EICPL lecture series on QCD for EIC

The physics of the QCD energy-momentum tensor (1/2)

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Outline

- Energy-momentum tensor
  - Definitions
  - Properties
  - Parametrization
- Four-momentum sum rule
  - Expectation value
  - Constraints on form factors
  - Mechanical equilibrium
- Mass decompositions

Energy-momentum tensor (EMT)

EMT is a key fundamental object

It is the conserved current associated with invariance under spacetime translations

$$\boxed{P^\mu = \int \mathrm{d}^3x\, T^{0\mu}(x)}$$

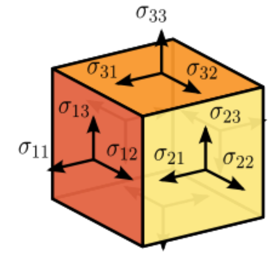
$$\phi(x+a) = e^{iP\cdot a}\phi(x)e^{-iP\cdot a} \quad \Leftrightarrow \quad i[P^\mu, \phi(x)] = \partial^\mu \phi(x)$$

It also plays the role of source for gravitation in the Einstein equations of GR

$$\boxed{R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu} = \kappa\, T_{\mu\nu}}$$

Energy-momentum tensor (EMT)

Mass, **spin** and **pressure** are all encoded in the EMT



Energy density

Momentum density

$$T^{\mu\nu} = \begin{bmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{bmatrix}$$

Shear stress

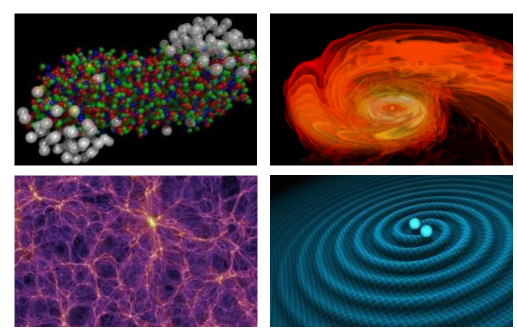
Normal stress **(pressure)**

Energy flux

Momentum flux

Central object for

- **Nucleon mechanical properties**
- **Quark-gluon plasma**
- **Relativistic hydrodynamics**
- **Stellar structure and dynamics**
- **Cosmology**
- **Gravitational waves**
- **Modified theories of gravitation**
- …



Energy-momentum tensor (EMT)

⚠ **The definition of the EMT is not unique**

Canonical EMT (Noether's theorem)

$$\boxed{T^{\mu\nu}_{\text{can}} = \sum_a \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\partial^\nu\phi_a - g^{\mu\nu}\mathcal{L}}$$

Usually neither symmetric (for fields with non-zero spin)

nor gauge invariant (when field gradients do not transform covariantly under gauge transformations)

However, P^μ **is unique and gauge invariant !**

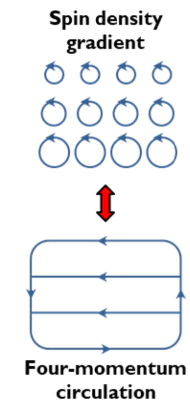
« Improved » EMT (relocalization of energy and momentum)

$$\boxed{T^{\mu\nu}_{\text{new}} = T^{\mu\nu}_{\text{can}} + \partial_\alpha G^{\alpha\mu\nu}} \qquad G^{\alpha\mu\nu} = -G^{\mu\alpha\nu}$$

$$\Rightarrow \quad P^\mu = \int \mathrm{d}^3x\, T^{0\mu}_{\text{new}} = \int \mathrm{d}^3x\, T^{0\mu}_{\text{can}}$$

Energy-momentum tensor (EMT)

	Gauge invariant	Symmetric
Canonical	✗	✗
Kinetic	✓	✗
Belinfante	✓	✓



Gauge invariance is a necessity

Symmetry under exchange of indices is only motivated by GR …

<u>NB:</u> In GR one *assumes* the symmetry of the metric

or the absence of torsion

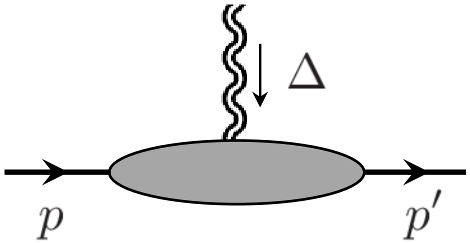
or the purely orbital form of AM

Kinetic EMT (most natural one in QFT)

$$J^{\mu\alpha\beta} = \underbrace{x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha}}_{\text{Orbital}} + \underbrace{S^{\mu\alpha\beta}}_{\text{Intrinsic}}$$

$$\begin{aligned} \partial_\mu T^{\mu\nu} &= 0 \\ \partial_\mu J^{\mu\alpha\beta} &= 0 \end{aligned} \quad \Rightarrow \quad \boxed{T^{\alpha\beta} - T^{\beta\alpha} = -\partial_\mu S^{\mu\alpha\beta}}$$

