Lensing of gravitational waves in Palatini $f(\hat{R})$ gravity

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Content of the talk:

- Lensing of GWs (Geometric Optics, Wave Optics, Estimates, Applications)
- WKB Approximation
- Palatini $f(\hat{R})$ gravity
- Eikonal analysis in Palatini $f(\hat{R})$
- Conclusion

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1. Biesiada, Marek, and **Sreekanth Harikumar**. 2021. "Gravitational Lensing of Continuous Gravitational Waves" Universe 7, no. 12: 502. https://doi.org/10.3390/universe7120502
2. Sreekanth H,Laur Järv , Aneta Wojnar, Margus Saal and Marek Biesiada "Gravitational Wave lensing in Palatini f(R) gravity " - in preparation.

The intervening matter in the line of sight between the source and observer will lens the gravitational wave signal.

Consequences:

- Magnification/de-magnification of signal amplitude (Frequency dependent modulation)
- Bias in estimated parameters (Chirp mass, Luminosity distance)
- GW beat patterns
- Multiple Signals
- GW Faraday rotation

Lensing Estimates:

- Einstein Telescope(10^4 to 10^5) detections per year of which 50 -100 lensed events per year
- Estimates for aLIGO Global network- 1 in 1500 events would be lensed
- Estimates for DECIGO : 5- 50 lensed events out of $10^4 - 10^5$ detections

Geometric Optics:
$$\lambda << R_s = \frac{2GM_L}{c^2}$$

Wave Optics: $\lambda \geq R_s \; ; h_{\mu\nu} = h e_{\mu\nu}$

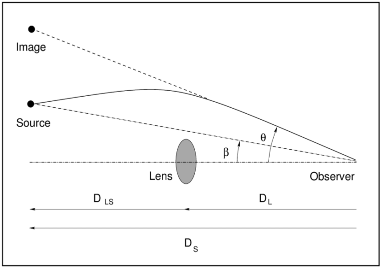
$$\partial_\mu \left(\sqrt{-g^{(L)}} g^{(L)\mu\nu} \partial_\nu h \right) = 0$$

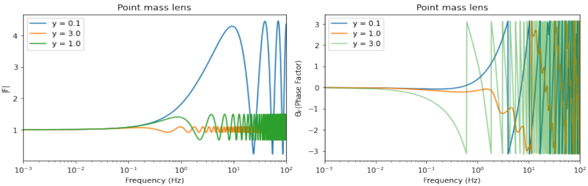
$$\left(\nabla^2 + 4\pi^2 f^2 \right) \tilde{h} = 16\pi^2 f^2 U \tilde{h}$$

Krichoff Integral

$$F(f,\beta) = \frac{1+z_l}{c} \frac{D_s}{D_l D_{ls}} \frac{f}{i} \int d^2\theta \exp[2\pi i f \Delta t(\theta,\beta)]$$

Lens equation $\beta = \theta - \alpha(\theta)$





$$F(f) = \frac{\tilde{h}^L}{\tilde{h}}$$ **(Amplification Factor)**

Timedelay Function

$$\Delta t = \frac{1+z_l}{c} \frac{D_l D_s}{D_{ls}} \left[\frac{(\theta - \beta)^2}{2} - \phi(\theta) + \phi_m(\beta) \right]$$

Interference

Beat Frequency

$$\omega_b = \frac{96}{5}\left(\frac{\omega_f}{2}\right)^{11/3}\mathcal{M}\Delta t$$

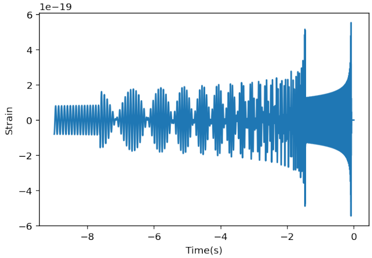
$$M_l(1+z_l) = \frac{c^3}{2G}\Delta t\left[\frac{r-1}{\sqrt{r}} + \ln r\right]^{-1}$$

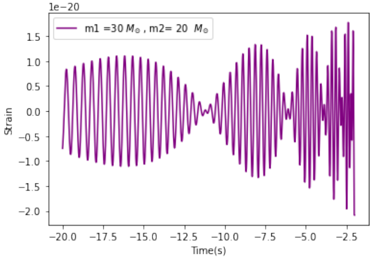
GW Beat Pattern

$$h = h_1(t) + h_2(t + \Delta t)$$

Applications

- Accurate H_0 measurements
- Estimation of lens mass





Gravitational wave interference via gravitational lensing: Measurements of luminosity distance, lens mass, and cosmological parameters. *Phys. Rev. D* **2020**, *101*, 064011. doi:black10.1103/PhysRevD.101.064011.

