism invariance strongly suggests a larger number.
- Hu, Katsuragawa, Qiu (2022): The 3+1 action has mixing of kinetic terms different than us, producing qualitatively different primary constraints.

Conclusions

- Teleparallel geometric frameworks, and gravity theories based on it, in particular TEGR and STEGR, are legitimate starting points for building modifications to gravity.
- The simplest nonlinear modifications, $f(\mathbb{T})$ and $f(\mathbb{Q})$ gravity, present genuine differences regarding their $f(\mathbb{R})$ counterpart.
- Full Hamiltonian analysis is necessary for non ambiguous identification of degrees of freedom, generators of gauge transformations, etc. of a theory.
- Special care must be given to their constraint structure. Nonlinear constraint effect, strong coupling, and unusual mathematical anomalies appear, that are theoretically challenging.