Gravitational form factors (GFFs)



Poincaré symmetry constrains the form of the EMT matrix elements

Symmetrized variables $P = \frac{p' + p}{2}, \qquad \Delta = p' - p, \qquad t = \Delta^2$

$$p'^2 = p^2 = M^2 \Rightarrow \boxed{P \cdot \Delta = 0}$$

Spin-0 target $T^{\mu\nu} = \sum_a T_a^{\mu\nu}$

$$\langle p'|T_a^{\mu\nu}(0)|p\rangle = 2M\left[\frac{P^\mu P^\nu}{M}A_a(t) + \frac{\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2}{M}C_a(t) + M g^{\mu\nu}\bar{C}_a(t)\right]$$

Non-conserved

$$0 = \langle p'|\partial_\mu T^{\mu\nu}(x)|p\rangle = i\Delta_\mu\langle p'|T^{\mu\nu}(x)|p\rangle \Rightarrow \sum_a \bar{C}_a(t) = 0$$

Translation symmetry $\phi(x+a) = e^{iP\cdot a}\phi(x)e^{-iP\cdot a}$

Gravitational form factors (GFFs)

Spin-1/2 target

$$\langle p', s'|T_a^{\mu\nu}(0)|p, s\rangle = \overline{u}(p', s')\Gamma_a^{\mu\nu}(P, \Delta)u(p, s)$$

$$\Gamma_a^{\mu\nu}(P,\Delta) = \frac{P^\mu P^\nu}{M} A_a(t) + \frac{\Delta^\mu \Delta^\nu - g^{\mu\nu}\Delta^2}{M} C_a(t) + M g^{\mu\nu} \bar{C}_a(t) + \frac{P^{\{\mu} i\sigma^{\nu\}\lambda}\Delta_\lambda}{2M} J_a(t) - \frac{P^{[\mu} i\sigma^{\nu]\lambda}\Delta_\lambda}{2M} S_a(t)$$

$$x^{\{\mu}y^{\nu\}} = x^\mu y^\nu + x^\nu y^\mu$$
$$x^{[\mu}y^{\nu]} = x^\mu y^\nu - x^\nu y^\mu$$

NB: Because of the Dirac equation, alternative but equivalent parametrizations may *look* quite different !

Gordon identity

$$\overline{u}(p', s')\gamma^\mu u(p, s) = \overline{u}(p', s')\left[\frac{P^\mu}{M} + \frac{i\sigma^{\mu\nu}\Delta_\nu}{2M}\right] u(p, s)$$

Four-momentum conservation

Expectation value

$$\langle P^\mu_a\rangle = \frac{\langle p|P^\mu_a|p\rangle}{\langle p|p\rangle} = \underbrace{\frac{\int \mathrm{d}^3r}{(2\pi)^3\delta^{(3)}(\mathbf{0})}}_{=1} \frac{\langle p|T^{0\mu}_a(0)|p\rangle}{2p^0}$$

$$\langle p|T^{\mu\nu}_a(0)|p\rangle = 2p^\mu p^\nu A_a(0) + 2M^2 g^{\mu\nu}\bar{C}_a(0)$$

$$\Rightarrow \quad \langle P^\mu_a\rangle = p^\mu A_a(0) + \frac{M^2}{p^0} g^{0\mu}\bar{C}_a(0)$$

Not a four-vector !

(unless state is massless)

Light-front version

$p^\pm = (p^0 \pm p^3)/\sqrt{2}$

$$\langle P^\mu_{a,\mathrm{LF}}\rangle = p^\mu A_a(0) + \frac{M^2}{p^+} g^{+\mu}\bar{C}_a(0) \quad \underset{g^{++}=0}{\Rightarrow} \quad \langle P^+_{a,\mathrm{LF}}\rangle = \underbrace{A_a(0)}_{=\langle x\rangle_a}\, p^+$$

Deep-inelastic scattering

Four-momentum sum rules

$$p^\mu = \sum_a \langle P^\mu_a\rangle \quad \Rightarrow \quad \boxed{\begin{aligned} &\sum_a A_a(0) = 1 \\ &\sum_a \bar{C}_a(0) = 0 \end{aligned}}$$

Why two sum rules ?