WKB Approximation

Eikonal ansatz

$$h_{\mu\nu} = \mathcal{R}\left\{\left[\xi^{(0)}_{\mu\nu} + \frac{1}{\omega}\xi^{(1)}_{\mu\nu} + \frac{1}{\omega^2}\xi^{(2)}_{\mu\nu} + ...\right]e^{i\omega\Phi}\right\}$$

$\omega \longrightarrow \infty$ defines the geometric optics limit.

$$\nabla^{\alpha}\nabla_{\alpha}h_{\mu\nu} - 2h_{\alpha\beta}R^{\alpha}{}_{\mu\nu}{}^{\beta} = e^{i\omega\Phi}\Big\{\omega^2[-k^{\beta}k_{\beta}\xi^{(0)}_{\mu\nu}]$$
$$+\omega[i(\nabla_{\beta}k^{\beta}\xi^{(0)}_{\mu\nu} + k^{\beta}\nabla_{\beta}\xi^{(0)}_{\mu\nu}) - k^{\beta}k_{\beta}\xi^{(1)}_{\mu\nu}]$$
$$+\omega^0[\nabla^{\beta}\nabla_{\beta}\xi^{(0)}_{\mu\nu} + i[\nabla_{\beta}k^{\beta}\xi^{(1)}_{\mu\nu} + k^{\beta}\nabla_{\beta}\xi^{(1)}_{\mu\nu}$$
$$+k_{\beta}\nabla^{\beta}\xi^{(1)}_{\mu\nu}]\Big\} - 2h_{\alpha\beta}R^{\alpha}{}_{\mu\nu}{}^{\beta}$$

Wave vector- k^{β}

$$k^{\alpha} = g^{\alpha\beta}\partial_{\beta}\Phi$$

$\Phi(x) \longrightarrow$ Phase function

Gauge condition

$$\nabla^{\mu}h_{\mu\nu} = e^{i\omega\Phi}\left\{\omega[ik^{\mu}\xi^{(0)}_{\mu\nu}] + \omega^0[ik^{\mu}\xi^{(1)}_{\mu\nu} + \nabla^{\mu}\xi^{(0)}_{\mu\nu}]\right\} = 0$$

Geometric Optics limit

$$e^{i\omega\Phi}\left\{\omega^2(-k^\beta k_\beta \xi^{(0)}_{\mu\nu})+\omega(i(\nabla_\beta k^\beta \xi^{(0)}_{\mu\nu}+k^\beta\nabla_\beta \xi^{(0)}_{\mu\nu})-k^\beta k_\beta \xi^{(1)}_{\mu\nu})\right\}=0$$

$$k_\beta k^\beta = 0$$

$$k^\mu \nabla_\mu k_\nu = 0$$

$$\frac{dx^\beta}{d\lambda^2}+\Gamma^\beta_{\alpha\mu}\frac{dx^\alpha}{d\lambda}\frac{dx^\mu}{d\lambda}=0,$$

From Gauge condition

$$k^\mu \xi^{(0)}_{\mu\nu} = 0$$

In the next order we obtain

$$2k^\beta \nabla_\beta \xi^{(0)}_{\mu\nu} + \nabla_\beta k^\beta \xi^{(0)}_{\mu\nu} = 0$$

$$\xi^{(0)}_{\mu\nu} = AA_{\mu\nu} \; ; \; A = \sqrt{\xi^*_{\mu\nu}\xi^{*\mu\nu}}$$

$$\nabla_\rho(k^\rho \mathcal{A}^2) = 0$$

$$k^\alpha \nabla_\alpha A_{\mu\nu} = 0$$

$$P^\mu = \hbar k^\mu \; ; \; N^\mu = \frac{A^2}{\hbar^2}P^\mu$$

In GO limit :

- Poplatization tensor is parallel propagated along the null vector k^μ
- Number of gravitons in a ray bundle is conserved $\nabla_\mu N^\mu = 0$