What is the meaning of $\bar{C}_a(0)$?

Mechanical equilibrium

Physical interpretation is simpler in target rest frame

$$\frac{\langle p_{\text{rest}}|\int \mathrm{d}^3x\, T_a^{\mu\nu}(x)|p_{\text{rest}}\rangle}{\langle p_{\text{rest}}|p_{\text{rest}}\rangle} = M \left(\begin{array}{c|ccc} A_a(0)+\bar{C}_a(0) & 0 & 0 & 0 \\ \hline 0 & -\bar{C}_a(0) & 0 & 0 \\ 0 & 0 & -\bar{C}_a(0) & 0 \\ 0 & 0 & 0 & -\bar{C}_a(0) \end{array}\right)$$

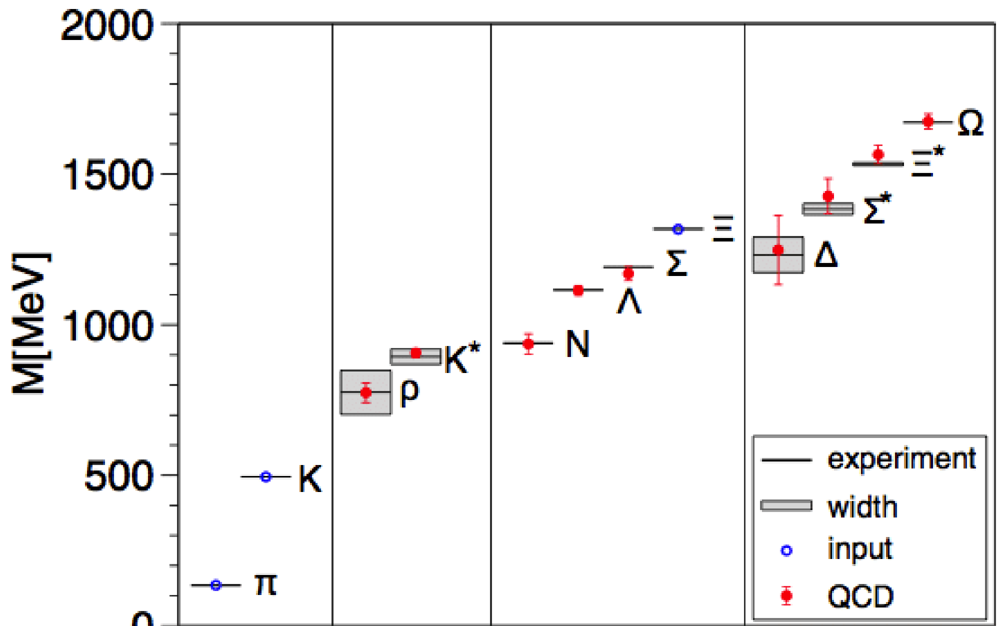
$$\Leftrightarrow \left(\begin{array}{c|ccc} \varepsilon_a & 0 & 0 & 0 \\ \hline 0 & p_a & 0 & 0 \\ 0 & 0 & p_a & 0 \\ 0 & 0 & 0 & p_a \end{array}\right) V$$

➡ $-\bar{C}_a(0)$ **measures the average stress (or pressure) exerted by subsystem** a

Mechanical equilibrium implies $\sum_a p_a = 0 \quad \Rightarrow \quad \sum_a \bar{C}_a(0) = 0$

Hadron spectroscopy

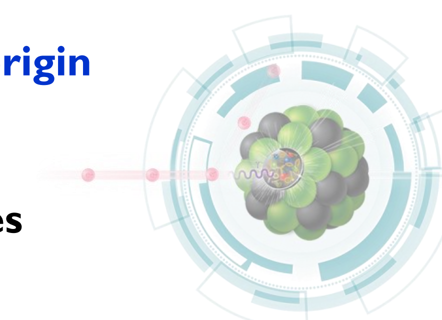
Lattice QCD reproduces very well the light hadron spectrum



[Durr *et al.*, Science 322 (2008)]

… but this does not tell us much about the origin of the hadron masses

One of the goals of the EIC is to provide clues for this fundamental question



What is mass ?

In relativity, there are essentially two equivalent definitions of mass

« Formal » definition $\boxed{p^\mu p_\mu = M^2}$

A global Lorentz-invariant quantity characterizing the physical system

Not additive ! $p^\mu = p^\mu_q + p^\mu_g \quad \Rightarrow \quad M^2 = p_q^2 + p_g^2 + 2\, p_q \cdot p_g$

« Physical » definition $\boxed{p^\mu u_\mu = M}$

Proper inertia (i.e. rest-frame energy) of the system

CM four-velocity $u^\mu = p^\mu / M$

Additive $p^\mu = p^\mu_q + p^\mu_g \quad \Rightarrow \quad M = p_q \cdot u + p_g \cdot u$

$$= \quad p_q^0 \quad + \quad p_g^0$$

Rest frame

What is mass ?

Poincaré symmetry tells us that $\langle p|T^{\mu\nu}(0)|p\rangle = 2p^\mu p^\nu$ $\langle p'|p\rangle = 2p^0(2\pi)^3\delta^{(3)}(\vec{p}\,'-\vec{p})$

« Formal » definition $\langle p|T^\mu{}_\mu(0)|p\rangle = 2M^2$

« Physical » definition $\langle p_{\rm rest}|T^{00}(0)|p_{\rm rest}\rangle = 2M^2$

In the literature one often introduces an *ad hoc* $\frac{1}{2M}$ normalization factor

Only expectation values correspond to physical quantities !

(State normalization is a pure human convention)

« Formal » definition

$$\frac{\langle p|\int \mathrm{d}\mathcal{V}\, T^\mu{}_\mu(x)|p\rangle}{\langle p|p\rangle} = M, \qquad \mathrm{d}\mathcal{V} = u^0\,\mathrm{d}^3x$$

Proper volume element

« Physical » definition

$$\langle H\rangle\big|_{\rm rest} = \frac{\langle p_{\rm rest}|\int \mathrm{d}^3x\, T^{00}(x)|p_{\rm rest}\rangle}{\langle p_{\rm rest}|p_{\rm rest}\rangle} = M$$

$$\langle P^\mu\rangle u_\mu = \frac{\langle p|\int \mathrm{d}^3x\, T^{0\mu}(x)|p\rangle}{\langle p|p\rangle}\, u_\mu = M$$

$u^\mu = p^\mu/M$

Mass decomposition

Trace decomposition

The behavior under spacetime dilations is determined by

$$j^\mu_D = T^{\mu\nu} x_\nu \qquad \Rightarrow \qquad \partial_\mu j^\mu_D = T^\mu{}_\mu$$

Quark mass and quantum corrections break conformal symmetry

$$T^\mu{}_\mu = \underbrace{\left[\frac{\beta(g)}{2g} G^2 + \gamma_m \, \overline{\psi} m \psi\right]}_{\text{Trace anomaly}} + \overline{\psi} m \psi$$

Quark mass matrix

$$\underbrace{\langle \textstyle\int \mathrm{d}\mathcal{V} \, \frac{\beta(g)}{2g} G^2 \rangle}_{\sim 90\%} + \underbrace{\langle \textstyle\int \mathrm{d}\mathcal{V} \, (1+\gamma_m) \, \overline{\psi} m \psi \rangle}_{\sim 10\%} = M$$

$\mu = 2\,\mathrm{GeV}$

~ 90% (to be measured)

~ 10% (measurement to be improved)

$\langle \overline{\psi} m \psi \rangle$ **Nucleon-meson scattering**

$\langle G^2 \rangle$ **Near-threshold heavy meson production**

Based on this picture, one often concludes that most of the hadron mass comes from gluons !