Beyond Geometric Optics $\longrightarrow \lambda \sim b >> R_s$

$$\boxed{\nabla_\beta \nabla^\beta \xi^{(0)}_{\mu\nu} + i[\nabla_\beta k^\beta \xi^{(1)}_{\mu\nu} + 2k^\beta \nabla_\beta \xi^{(1)}_{\mu\nu}] - 2\xi^{(0)}_{\alpha\beta} R^\alpha{}_{\mu\nu}{}^\beta = 0}$$

$$\nabla_\beta k^\beta \xi^{(1)}_{\mu\nu} + 2k^\beta \nabla_\beta \xi^{(1)}_{\mu\nu} = S^{(0)}_{\mu\nu}$$

Source Tensor:

$$S^{(0)}_{\mu\nu} = -i[2\xi^{(0)}_{\alpha\beta} R^\alpha{}_{\mu\nu}{}^\beta - \nabla^\beta_\beta \xi^{(0)}_{\mu\nu}]$$

Gravitational waves are not transverse in the beyond geometric optics limit

$$k^\mu \xi^{(1)}_{\mu\nu} = i\nabla^\mu \xi^{(0)}_{\mu\nu}$$

$$k^\mu \xi^{(1)}_{\mu\nu} = S^{g(0)}_\mu$$

(Gauge condition) $\nabla^\mu h_{\mu\nu} = e^{i\omega\Phi}\left\{\omega[ik^\mu \xi^{(0)}_{\mu\nu}] + \omega^0[ik^\mu \xi^{(1)}_{\mu\nu} + \nabla^\mu \xi^{(0)}_{\mu\nu}]\right\}$

(Eikonal expansion)

$$\nabla^\alpha \nabla_\alpha h_{\mu\nu} - 2h_{\alpha\beta} R^\alpha{}_{\mu\nu}{}^\beta = e^{i\omega\Phi}\Big\{\omega^2[-k^\beta k_\beta \xi^{(0)}_{\mu\nu}] + \omega[i(\nabla_\beta k^\beta \xi^{(0)}_{\mu\nu} + k^\beta \nabla_\beta \xi^{(0)}_{\mu\nu}) - k^\beta k_\beta \xi^{(1)}_{\mu\nu}]$$

$$+\omega^0[\nabla^\beta \nabla_\beta \xi^{(0)}_{\mu\nu} + i[\nabla_\beta k^\beta \xi^{(1)}_{\mu\nu} + k^\beta \nabla_\beta \xi^{(1)}_{\mu\nu} + k_\beta \nabla^\beta \xi^{(1)}_{\mu\nu}]\Big\} - 2h_{\alpha\beta} R^\alpha{}_{\mu\nu}{}^\beta$$

Palatini $f(\hat{R})$ gravity

Action:

$$S[g,\Gamma,\psi_m] = \frac{1}{2\kappa}\int \sqrt{-g}f(\hat{R})d^4x + S_{\text{matter}}[g,\psi_m]$$

Field Equations

$$f'(\hat{R})\hat{R}_{\mu\nu} - \frac{1}{2}f(\hat{R})g_{\mu\nu} = \kappa T_{\mu\nu}$$

$$\nabla_\beta(\sqrt{-g}f'(\hat{R}(T))g^{\mu\nu}) = 0$$

$$\hat{g}_{\mu\nu} = f'(\hat{R}(T))g_{\mu\nu}$$

Trace Equation:

$$f'(\hat{R})\hat{R} - 2f(\hat{R}) = \kappa T$$

$$\hat{\Gamma}^\alpha_{\mu\nu} = \frac{\hat{g}^{\mu\beta}}{2}[\partial_\mu\hat{g}_{\beta\nu}+\partial_\nu\hat{g}_{\beta\mu}-\partial_\beta\hat{g}_{\mu\nu}]$$

Sarmah, L., Kalita, S. & Wojnar, A. Stability criterion for white dwarfs in Palatini *Physical Review D*. **105** (2022,1), https://doi.org/10.1103252Fphysrevd.105.024028

GWs Palatini f(R) gravity

Scalar Tensor Representation

$$\hat{G}_{\mu\nu} = \frac{1}{f'}[\kappa T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} U(f')]$$

$$\hat{R}_{\mu\nu} - \frac{1}{2}\hat{g}_{\mu\nu}\hat{R} = 0$$

$$\hat{g}_{\mu\nu} = \hat{g}^{(B)}_{\mu\nu} + \hat{h}_{\mu\nu}$$

Linearzed Field Equation:

$$\boxed{\hat{\nabla}_\alpha \hat{\nabla}^\alpha h_{\mu\nu} - 2\hat{R}^{\tau}_{\rho\mu\nu}\hat{h}^{\rho}_{\tau} = -2\kappa T^{(1)}_{\mu\nu}}$$

$$T^{(1)}_{\mu\nu} = i\omega \frac{c^4}{16\pi G} t_{\mu\nu} e^{i\omega\phi}.$$

Eikonal Expansion

$$\begin{aligned}\hat{\nabla}^\alpha \hat{\nabla}_\alpha h_{\mu\nu} \; -2h_{\alpha\beta}\hat{R}^{\alpha}{}_{\mu\nu}{}^{\beta} = e^{i\omega\Phi}\Big\{ & \omega^2[-k^\beta k_\beta \xi^{(0)}_{\mu\nu}] \\ & +\omega[i(\hat{\nabla}_\beta k^\beta \xi^{(0)}_{\mu\nu} + k^\beta \hat{\nabla}_\beta \xi^{(0)}_{\mu\nu}) - k^\beta k_\beta \xi^{(1)}_{\mu\nu}] \\ & +\omega^0[\hat{\nabla}_\beta \hat{\nabla}^\beta \xi^{(0)}_{\mu\nu} + i[\hat{\nabla}_\beta k^\beta \xi^{(1)}_{\mu\nu} + k^\beta \hat{\nabla}_\beta \xi^{(1)}_{\mu\nu} \\ & +k^\beta \hat{\nabla}_\beta \xi^{(1)}_{\mu\nu}]\Big\} - 2h_{\alpha\beta}\hat{R}^{\alpha}{}_{\mu\nu}{}^{\beta}\end{aligned}$$

$$\begin{aligned}\hat{\Gamma}^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} \; & + \tfrac{1}{2} g^{(B)\alpha\rho}[\hat{\nabla}_\mu h_{\rho\nu} + \hat{\nabla}_\nu h_{\rho\mu} - \hat{\nabla}_\rho h_{\mu\nu}] \\ & + \tfrac{1}{2f'}\left[\delta^\alpha_\nu \partial_\mu f' + \delta^\alpha_\mu \partial_\nu f' - g^{(B)}_{\mu\nu}\partial^\alpha f' - h_{\mu\nu}\partial^\alpha f' + g^{(B)}_{\mu\nu} h^{\alpha\beta}\partial_\beta f'\right]\end{aligned}$$

Geometric optics limit in Palatini

Leading order:

Null geodesics enjoy conformal invariance

$$\boxed{k^{\mu}\hat{\nabla}_{\mu}k_{\nu}=0}$$

Next to leading order:

$$\boxed{2k^{\beta}\hat{\nabla}_{\beta}\xi^{(0)}_{\mu\nu}+\hat{\nabla}_{\beta}k^{\beta}\xi^{(0)}_{\mu\nu}=\frac{1}{f'}t_{\mu\nu}}$$

$\xi^{(0)}_{\mu\nu}=\mathcal{A}\mathcal{A}_{\mu\nu}$ where $\mathcal{A}$ is the amplitude, defined as $\mathcal{A}=\sqrt{\xi^{*}_{\mu\nu}\xi^{*\mu\nu}}$

$$k^{\alpha}\hat{\nabla}_{\alpha}\mathcal{A}_{\mu\nu}=\frac{1}{2\mathcal{A}\gamma(f')}\Big[t_{\mu\nu}-\mathcal{A}_{\mu\nu}(t_{\rho\sigma}\mathcal{A}^{\rho\sigma})\Big]$$

$$\hat{\nabla}_{\rho}(k^{\rho}\mathcal{A}^{2})=\mathcal{A}t_{\mu\nu}\mathcal{A}^{\mu\nu}$$

Non-conservation of graviton number density:

$$\hat{\nabla}_{\mu}N^{\mu}=\mathcal{A}t_{\mu\nu}\mathcal{A}^{\mu\nu}$$

$$\xi_{\mu\nu}=\mathcal{C}_{kk}\Theta^{kk}_{\mu\nu}+\mathcal{C}_{ll}\Theta^{ll}_{\mu\nu}+\mathcal{C}_{mm}\Theta^{mm}_{\mu\nu}+\mathcal{C}_{nn}\Theta^{nn}_{\mu\nu}+\mathcal{C}_{kl}\Theta^{kl}_{\mu\nu}+\mathcal{C}_{km}\Theta^{km}_{\mu\nu}+\mathcal{C}_{kn}\Theta^{kn}_{\mu\nu}+\mathcal{C}_{ml}\Theta^{ml}_{\mu\nu}+\mathcal{C}_{nl}\Theta^{ml}_{\mu\nu}+\mathcal{C}_{mn}\Theta^{mn}_{\mu\nu}$$