Mass decomposition

Trace decomposition

The physical interpretation of the « quark » and « gluon » contributions is however not so clear …

It is clearer to work in the rest frame

$$\underbrace{\langle \int \mathrm{d}^3x\, T^\mu{}_\mu \rangle}_{=M} = \underbrace{\langle \int \mathrm{d}^3x\, T^{00} \rangle}_{=M} - \sum_i \underbrace{\langle \int \mathrm{d}^3x\, T^{ii} \rangle}_{=0}$$

Mechanical equilibrium

$$T^{\mu\nu} = \sum_a T^{\mu\nu}_a$$

$$\Rightarrow \quad \underbrace{\langle \int \mathrm{d}^3x\, T^\mu_{a\mu} \rangle}_{\text{Can be negative !}} = \underbrace{\langle \int \mathrm{d}^3x\, T^{00}_a \rangle}_{=\langle H_a \rangle} - \sum_i \underbrace{\langle \int \mathrm{d}^3x\, T^{ii}_a \rangle}_{\neq 0} \qquad a = q, g$$

Partial pressure-volume work

The « gluon » contribution is enhanced because the gluon pressure-volume work is negative (attractive forces)

Mass decomposition

Energy decomposition

Renormalized QCD operators $T^{\mu\nu} = T_q^{\mu\nu} + T_g^{\mu\nu}$

$$T_q^{\mu\nu} = \overline{\psi}\gamma^\mu \tfrac{i}{2}\overleftrightarrow{D}^\nu\psi$$
$$T_g^{\mu\nu} = -G^{\mu\lambda}G^\nu{}_\lambda + \tfrac{1}{4}\,g^{\mu\nu}\,G^2$$

Rest-frame energy

$$\begin{aligned} M &= \langle H_q\rangle + \langle H_g\rangle \\ &= \underbrace{\langle\textstyle\int \mathrm{d}^3x\,\overline{\psi}\gamma^0 iD^0\psi\rangle}_{[A_q(0)+\bar{C}_q(0)]\,M} + \underbrace{\langle\textstyle\int \mathrm{d}^3x\,\tfrac{1}{2}(\vec{E}^2+\vec{B}^2)\rangle}_{[A_g(0)+\bar{C}_g(0)]\,M} \end{aligned}$$

$$A_a(0) = \langle x\rangle_a$$
$$\bar{C}_a(0) = f_a(\langle x\rangle_q, \langle\overline{\psi}m\psi\rangle)$$

Known but scheme and scale-dependent !

Refinement

$$M = \underbrace{\langle\textstyle\int \mathrm{d}^3x\,\overline{\psi}\vec{\gamma}\cdot i\vec{D}\psi\rangle}_{\sim 21\%\,(\overline{\text{MS}}),\ \sim 38\%\,(\text{D2})} + \underbrace{\langle\textstyle\int \mathrm{d}^3x\,\overline{\psi}m\psi\rangle}_{\sim 8\%\,(\overline{\text{MS}}),\ \sim 8\%\,(\text{D2})} + \underbrace{\langle\textstyle\int \mathrm{d}^3x\,\tfrac{1}{2}(\vec{E}^2+\vec{B}^2)\rangle}_{\sim 71\%\,(\overline{\text{MS}}),\ \sim 54\%\,(\text{D2})}$$

$\mu = 2\,\text{GeV}$

Mass decomposition

Ji's decomposition

It combines features from both trace and energy decompositions

Step 1 $$T^{\mu\nu} = \bar{T}^{\mu\nu} + \hat{T}^{\mu\nu}$$

$$\bar{T}^{\mu\nu} = T^{\mu\nu} - \frac{1}{4}\, g^{\mu\nu} T^{\alpha}{}_{\alpha} \qquad \text{Twist-2}$$

$$\hat{T}^{\mu\nu} = \frac{1}{4}\, g^{\mu\nu} T^{\alpha}{}_{\alpha} \qquad \text{Twist-4}$$

Poincaré symmetry ensures that this separation is scheme and scale-independent !

Step 2

$$\bar{T}^{\mu\nu} = \bar{T}^{\mu\nu}_q + \bar{T}^{\mu\nu}_g$$

$$\hat{T}^{\mu\nu} = \hat{T}^{\mu\nu}_m + \hat{T}^{\mu\nu}_{\mathrm{a}}$$

$$\hat{T}^{\mu\nu}_m = \frac{1}{4}\, g^{\mu\nu}\, \overline{\psi} m \psi$$

$$\hat{T}^{\mu\nu}_{\mathrm{a}} = \frac{1}{4}\, g^{\mu\nu} \left[\frac{\beta(g)}{2g}\, G^2 + \gamma_m\, \overline{\psi} m \psi\right]$$