$\mathcal{C}_{AB} \longrightarrow$ **Amplitude** ; $\Theta^{AB}_{\mu\nu} \longrightarrow$ **New basis tensors**

Lorentz gauge condition

$$k^{\mu}\xi^{(0)}_{\mu\nu} = 0$$

Traceless condition

$$g^{\mu\nu}\xi_{\mu\nu} = 0$$

Source terms vanish in the geometric optics limit

$$\boxed{\xi^{(0)}_{\mu\nu} = \mathcal{C}^{(0)}_{mm} m_{\mu} m_{\nu} + \mathcal{C}^{(0)}_{ll} l_{\mu} l_{\nu}}$$

$A,B=\{k^{\mu},m^{\mu},l^{\mu},n^{\mu}\}$

$$m_{\mu} l^{\mu}=1, \qquad k_{\mu} n^{\mu}=-1$$

Basis tensor

$$\Theta^{AB}_{\mu\nu}=\frac{1}{2}(A_{\mu}B_{\nu}+A_{\nu}B_{\mu})$$

$$(\Theta_{ml})_{\mu\nu}(\Theta_{ml})^{\mu\nu}=\frac{1}{2}$$

$$(\Theta_{nk})_{\mu\nu}(\Theta_{nk})^{\mu\nu}=\frac{1}{2}$$

$$(\Theta_{km})_{\mu\nu}(\Theta_{nl})^{\mu\nu}=-\frac{1}{2}$$

$$(\Theta_{kl})_{\mu\nu}(\Theta_{nm})^{\mu\nu}=-\frac{1}{2}$$

$$(\Theta_{kk})_{\mu\nu}(\Theta_{nn})^{\mu\nu}=1$$

$$(\Theta_{mm})_{\mu\nu}(\Theta_{ll})^{\mu\nu}=1$$

Dalang, C., Cusin, G. & Lagos, M. Polarization distortions of lensed gravitational waves. *Physical Review D*. **105** (2022,1), https://doi.org/10.1103 252Fphysrevd.105.024005

Evolution of GW amplitude

(Geometric Optics term with source)

$$2\hat{\mathcal{D}}\mathcal{C}_{AB} + \hat{\nabla}_\alpha \hat{k}^\alpha \mathcal{C}_{AB} = \frac{2}{f'} t_{\mu\nu} \hat{\Theta}^{\mu\nu}_{AB}$$

$$\hat{\mathcal{D}} = \tilde{k}^\mu \hat{\nabla}_\mu = \frac{D}{d\nu}$$

Palatini f(R)

$$\mathcal{C}_{AB}(\nu) = \frac{D(\nu_s)\mathcal{C}_{AB}(\nu_s)}{D(\nu)} - \frac{1}{2D(\nu)} \int_{\nu_s}^{\nu} D(\nu)[\gamma^\alpha_{\alpha\mu} k^\mu - k^\alpha \partial_\alpha \ln f'] \mathcal{C}_{AB} d\nu + \frac{1}{2D(\nu)} \int_{\nu_s}^{\nu} D(\nu) t_{\mu\nu} \Theta^{\mu\nu}_{AB} d\nu$$

Optical Scalars:

$$\mathcal{S} = \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix} + \begin{pmatrix} -\sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_1 \end{pmatrix}$$

where θ and $\sigma = \sigma_1 + i\sigma_2$ are the optical scalars.

$$\nabla_\mu k^\mu = tr\mathcal{S} = 2\theta$$

where $2\theta = \frac{2}{D_A}\frac{dD_A}{d\nu}$

In GR

$$\hat{\mathcal{C}}^{(0)}_{AB}(\nu) = \frac{\mathcal{C}^{(0)}_{AB}(\nu_s)D(\nu_s)}{D(\nu)}$$

Amplitudes evolve independently and no extra polarizations arise in GO

Conclusion:

- Evolution of GW amplitude is theory dependent
- Lensing induce errors in the estimated parameters
- Modified gravity affects the evolution of gravitational wave amplitude and also the polarizations in higher order corrections.
- The gravitational potential of structures in the line of sight depends on the theory of gravity and hence GW lensing
- Vector and scalar polarizations arise in beyond geometric optics along with the effects of Palatini $f(\hat{R})$.