Rest-frame energy

$$M = \langle \textstyle\int \mathrm{d}^3x\, (\bar{T}^{00}_q - \frac{3}{4}\, \overline{\psi} m \psi)\rangle + \langle \textstyle\int \mathrm{d}^3x\, \overline{\psi} m \psi\rangle + \langle \textstyle\int \mathrm{d}^3x\, \bar{T}^{00}_g\rangle + \langle \textstyle\int \mathrm{d}^3x\, \hat{T}^{00}_{\mathrm{a}}\rangle$$

« Quantum anomalous energy »

Mass decomposition

Ji's decomposition

The physical interpretation of the « quark » and « gluon » contributions is however not so clear …

$$T^{00}_a = \underbrace{\bar{T}^{00}_a}_{=\frac{3}{4}T^{00}_a + \frac{1}{4}\sum_i T^{ii}_a} + \underbrace{\hat{T}^{00}_a}_{=\frac{1}{4}T^{00}_a - \frac{1}{4}\sum_i T^{ii}_a} \qquad a = q, g$$

$$\sum_i \langle \int \mathrm{d}^3x\, T^{ii}_a \rangle \neq 0$$

Partial pressure-volume work

Also, it is tempting to write

$$\bar{T}^{00}_q - \frac{3}{4}\overline{\psi}m\psi \stackrel{?}{=} \overline{\psi}\vec{\gamma}\cdot i\vec{D}\psi$$

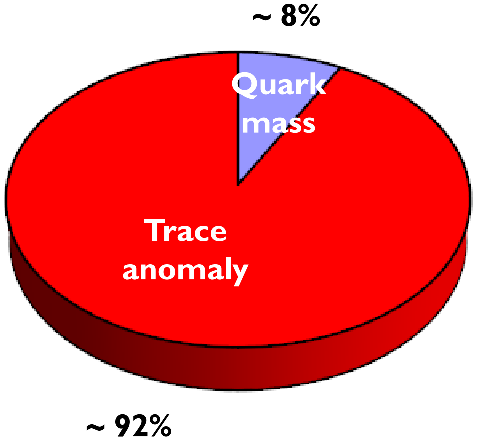
$$\bar{T}^{00}_g \stackrel{?}{=} \frac{1}{2}(\vec{E}^2 + \vec{B}^2)$$

… but there is no scheme where both are simultaneously justified !

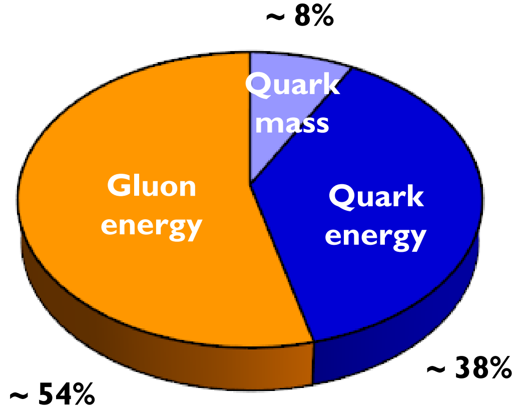
Mass decomposition (in D2 scheme)

Trace decomposition

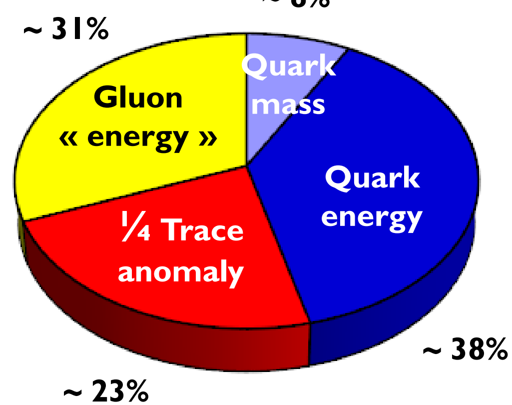
$\mu = 2\,\mathrm{GeV}$



Energy decomposition



Ji's decomposition



	Trace decomposition	Energy decomposition	Ji's decomposition
Scale dependence	Simple	Complicated	Simple
Physical interpretation	Complicated	Simple	Complicated

Some references

- **Leader, Lorcé, PR541 (2014) 3, 163**
- **Polyakov, Schweitzer, IJMPA33 (2018) 26, 1830025**
- **Ji, FP (*Beijing*) 16 (2021) 6, 64601**
- **Ji, Liu, Schäfer, NPB971 (2021) 115537**
- **Lorcé, Metz, Pasquini, Rodini, JHEP11 (2021) 121**

… and references therein !